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On G-foliations

by ROBERT WOLAK (Kraków)

Franciszek Leja in memoriam

Abstract. In this short note the author looks at how properties of a pseudogroup of automorphisms of a given G-structure influence topological and geometrical properties of G-foliations modelled on this G-structure.

0. Introduction. All the geometrical objects considered in this paper are smooth, i.e., of C^∞ differentiability.

Let N be a q-manifold and Γ a pseudogroup of diffeomorphisms of the manifold N. For any open subset U of N by $\Gamma(U)$ we denote the set $\{f \in \Gamma\colon \operatorname{dom} f = U\}$. We say that the pseudogroup Γ has the *property* E_k, k any integer, if for any point x of the manifold the spaces $\{j_x^k f\colon f \in \Gamma(U_n)\}$ are equal for some sequence of open subsets U_n such that $U_0 = N$ and $\bigcap U_n = \{x\}$.

EXAMPLES. 1. Any pseudogroup of diffeomorphisms generated by a group of global diffeomorphisms has the property E_k for any k.

2. Let $B(N, G)$ be a regular G-structure on a simple connected compact manifold N for a group G of finite type k. The pseudogroup Γ generated by the flows of infinitesimal automorphisms of this G-structure has the property E_k. In fact, the sheaf of germs of infinitesimal automorphisms of the G-structure $B(N, G)$ is constant (cf. [4]). Thus any germ of an infinitesimal automorphism can be extended to a global infinitesimal automorphism of $B(N, G)$, hence any diffeomorphism from some flow, with the domain small enough, can be extended to a global diffeomorphism, so the pseudogroup has the property E_k.

Let G be a closed subgroup of the linear group $\mathrm{GL}(q)$ of finite type k, let $B(N, G)$ be a G-structure on the manifold N. Let Γ be a pseudogroup of automorphisms of the G-structure $B(N, G)$ having the property E_k. Let F be a Γ-foliation on an n-manifold M in the sense of Haefliger (cf. [5]). Then we have the following

THEOREM 1. *The lifted foliation $\tilde{F}$ to the universal covering $\tilde{M}$ of the manifold M is simple, i.e., defined by a global submersion.*

THEOREM 2. *Assume that the manifold M is compact. The growth of the leaves of the foliation F is dominated by the growth of the fundamental group $\pi_1(M)$ of the manifold M.*

THEOREM 3. *The leaves of a complete foliation F have the common universal covering space. The space of leaves of the foliation F is homeomorphic to the orbit space of some action of a group on a covering space of the manifold N.*

We would like to express our gratitude to Robert A. Blumenthal for providing the preprints of papers of his which have been the inspiration of this note.

1. G-foliations. Let F be a G-foliation on the manifold M (cf. [2]). Then the normal bundle $N(M, F)$ of the foliation F admits a reduction of the structure group to the group G. Denote by $B(M, G; F)$ the reduction to the group G of the linear frame bundle $L(M; F)$ of the normal bundle $N(M; F)$.

On $L(M; F)$ and $B(M, G; F)$ we define an R^q-valued 1-form θ, the fundamental form, as follows (cf. [7], [8]):

$$T_p(L(M; F)) \xrightarrow{d\pi} T_{\pi p} M \to N_{\pi p}(M; \mathrm{F}) \xrightarrow{p^{-1}} R^q,$$

where $\pi\colon L(M; F) \to M$ is the natural projection. Then

$$R_a^* \theta = a^{-1} \theta \quad \text{for any } a \in G,$$

and

$$L_{A^*}\theta = -A \cdot \theta \quad \text{for any } A \in \mathrm{Lie}(G) = \ .$$

On the manifold $B(M, G; F)$ one defines a foliation F_1 of dimension $n-q$ as follows:

$$F_1 = \{X \in TB(M, G; F)\colon i_X \theta = 0,\ i_X d\theta = 0\}$$

(cf. [7], [8]).

