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Restricted homogeneity implies bi-additivity

by T. M. K. DAVISON (Hamilton, Ontario, Canada)

Abstract. Let R be a commutative ring with identity, and let M be an R-module. Suppose $F\colon M \times M \to R$ satisfies

$$F(x, y)+F(x+y, z) = F(x, y+z)+F(y, z) \tag{1}$$

and

$$F(0, 0) = 0. \tag{2}$$

Two results are proved.

THEOREM 1. *Let r, s be fixed elements of R such that rs is not a zero divisor, and $r-s$ is an invertible element of R. If F satisfies* (1), (2) *and*

$$F(rx, sy) = rsF(x, y), \tag{3}$$

then F is additive in each variable.

THEOREM 2. *Let R be the ring Z of rational integers. Let r, s be distinct non-zero integers. If F satisfies* (1), (2) *and* (3), *then F is bi-additive.*

Theorem 2 yields the result due to Jordan and von Neumann, that if $f\colon M \to Z$ satisfies the parallelogram law, then $F(x, y) = f(x+y)-f(x)-f(y)$ is bi-additive.

A commonly accepted definition of quadratic forms is the following (see e.g. Jacobson [1], Definition 6.1), where R is a commutative ring with identity, and M is a (unitary) R-module. A function $f\colon M \to R$ is a *quadratic form* if

(i) $f(rx) = r^2 f(x)$ for all $r \in R$, $x \in M$,

(ii) $F\colon M \times M \to R$ is bilinear, where, for all $x, y \in M$,

$$F(x, y) := f(x+y)-f(x)-f(y). \tag{1}$$

The second requirement can be separated into two parts: that F be homogeneous (of degree one) in each variable, and that F be bi-additive. It will be shown that suitable homogeneity of F, and the fact (a consequence of its definition by (1)) that F satisfies

$$F(x, y)+F(x+y, z) = F(x, y+z)+F(y, z) \tag{2}$$

implies that F is bi-additive.

The particular situation which motivates and illuminates our treatment is the deduction of bi-additivity from the parallelogram law (cf. Jordan–von Neumann [2], Theorem 1). Suppose $f\colon M \to R$ satisfies

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{3}$$

and 2 is not a zero divisor in R, then F satisfied the partial homogeneity condition

$$F(x, -y) = -F(x, y). \tag{4}$$

Bi-additivity of F is proved in

PROPOSITION 1. *Let R be a ring in which 2 is not a zero divisor. Let M be an R-module and suppose $F\colon M\times M \to R$ satisfies* (2) *and* (4). *Then F is bi-additive.*

Proof. Substitute $-z$ for z in (2) to obtain

$$F(x, y)+F(x+y, -z) = F(x, y-z)+F(y, -z). \tag{5}$$

Add (2) and (5), and use (4) to deduce that

$$2F(x, y) = F(x, y+z)+F(x, y-z). \tag{6}$$

In (6) interchange y, and z, and add the resulting equation to (6) to deduce (using (4) again) that

$$2F(x, y)+2F(x, z) = 2F(x, y+z). \tag{7}$$

Since 2 is not a zero divisor, (7) yields the additivity of F in the second variable. A further use of (2) yields the additivity of F in the first variable. Hence F is bi-additive.

This result is generalized in

PROPOSITION 2. *Let r, s be fixed elements of R such that rs is not a zero divisor. If $F\colon M\times M \to R$ satisfies* (2)

$$F(rx, sy) = rsF(x, y) \tag{8}$$

and

$$F(0, 0) = 0 \tag{9}$$

then

$$F(x+ty, z) = F(x, z)+F(ty, z) \tag{10}$$

for all $x, y, z \in M$, where $t := r-s$.

Remark. If $rs-1$ is not a zero divisor, then (9) is a consequence of (8). If F satisfies (2) and (8), then $F-F(0, 0)$ satisfies (2), (8) and (9).

Proof. In (2) replace x by rx, y by rsy, and z by sz to obtain

$$F(rx, rsy)+F(rx+rsy, sz) = F(rx, rsy+sz)+F(rsy, sz). \tag{11}$$

Simplify (11) using (8) and cancel the rs factors resulting to deduce that

$$F(x, ry)+F(x+sy, z) = F(x, ry+z)+F(sy, z). \tag{12}$$

However, writing ry for y i (2) yields

$$F(x, ry)+F(x+ry, z) = F(x, ry+z)+F(ry, z). \tag{13}$$

Subtracting (12) from (13), and using the notation $t := r-s$ we have

$$F(x+sy+ty, z)-F(x+sy, z) = F(sy+ty, z)-f(sy, z). \tag{14}$$

Set $x = -sy$ in (14), and use the fact that $F(x', 0) = F(0, z') = 0$ by (9) and (2), to obtain $F(sy+ty, z)-F(sy, z) = F(ty, z)$; so (14) can be rewritten

$$F(x+sy+ty, z)-F(x+sy, z) = F(ty, z). \tag{15}$$

Finally in (15) replace $x+sy$ by x to deduce (10).

