

Approximation and interpolation of entire functions

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1. Introduction. Let E be a bounded closed set in the complex z -plane C . Denote by $\mathcal{C}(E)$ the algebra of all continuous complex functions on E with the norm

$$\|f\| = \sup_{z \in E} |f(z)| \quad \text{for } f \in \mathcal{C}(E).$$

Let P_n denote the set of the polynomials in z of degree $\leq n$. It is known that for every $f \in \mathcal{C}(E)$ there is exactly one polynomial $\pi_n \in P_n$ such that

$$\mathcal{E}_n(f, E) = \inf_{g \in P_n} \|f - g\| = \|f - \pi_n\|.$$

Batyrev [1] proved the following

THEOREM. *A function f , analytic on a bounded closed set E with the positive transfinite diameter d , and with the simply connected $C - E$, can be prolonged to an entire function of order ϱ ($0 < \varrho < \infty$) and of type σ if and only if*

$$\limsup_{n \rightarrow \infty} n^{1/\varrho} (\mathcal{E}_n(f, E))^{1/n} = d(e\varrho\sigma)^{1/\varrho}.$$

Batyrev has proved this theorem using Faber polynomials. His method cannot be applied in the case where the set $C - E$ is not simply connected or if the function f is not analytic on E .

The main object of this paper is to prove the Batyrev theorem in the case of any compact $E \subset C$ with $d(E) > 0$ and of any function f defined and bounded on E . Moreover (see Section 5), we give formulas expressing the type and the order of an entire function f in terms of the coefficients of the expansion of f into a Newton interpolating series. These formulas generalize the classical formulas expressing the order and type of f in terms of the coefficients of type Taylor series of f .

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2. Extremal polynomials. Let $\zeta^{(n)} = \{\zeta_{n0}, \zeta_{n1}, \dots, \zeta_{nn}\}$ be a system of $n+1$ points of a bounded closed set E in the complex plane C . Put

$$(2.0) \quad V(\zeta^{(n)}) = \prod_{0 \leq j < k \leq n} |\zeta_{nj} - \zeta_{nk}|,$$

$$\Delta^{(j)}(\zeta^{(n)}) = \prod_{\substack{k=0 \\ k \neq j}}^n |\zeta_{nj} - \zeta_{nk}|, \quad j = 0, 1, \dots, n.$$

DEFINITION 2.1. A system of points of E

$$(2.1) \quad \eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \dots, \eta_{nn}\}$$

satisfying the relations

$$(a) \quad V_n = V(\eta^{(n)}) = \sup_{\zeta^{(m)} \subset E} V(\zeta^{(n)}),$$

$$(b) \quad \Delta^{(0)}(\eta^{(n)}) \leq \Delta^{(j)}(\eta^{(n)}), \quad j = 1, 2, \dots, n$$

is called the n -th extremal system of E . Polynomials

$$(2.2) \quad L^j(z, \eta^{(n)}) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{z - \eta_{nk}}{\eta_{nj} - \eta_{nk}}, \quad j = 0, 1, \dots, n$$

are called the n -th Lagrange extremal polynomials, and the limit

$$d = d(E) = \lim_{n \rightarrow \infty} V_n^{2/n(n+1)} = \lim_{n \rightarrow \infty} (\Delta^{(0)}(\eta^{(n)}))^{1/n}$$

is called the transfinite diameter of E .

The following theorem is known [2]:

If $d(E) > 0$, then there exists a finite limit

$$(2.3) \quad \lim_{n \rightarrow \infty} |L^0(z, \eta^{(n)})|^{1/n} = L(z) \geq 1$$

for every z in the unbounded component E_∞ of $C - E$, the convergence in (2.3) is uniform on every bounded closed subset of E , and L is a modulus of an analytic function φ in E_∞ which has a univalent branch

$$(2.4) \quad \varphi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \dots, \quad |\gamma| = \frac{1}{d(E)},$$

in a neighbourhood of infinity. Moreover, $\text{Log } L(z)$ is the generalized Green function for E_∞ with pole at ∞ .