Let us choose a splitting s of the exact sequence

$$0 \to F \to TM \overset{s}{\leftrightarrows} N(M, F) \to 0.$$

Take any q-dimensional subspace H_p at a point p in $T_p B(M, G; F)$ such that its projection $d\pi H_p$ onto $T_{\pi p} M$ is equal to $s(N_{\pi p}(M, F))$. From now on we shall consider only such subspaces and call them the horizontal subspaces.

Let us take two horizontal subspaces H_1 and H_2 at a point p of $B(M, G; F)$. Take any vector v of R^q. Then there exist the unique vectors $X_1 \in H_1$ and $X_2 \in H_2$ such that $\theta(X_1) = \theta(X_2) = v$. Thus $\theta(X_1 - X_2) = 0$ and $d\pi(X_1 - X_2) = 0$. Therefore $X_1 - X_2 = A^*$ for some vector $A \in \underline{g}$. In this way, for any two horizontal subspaces H_1 and H_2 at a point p we define a linear mapping $S_{H_1 H_2}\colon R^q \to \underline{g}$ by putting $S_{H_1 H_2}(v) = A$ such that if $X_1 \in H_1$, $X_2 \in H_2$ and $\theta(X_1) = \theta(X_2) = v$, then $X_1 - X_2 = A^*$.

If a linear mapping $S\colon R^q \to \underline{g}$ and a horizontal space H at a point p are given we define another horizontal space H' at p in the following way:

$$H' = \{X' = X + S(v)^*\colon\ \theta(X) = v,\ X \in H\}.$$

Then

$$X' - X = S(v)^* \quad \text{and} \quad S_{H'H} = S.$$

For any horizontal subspace H we can define the following mapping $c_H\colon R^q \wedge R^q \to R^q$.

$$c_H(u \wedge v) = \langle X \wedge Y, d\theta\rangle \quad \text{for } X;\, Y \in H \text{ such that } \theta(X) = u,\ \theta(Y) = v.$$

Then comparing the mappings c_H and $c_{H'}$ defined for two different horizontal subspaces H and H' at a given point p we have the following:

$$\begin{aligned} c_H(u \wedge v) - c_{H'}(u \wedge v) &= \langle X \wedge Y, d\theta\rangle - \langle X' \wedge Y', d\theta\rangle \\ &= \langle (X - X') \wedge Y, d\theta\rangle - \langle X' \wedge (Y' - Y), d\theta\rangle \\ &= \langle S_{HH'}(u)^* \wedge Y, d\theta\rangle - \langle X' \wedge S_{H'H}(v)^*, d\theta\rangle \\ &= -\tfrac{1}{2}\big(S_{HH'}(u)\cdot\theta(Y) + S_{H'H}(v)\cdot\theta(X)\big) \\ &= \tfrac{1}{2}\big(S_{H'H}(u)(v) - S_{H'H}(v)(u)\big) = \partial S_{H'H}(u \wedge v), \end{aligned}$$

where ∂ is the antisymmetrization operator.

Therefore for any $p \in B(M, G; F)$ the mappings c_H define the unique class $c(p)$ in $\mathrm{Hom}(R^q \wedge R^q, R^q)/\partial\,\mathrm{Hom}(R^q, \underline{g})$. We call this tensor the structure tensor of the transverse G-structure $B(M, G; F)$. Since $\partial\,\mathrm{Hom}(R^q; \underline{g})$ is a vector subspace of $\mathrm{Hom}(R^q \wedge R^q, R^q)$, we can choose a supplementary subspace C to $\partial\,\mathrm{Hom}(R^q, \underline{g})$ in $\mathrm{Hom}(R^q \wedge R^q, R^q)$. This choice, at a point p of $B(M, G; F)$, distinguishes a family of horizontal subspaces H such that $c_H \in C$. If H_1 and H_2 are two such horizontal subspaces at a point p

$$c_{H_1}(u \wedge v) - c_{H_2}(u \wedge v) = \partial S_{H_2 H_1}(u \wedge v) = 0,$$

so

$$S_{H_1 H_2} \in g^{(1)} - g^{(1)} \quad \text{the first prolongation of the Lie algebra } \underline{g}.$$