Corollary. *Suppose F satisfies* (2), (8), *and* (9); *and $r-s$ is an invertible element of R, then F is bi-additive. In particular, if R is a field with at least* 3 *elements and F satisfies* (2), (8) *and* (9) *for a pair $r \neq 0$, $s \neq 0$, $r \neq s$, then F is bi-additive.*

Proof. Write $t^{-1}y$ for y in (10).

If one considers the corollary applied to $r = 1$, $s = -1$ one sees that one has to assume that 2 is invertible to deduce the additivity of F, whereas in Proposition 1 all one requires in that 2 not be a zero divisor. It seems that the fact that r and s are in the subring of R generated by 1 is critical, as is shown in the next (and final) result.

Proposition 3. *Let r, s be fixed elements of Z* (the ring of rational integers) *such that $rs \neq 0$, and $r \neq s$. If F: $M \times M \to Z$ satisfies* (2), (8) *and* (9), *then F is bi-additive.*

Proof. We can assume that $|rs| > 1$, as $rs = 1$ violates the assumption $r \neq s$, and $rs = -1$ is taken care of in Proposition 1.

We exploit (10) by writing ($t := r-s$ as usual),

$$rsF(x, y) = F(rx, sy) = F(sx+tx, sy) = F(sx, sy)+F(tx, sy). \tag{16}$$

Now by (2)

$$F(tx, y)+F(tx+y, z) = F(tx, y+z)+F(y, z)$$

so using (10) here again

$$F(tx, y)+F(tx, z) = F(tx, y+z). \tag{17}$$

Hence by (17), and the fact that $s \in Z$

(18) $$F(tx, sy) = sF(tx, y).$$

Indeed, for all $m, n \in Z$,

(19) $$F(mtx, ny) = mnF(tx, y).$$

The outcome of all this is that (16) may be rewritten as

(20) $$F(sx, sy) = rsF(x, y) - sF(tx, y).$$

For each $n \geqslant 0$ define $\alpha_n, \beta_n \in Z$ by $\alpha_0 = 1$, $\beta_0 = 0$ and

(21) $$\alpha_{n+1} = rs \cdot \alpha_n, \quad \beta_{n+1} = rs \cdot \beta_n + s^{2n+1}$$

for $n \geqslant 1$. Then it is an easy exercise to use (20) and (21) to prove by induction, that for all $n \geqslant 0$

(22) $$F(s^n x, s^n y) = \alpha_n F(x, y) - \beta_n F(tx, y).$$

It is also easy to prove from (21) that

(23) $$\alpha_n - t\beta_n = s^2 n.$$

Since $t \neq 0$, there are by Euler's theorem (or the fact that Z/tZ is a finite ring), positive integers $m > n$ such that $s^m \equiv s^n \bmod t$ say $s^m = s^n + ut$. Then on the one hand by (22)

(24) $$F(s^m x, s^m y) = \alpha_m F(x, y) - \beta_m F(tx, y)$$

and on the other hand, using (10),

$$\begin{aligned} F(s^m x, s^m y) &= F(s^n x + utx, s^n y + uty) \\ &= F(s^n x, s^n y) + F(s^n x, uty) + F(utx, s^n y) + F(utx, uty) \\ &= \alpha_n F(x, y) - \beta_n F(tx, y) + us^n F(x, ty) + \\ &\qquad + us^n F(tx, y) + u^2 tF(tx, y) \end{aligned}$$

by repeated use of (17). Moreover, $F(tx, ty) = tF(tx, y)$ and $F(tx, ty) = tF(x, ty)$, by (19). Since $t \neq 0$, we see that $F(tx, y) = F(x, ty)$. Thus our second evaluation of $F(s^m x, s^m y)$ is $\alpha_n F(x, y) - (\beta_n - 2us^n - u^2 t) = F(tx, y)$. Equating the two evaluations, and rearranging, we obtain

$$(\beta_m - \beta_n + 2us^n + u^2 t) F(tx, y) = (\alpha_m - \alpha_n) F(x, y).$$

Multiply both sides of this by t, and use (23) to deduce.

(25) $$(\alpha_m - \alpha_n) F(tx, y) = (\alpha_m - \alpha_n) tF(x, y).$$

Finally, $\alpha_m - \alpha_n = (rs)^m - (rs)^n \neq 0$ since $|rs| > 1$ and $m > n$. So we deduce from (25) that

$$F(tx, y) = tF(x, y). \tag{26}$$

Now use (17) and (26) to infer that

$$tF(x, y+z) = tF(x, y) + tF(x, z);$$

cancelling t (since t is not a zero divisor) yields the desired result.

References

[1] N. Jacobson, *Basic Algebra I*, W. H. Freeman, San Francisco 1974.
[2] P. Jordan, and J. von Neumann, *On inner products in linear metric space*, Ann. of Math. 36 (1935), 719–723.

DEPARTMENT OF MATHEMATICAL SCIENCES
MCMASTER UNIVERSITY, HAMILTON, ONTARIO, CANADA

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