Let $|E|$ denote the diameter of a compact set E ,

$$|E| = \sup\{|z - w| : z, w \in E\}.$$

We note that

$$\frac{\text{dist}(z, E)}{|\Delta^{(0)}(\eta^{(n)})|^{1/n}} \leq |L^0(z, \eta^{(n)})|^{1/n} \leq \frac{\text{dist}(z, E) + |E|}{|\Delta^{(0)}(\eta^{(n)})|^{1/n}} \quad \text{for } z \in C, \quad n = 1, 2, \dots$$

In consequence we have

LEMMA 2.1. *If $d(E) > 0$, then*

$$(2.5) \quad \text{dist}(z, E) \leq d(E)L(z) \leq \text{dist}(z, E) + |E| \quad \text{for } z \in C - E.$$

DEFINITION 2.2. Let z_0 be a fixed point of E . A sequence $\{z_n\}$ of points of E defined by

$$|(z_{n+1} - z_0) \dots (z_{n+1} - z_n)| = \sup_{z \in E} |(z - z_0) \dots (z - z_n)| \quad \text{for } n = 1, 2, \dots,$$

is called the *extremal sequence* of E .

One can prove [3] that if $\{z_n\}$ is an extremal sequence of E , then

$$(2.6) \quad \lim_{n \rightarrow \infty} \left(\sup_{z \in E} |(z - z_0) \dots (z - z_{n-1})| \right)^{1/n+1} = d(E)$$

and if $d(E) > 0$, then

$$(2.7) \quad \lim_{n \rightarrow \infty} |(z - z_0) \dots (z - z_{n-1})|^{1/n+1} = dL(z) \quad \text{for } z \in E_\infty.$$

Equation (2.6) remits from the following [6]

THEOREM 2.1. *Let $\{\zeta_{n1}, \dots, \zeta_{nn}\}$ be a system of n points of a bounded closed set E with a positive transfinite diameter d . If we introduce the notation*

$$(2.8) \quad M_n = \max_{z \in E} |(z - \zeta_{n1}) \dots (z - \zeta_{nn})|,$$

then

$$\lim_{n \rightarrow \infty} M_n^{1/n} = d$$

is a sufficient condition that

$$(2.9) \quad \lim_{n \rightarrow \infty} |(z - \zeta_{n1}) \dots (z - \zeta_{nn})|^{1/n} = dL(z)$$

should hold uniformly on any closed limited point set interior to E_∞ .

3. Order and type of an entire function. Let f be an entire function and let

$$(3.1) \quad M(r) = \sup_{|z|=r} |f(z)|, \quad r > 0,$$

be its supremum modulus.

DEFINITION 3.1. We call $\rho = \rho(f)$ the *order* of f if

$$(3.2) \quad \rho(f) = \limsup_{r \rightarrow \infty} \frac{\ln \ln M(r)}{\ln r}.$$

We have $0 \leq \rho(f) \leq \infty$. If $0 < \rho(f) < \infty$, we say that f is of type $\sigma = \sigma(f)$ if

$$(3.3) \quad \sigma(f) = \limsup_{r \rightarrow \infty} \frac{\ln M(r)}{r^\rho}.$$

Replacing in Definition 3.1 the circle $|z| = r$ by the curve

$$(3.4) \quad E_r = \{z \in C: d(E)L(z) = r\} \quad \text{for } r > 1,$$

and $M(r)$ by

$$(3.5) \quad \tilde{M}(r) = \sup_{z \in E_r} |f(z)| \quad \text{for } r > 1,$$

we shall prove the following

LEMMA 3.1. *If f is an entire function of order ρ and type σ , and $d(E) > 0$, then*

$$\rho = \limsup \frac{\ln \ln \tilde{M}(r)}{\ln r}.$$

If $0 < \rho < \infty$, then

$$\sigma = \limsup \frac{\ln \tilde{M}(r)}{r^\rho}.$$

Proof. Let z_0 be a fixed point of the set E , and $r > 1$. For every point $z \in E_r$ there exists a $z_1 = z_1(z) \in E$ such that

$$|z - z_1| = \text{dist}(z, E).$$

By the triangle inequality and by (2.5) we have

$$|z - z_0| \leq |z - z_1| + |z_1 - z_0| \leq r + |E| \quad \text{for } z \in E_r, r > 1,$$

and

$$r - |E| \leq |z - z_1|, \quad |E| \geq |z_1 - z_0|.$$

Observe that

$$r - 2|E| - |z_0| \leq |z| \leq r + |E| + |z_0| \quad \text{for } z \in E_r, r > 1.$$