Thus the choice of C defines a $G^{(1)}$-structure on $L\big(B(M, G; F), F_1\big)$, where

$$G^{(1)} = \left\{\begin{pmatrix}\mathrm{id} & 0\\ h & \mathrm{id}\end{pmatrix} \in \mathrm{GL}(R^q + \underline{g})\colon\ h \in g^{(1)}\right\}.$$

We call this $G^{(1)}$-structure the *first prolongation of the transverse G-structure* $B(M, G; F)$ and denote it by $B^1(M, G; F)$. By induction, we define further prolongations, i.e., $B^{k+1}(M, G; F)$ is the first prolongation of the transverse G^k-structure $B^k(M, G; F)$. If the group G is of finite type k, after a finite number k of steps, we get an $\{e\}$-structure $B^k(M, G; F)$, so the foliation F_k is transversely parallelisable.

Remark. Let a transverse G-structure be projectible (cf. [7], [8]). With a suitable choice of the splitting, if the diagram

$$\begin{tikzcd} B(M, G; F) \arrow[r] \arrow[d] & B(N, G) \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

is commutative, we have the following equality for the structure tensors: $f^* c_N = c_M$.

2. Proof of Theorem 1. The proof is based on the following lemma:

LEMMA 1. *Let G be a closed subgroup of the group $\mathrm{GL}(q)$ of finite type and let Γ be a pseudogroup of automorphisms of a G-structure $B(N, G)$. Then the restriction mapping $\Gamma(U_m) \to \Gamma(U_{m+1})$ is bijective. In particular, any automorphism of the G-structure belonging to Γ can be extended to a global one.*

Proof. Let $f \in \Gamma(U_{m+1})$. Then the lift $\bar{f}$ to $B^k(N, G)$ preserves the $\{e\}$-structure on $B^k(N, G)$, i.e., the parallelism $\{X_1, \ldots, X_r\}$ of $B^k(N, G)$ (cf. [10]). Thus

$$(*) \qquad \bar{f} \exp tX_i = \exp tX_i \bar{f} \quad \text{for any } t \text{ and } i = 1, \ldots, r$$

whenever $\exp tX_i$ is defined.

Let g be an element of $\Gamma(U_m)$ such that $j_x^k g = j_x^k f$. Then for the lifted mapping $\bar{g}$ to $B^k(N, G)$ we have the equality $\bar{g}(p) = \bar{f}(p)$ for any p over x. The set $A = \{p \in B^k(N, G)\colon \bar{f}(p) = \bar{g}(p)\}$ is nonempty and closed. It is open as well because of $(*)$. The uniqueness is proved in the same way.

Let $\{U_i, f_i, g_{ij}\}$ be a Γ-cocycle defining the given Γ-foliation. For each g_{ij}, by Lemma 1, there exists the unique global diffeomorphism $\bar{g}_{ij}$ of the pseudogroup Γ extending g_{ij}.

Let $\underline{H} = \{(L_g f_i)_x \text{ for } x \in U_i,\ g \in \Gamma\}$, where $(\ )_x$ denotes the germ at x. The space $\underline{H} \to M$ with the natural projection w and the sheaf topology admits a C^∞-manifold structure of dimension n. Any connected component of $\underline{H}$ is a cover space of M. Let us take such a component $\hat{M}$. Then there exists a natural equivariant submersion, denoted by f, of $\hat{M}$ into N

$$\hat{M} \ni (L_g f_i)_x \mapsto g f_i(x) \in N.$$

The lifted foliation $\hat{F}$ to $\hat{M}$ is just the foliation defined by this submersion.

3. Proof of Theorem 2. Lemma 1 and the way the space $\underline{H}$ has been constructed allows us to use precisely the methods of R. Blumenthal (cf. [1]). We leave it to the reader to fill the details.