Let $R > 1$ be such that

$$r - 2|E| - |z_0| \geq \frac{1}{2}r \quad \text{and} \quad r + |E| + |z_0| \leq 2r \quad \text{for } r > R.$$

Hence for $r > R$ we have

$$\frac{\ln \ln M(\frac{1}{2}r)}{\ln r} \leq \frac{\ln \ln \tilde{M}(r)}{\ln r} \leq \frac{\ln \ln M(2r)}{\ln r}$$

and, if $0 < \rho < \infty$,

$$\frac{\ln M(r-a)}{r^\rho} \leq \frac{\ln M(r)}{r^\rho} \leq \frac{\ln M(r+b)}{r^\rho},$$

where

$$a = 2|E| + |z_0|, \quad b = |E| + |z_0|.$$

After passing to the upper limit the proof of Lemma 3.1 is completed.

LEMMA 3.2. *If a function f is defined and bounded on a compact set $E \subset C$ containing infinitely many points, then*

$$\mathcal{E}_n(f, E) \leq \|f - L_n\| \leq (n+2) \mathcal{E}_n(f, E)$$

and

$$\|L_n - L_{n-1}\| \leq 2(n+2) \mathcal{E}_{n-1}(f, E) \quad \text{for } n = 2, 3, \dots,$$

where

$$L_n(z) = \sum_{j=0}^n L^{(j)}(z, \eta^{(n)}) f(\eta_{nj})$$

is the n -th Lagrange interpolation polynomial for f with nodes at extremal points η_{nj} ($j = 0, \dots, n$) of E defined by (2.1).

Proof. Let π_n and P_n ($n = 1, 2, \dots$) be polynomials of degree $\leq n$ such that

$$\mathcal{E}_n(f, E) = \|f - \pi_n\|$$

and

$$f(\eta_{nj}) - \pi_n(\eta_{nj}) = P_n(\eta_{nj}) \quad \text{for } n = 1, 2, \dots, j = 0, 1, \dots, n.$$

Since

$$L_n(\eta_{nj}) = \pi_n(\eta_{nj}) + P_n(\eta_{nj}) \quad \text{for } j = 0, 1, \dots, n,$$

we have

$$L_n(z) = \pi_n(z) + P_n(z) \quad \text{for } z \in C.$$

Therefore

$$\begin{aligned} \|f - L_n\| &\leq \|f - \pi_n\| + \|P_n\| \\ &\leq \mathcal{E}_n(f, E) + \mathcal{E}_n(f, E) \sum_{j=0}^n \|L^{(j)}\| \leq (n+2) \mathcal{E}_n(f, E). \end{aligned}$$

Hence and the sequence $\{\mathcal{E}_n(f, E)\}$ being decreasing, we have

$$\|L_n - L_{n-1}\| \leq \|f - L_n\| + \|f - L_{n-1}\| \leq 2(n+2) \mathcal{E}_{n-1}(f, E) \quad \text{for } n = 2, 3, \dots$$

This concludes the proof of Lemma 3.2.

LEMMA 3.3. *Let $\{P_n\}$ be a sequence of polynomials in z of the respective degrees $\leq n$. If the transfinite diameter of the compact $E \subset C$ is positive, and if there exist numbers $K, \mu, N > 0$ such that*

$$\|P_n\| \leq d^n \left(\frac{eK\mu}{n} \right)^{n/\mu} \quad \text{for } n \geq N,$$

then

$$f(z) = \sum_{n=1}^{\infty} P_n(z) \quad \text{for } z \in C$$

is an entire function, and for all $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that

$$\tilde{M}(r) \leq \exp((K + \varepsilon)r^\mu) \quad \text{for } r \geq R.$$

Proof. By the Bernstein-Walsh inequality (see Theorem 2.2 and Corollary 9.1 in [5]) we have

$$|P_n(z)| \leq \|P_n\| L^n(z) \quad \text{for } z \in C, n = 1, 2, \dots$$

Hence

$$\tilde{M}(r) \leq \sum_{n=1}^{\infty} \frac{\|P_n\|}{d^n} r^n \quad \text{for } r > 1.$$

Take $r_0 > 1$ in such a way that

$$n(r) = 2^\mu (eK\mu) r^\mu > N \quad \text{for } r \geq r_0.$$

Then

$$\left(\frac{eK\mu}{n}\right)^{n/\mu} r^n \leq \frac{1}{2^n} \quad \text{for } n \geq n(r)$$

and

$$\tilde{M}(r) \leq \sum_{n=1}^N \frac{\|P_n\|}{d^n} r^n + \sum_{n=N+1}^{n(r)} \left(\frac{eK\mu}{n}\right)^{n/\mu} r^n + 1 \quad \text{for } r \geq r_0.$$

It is clear that for a fixed r the maximum value of the expression $\left(\frac{eK\mu}{n}\right)^{n/\mu} r^n$ is reached for $r = \mu K r^\mu$ and is equal to $\exp(Kr^\mu)$. It follows that if $r \geq r_0$, then

$$\begin{aligned} \tilde{M}(r) &< r^N \sum_{n=1}^N \frac{\|P_n\|}{d^n} + [n(r) + 1 - N] e^{Kr^\mu} \\ &< r^N \sum_{n=1}^N \frac{\|P_n\|}{d^n} + (2^\mu e\mu K r^\mu + 1 - N) e^{Kr^\mu} \\ &= e^{Kr^\mu} \left(2^\mu e\mu K r^\mu + 1 - N + r^N e^{-Kr^\mu} \sum_{n=1}^N \frac{\|P_n\|}{d^n} \right). \end{aligned}$$

Hence, by a standard argument, for every $\varepsilon > 0$ there exists an $R > r_0$ such that for all $r > R$

$$2^\mu e\mu K r^\mu + 1 - N + r^N e^{-Kr^\mu} \sum_{n=1}^N \frac{\|P_n\|}{d^n} < e^{\varepsilon r^\mu}.$$

The proof is completed.

4. Generalizations of the Batyrev theorem. Let E be a bounded closed set in the complex plane C with the positive transfinite diameter d , and

let f be an entire function of order ρ and type σ . If we preserve the notation of the former section and set

$$(4.0) \quad \lambda_n^{(1)} = \mathcal{E}_n(f, E), \quad \lambda_n^{(2)} = \|L_n - L_{n-1}\|, \quad \lambda_n^{(3)} = \|f - L_n\|$$

for $n = 1, 2, \dots$,

then we can prove the following

THEOREM 4.1. *If $0 < \rho < \infty$, then*

$$\limsup_{n \rightarrow \infty} n^{1/\rho} (\lambda_n^{(j)})^{1/n} = d(e\rho\sigma)^{1/\rho} \quad \text{for } j = 1, 2, 3.$$

If $\rho = 0$, then for every $a > 0$

$$\limsup_{n \rightarrow \infty} n^a (\lambda_n^{(j)})^{1/n} = 0 \quad \text{for } j = 1, 2, 3.$$

Proof. Let $R > 1$ be so large that the function φ defined by (2.4) is univalent in $|z| \geq R$. Then $g(z) = d\varphi(z)$ is also univalent in $|z| \geq R$, $g'(\infty) = 1$ and $dL(z) = |g(z)|$. Hence

$$E_r = \{z \in C: |g(z)| = r\}, \quad r > 1,$$

and there exists an $m_0 > 0$ such that

$$g'(z) \geq m_0 \quad \text{for } |z| \geq R.$$

Therefore

$$(4.1) \quad \int_{E_r} |d\zeta| = \int_{|z|=r} \frac{|dz|}{|g'(z)|} \leq \frac{2\pi r}{m_0} \quad \text{for } r \geq R.$$