4. Complete G-foliations.

DEFINITION. Let G be a closed subgroup of $\mathrm{GL}(q)$ of finite type k. We say that a G-foliation is *complete* if the transverse parallelism of $B^k(M, G; F)$ is complete and foliation preserving.

LEMMA 2. *Let the k-th structure tensor of $B(N, G)$ be zero, and let the transverse parallelism be complete. Then the G-foliation F is complete.*

Proof. Since $(g^{(k-1)})^{(1)} = g^{(k)} = 0$, the connections are determined by their torsion tensors (cf. [4]). Therefore the torsion-free connection on $B^{k-1}(N, G)$ is unique, and therefore it determines a basic connection on $B^{k-1}(M, G; F)$. Thus the fundamental horizontal and vertical vector fields on $B^{k-1}(M, G; F)$ define a transverse parallelism. One can easily prove, using the same methods as in [7], that they are infinitesimal automorphisms of the foliation F_k.

Proof of Theorem 3. By Theorem 1 there exists a cover $\hat{M}$ of the manifold M such that the lifted foliation $\hat{F}$ is defined by a global submersion $f\colon \hat{M} \to N$. Thus the foliation $\hat{F}_k$ of $B^{k-1}(\hat{M}, G; \hat{F})$ is defined by the submersion f^k

$$\begin{array}{ccc} B^{k-1}(\hat{M}, G; \hat{F}) & \xrightarrow{f^k} & B^{k-1}(N, G) \\ \downarrow & & \downarrow \\ \hat{M} & \xrightarrow{f} & N \end{array}$$

The space $B^{k-1}(\hat{M}, G; \hat{F})$ is a cover space of $B^{k-1}(M, G; F)$ and the foliation $\hat{F}_k$ is the lifted foliation of F_k. The foliation $\hat{F}_k$ has closed leaves and is transversely parallelisable by its global infinitesimal automorphisms, since the foliation F_k is. These vector fields are complete as well. By Lemma 1 of [7] $B^{k-1}(\hat{M}, G; \hat{F})$ is a locally trivial fibre bundle over the Hausdorff manifold $B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k$. Thus the leaves of the foliation F have a common universal covering space.

Since the transverse parallelism of $\hat{F}_k$ is by infinitesimal automorphisms, it projects to a complete parallelism of $B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k$ and is mapped by the induced mapping $\bar{f}_k$ onto the parallelism of $B^{k-1}(N, G)$. On each manifold $B^{k-1}(N, G)$ and $B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k$ we define a connection which preserves the parallelism. Applying Theorem 3 of [6] we obtain that the mapping $\bar{f}_k\colon B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k \to B^{k-1}(N, G)$ is a covering mapping. Therefore we have the following commutative diagram

$$\begin{array}{ccccc} B^{k-1}(\hat{M}, G; \hat{F}) & \longrightarrow & B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k & \xrightarrow{\bar{f}_k} & B^{k-1}(N, G) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ \hat{M} & \longrightarrow & \hat{M}/\hat{F} & \xrightarrow{\bar{f}} & N \end{array}$$

where $\hat{M}/\hat{F}$ is a T_1 manifold according to Palais [9]. Hence the mapping $\bar{f}$ is a local homeomorphism. The next step is to show that $\bar{f}$ is a covering. To prove this we need only to show that $\bar{f}$ has the property of lifting of curves. Since $B^{k-1}(N, G) \to N$ is a principal fibre bundle, for any curve γ in N there is a horizontal lift $\tilde{\gamma}$ of the curve γ to $B^{k-1}(N, G)$. Because $\bar{f}_k$ is a covering mapping, the curve $\tilde{\gamma}$ can be lifted to a curve $\bar{\gamma}$ in $B^{k-1}(\hat{M}, G; \hat{F})/\hat{F}_k$. Thus the curve $\bar{\pi}\bar{\gamma}$ is a lift of γ to $\hat{M}/\hat{F}$. The choice of a point y in $\hat{f}^{-1}(\gamma(0))$ forces the following choices in the two liftings executed: $\bar{\pi}^{-1}(y) \ni \bar{y}$ and $\bar{f}_k(\bar{y}) \in \pi_0^{-1}(\gamma(0))$. Therefore the manifold $\hat{M}/\hat{F}$ is a Hausdorff manifold. We denote the manifold $\hat{M}/\hat{F}$ by $\hat{N}$.