Put

$$w_n(z) = (z - \eta_{n1}) \dots (z - \eta_{nn}), \quad n = 1, 2, \dots,$$

where $\eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \dots, \eta_{nn}\}$ is of n -th extremal points system of E . Put

$$M_n = \sup_{z \in E} |w_n(z)| = (\Delta^{(0)}(\eta^{(n)}))^{1/n}.$$

One can easily check that

$$(4.2) \quad \lim_{n \rightarrow \infty} \frac{|w_n(\zeta)|}{|g(\zeta)|^n} = 1 \quad \text{for } n = 1, 2, \dots$$

Since $\lim M_n^{1/n} = d$, and $d > 0$, so by Theorem 2.1 for any fixed $r_0 > 1$ there exists a finite limit

$$\lim_{n \rightarrow \infty} |w_n(\zeta)|^{1/n} = r_0 \quad \text{for } \zeta \in E_{r_0}$$

and convergence is uniform on E_{r_0} . Hence for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon, r_0)$ such that

$$(4.3) \quad \left(\frac{|w_n(\zeta)|}{M_n |g(\zeta)|^n} \right)^{1/n} \geq \frac{e^{-\varepsilon}}{d} \quad \text{for } n \geq n_0, \zeta \in E_{r_0}.$$

From (4.2) and by the extremum principle applied to the harmonic function

$$\frac{1}{n} \ln \frac{|w_n(\zeta)|}{M_n |g(\zeta)|^n},$$

inequality (4.3) holds true for $\zeta \in E_r$, $r \geq r_0$. Hence for $r \geq r_0$ and $n \geq n_0$ we get

$$(4.4) \quad \left(\frac{|w_n(\zeta)|}{M_n} \right)^{1/n} \geq \frac{r}{d} e^{-\sigma} \quad \text{for } \zeta \in E_r.$$

If r is sufficiently large, say $r > r_1$, then E_r is a union of a finite number of mutually disjoint analytic Jordan curves; therefore (see [6]),

$$f(z) - L_n(z) = \frac{1}{2\pi i} \int_{E_r} \frac{w_n(z)(z - \eta_{n0})}{w_n(\zeta)(\zeta - \eta_{n0})} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in E.$$

Hence and from (4.1), (4.4) and (2.5) we get

$$(4.5) \quad \|f - L_n\| \leq \lambda \frac{\tilde{M}(r)}{r^n} (de^\sigma)^n,$$

where

$$\lambda = \frac{2r_0 |E|}{m_0(r_0 - |E|)}, \quad r > R_1 = \max(r_0, r_1, |E|), \quad n \geq n_0.$$

Let $K > \sigma$. By the definition of the type of f there exists an $R_2 = R_2(K) > R_1$ such that

$$\tilde{M}(r) \leq e^{Kr^\sigma} \quad \text{for } r \geq R_2.$$

Let $N > n_0$ be so large that

$$\left(\frac{n}{K\varrho} \right)^{1/\sigma} > R_2 \quad \text{for } n \geq N.$$

Putting $r = \left(\frac{n}{K\varrho} \right)^{1/\sigma}$ in (4.5) gives

$$(4.6) \quad \|f - L_n\| \leq \lambda (de^\sigma)^n \left(\frac{eK\varrho}{n} \right)^{n/\sigma} \quad \text{for } n \geq N.$$

Hence

$$\limsup_{n \rightarrow \infty} n^{1/\sigma} \|f - L_n\|^{1/n} \leq d(eK\varrho)^{1/\sigma};$$

therefore, by Lemma 3.2, we have

$$(4.7) \quad \limsup_{n \rightarrow \infty} n^{1/\sigma} (\lambda_n^{(j)})^{1/n} \leq d(e\sigma\varrho)^{1/\sigma} \quad \text{for } j = 1, 2, 3.$$

Suppose that

$$\limsup_{n \rightarrow \infty} n^{1/\varrho} (\lambda_n^{(2)})^{1/n} < d(e\sigma\varrho)^{1/\varrho}.$$

Then, by Lemma 3.4, the type of f is smaller than σ . We have got a contradiction which shows that

$$\limsup_{n \rightarrow \infty} n^{1/\varrho} (\lambda_n^{(2)})^{1/n} = d(e\sigma\varrho)^{1/\varrho}.$$

Hence by Lemma 3.2 and from (4.7) we get

$$\limsup_{n \rightarrow \infty} n^{1/\varrho} (\lambda_n^{(j)})^{1/n} = d(e\sigma\varrho)^{1/\varrho} \quad \text{for } j = 1, 2, 3.$$

The proof of the first part of Theorem 4.1 is completed.