Since the mapping $\hat{f}$ is a covering mapping, we can lift the pseudogroup Γ to $\hat{N}$ – denote the lifted pseudogroup by $\hat{\Gamma}$. If the pseudogroup Γ has the property E_k, then the lifted pseudogroup has this property as well. The foliation F can be considered as a $\hat{\Gamma}$-foliation.

Let us choose a point x_0 of the manifold M. Any loop at x_0 defines an element of $\hat{\Gamma}(\hat{N})$ in the following way. The loop γ can be covered by a finite number of sets $U_{i_1}, \ldots, U_{i_s}$, $U_{i_1} = U_{i_s}$. We can choose a sequence of numbers $t_i \in [0, 1]$, $i = 0, \ldots, s$, such that $t_0 = 0$, $\gamma(t_i) \in U_{i_i} \cap U_{i_{i+1}}$, $i = 1, \ldots, s-1$, $t_s = 1$. Then

$$f_{i_1}(\gamma(t_1)) = g_{i_1 i_2} f_{i_2}(\gamma(t_1)).$$

Because of the uniqueness of the extension of $g_{i_1 i_2}$, the choice of the global automorphisms $g_{i_1 i_2}$ does not depend on a choice of t_1. Doing it step by step we get

$$f_{i_j}(\gamma(t_j)) = g_{i_j i_{j+1}} f_{i_{j+1}}(\gamma(t_j)).$$

Thus

$$g_{i_1 i_2} \cdots g_{i_{j-1} i_j} f_{i_j}(\gamma(t_j)) = g_{i_1 i_2} \cdots g_{i_j i_{j+1}} f_{i_{j+1}}(\gamma(t_j)),$$

and the curve $\tilde{\gamma}$: $t \to (g_{i_1 i_2} \cdots g_{i_{j-1} i_j} f_{i_j})(\gamma(t))$, $t \in [t_{j-1}, t_j]$, is the lift of the curve γ to $\underline{H}$ at $(f_{i_1})(\gamma(0))$. Therefore $\tilde{\gamma}(1) = (g_{i_1 i_2} \cdots g_{i_s i_1} f_{i_1})(\gamma(1))$. If γ' is a loop at x_0 homotopic to γ we get

$$(g_{i'_1 i'_2} \cdots g_{i'_t i'_1})(f_{i_1}(x_0)) = (g_{i_1 i_2} \cdots g_{i_s i_1})(f_{i_1}(x_0)).$$

By Lemma 1, $g_{i'_1 i'_2} \cdots g_{i'_t i'_1} = g_{i_1 i_2} \cdots g_{i_s i_1}$. Thus the correspondence defined above induces a homomorphisms h of the groups

$$h\colon \pi_1(M, x_0) \to \operatorname{Aut}(B(\hat{N}, G))$$

which is defined up to conjugation.

If we denote the $\operatorname{im} h$ by K, using the standard argument (cf. [1], [3]) we can show that the space of leaves of the foliation F is homeomorphic to the space $\hat{N}/K$.

Corollary. *Let L be a leaf of the foliation F. Denote the corresponding orbit on $\hat{N}$* by *K_L. Then*

(i) *L is proper iff K_L is discrete,*

(ii) *L is closed iff K_L is discrete and closed,*

(iii) *L is dense iff K_L is dense.*

The proof is straightforward.

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INSTYTUT MATEMATYKI
UNIWERSYTETU JAGIELLOŃSKIEGO
KRAKÓW

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