If $\varrho = 0$, then by the definition of the order of f for every $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that

$$\tilde{M}(r) \leq e^{r^\varepsilon} \quad \text{for all } r \geq R.$$

In the same way as in the proof of (4.6) we have

$$\lambda_n^{(3)} \leq (de^\varepsilon)^n \left(\frac{e\varepsilon}{n} \right)^{n/\varepsilon} \quad \text{for } n \geq N_1 = N_1(\varepsilon).$$

Hence, and by Lemma 3.2

$$(4.8) \quad \limsup_{n \rightarrow \infty} n^{1/\varepsilon} (\lambda_n^{(j)})^{1/n} < \infty \quad \text{for every } \varepsilon > 0, j = 1, 2, 3.$$

So, if there existed $\alpha > 0$ such that

$$\limsup_{n \rightarrow \infty} n^\alpha (\lambda_n^{(3)})^{1/n} > 0,$$

we would have

$$\limsup_{n \rightarrow \infty} n^{\alpha+1} (\lambda_n^{(3)})^{1/n} = \infty.$$

This is impossible because of (4.8). The proof is completed.

THEOREM 4.2. *If $d(E) > 0$, then the order of f is given by*

$$(4.9) \quad \varrho = \limsup_{n \rightarrow \infty} \frac{\ln n}{d \left| \ln \frac{d}{(\lambda_n^{(j)})^{1/n}} \right|} \quad \text{for } j = 1, 2, 3.$$

Proof. We denote the right-hand member of (4.9) by γ_j ($j = 1, 2, 3$) and we start by showing that $\varrho \leq \gamma_j$ ($j = 1, 2, 3$). If γ_2 is finite and $\varepsilon > 0$ is fixed, there exists an n_ε such that

$$\frac{\ln n}{d \left| \ln \frac{d}{(\lambda_n^{(j)})^{1/n}} \right|} \leq \gamma_2 + \varepsilon, \quad n > n_\varepsilon.$$

From this we get

$$n^{1/\mu}(\lambda_n^{(2)})^{1/n} \leq d \left(e \frac{1}{e\mu} \mu \right)^{1/\mu}, \quad \text{where } \mu = \gamma_2 + \varepsilon, \quad n > n_\varepsilon.$$

Then, because of Lemma 3.4, there exists an $R > 0$ such that

$$\tilde{M}(r) \leq e^{\left(\frac{1}{e\mu} + \varepsilon\right)r^\mu} \quad \text{for } r > R.$$

Therefore $\varrho \leq \gamma_2 + \varepsilon$ for every $\varepsilon > 0$. Hence $\varrho \leq \gamma_2$ provided γ_2 is finite. But the inequality is trivially true if $\gamma_2 = \infty$, so by Lemma 3.2 we have $\varrho \leq \gamma_j$ ($j = 1, 2, 3$), regardless of the value of γ_j .

Suppose that ϱ is finite. Then for any fixed $\varepsilon > 0$ there is a finite $R_1(\varepsilon)$ such that

$$\tilde{M}(r) \leq e^{r^{\varrho+\varepsilon}} \quad \text{for } r > R_1(\varepsilon).$$

Then, because of (4.6), we get

$$\lambda_n^{(3)} \leq \lambda (de^\varepsilon)^n \left(\frac{e\alpha}{n} \right)^{n/\alpha} \quad \text{for } n \geq N(\varepsilon),$$

where $\alpha = \varrho + \varepsilon$.

Therefore by Lemma 3.2 we have

$$\limsup_{n \rightarrow \infty} n^{1/\alpha} (\lambda_n^{(j)})^{1/n} \leq d(e\alpha)^{1/\alpha} \quad \text{for } j = 1, 2, 3.$$

So there exists an N_1 such that

$$n^{1/\alpha} (\lambda_n^{(j)})^{1/n} \leq d(e\alpha + \varepsilon)^{1/\alpha} \quad \text{for } n \geq N_1, \quad j = 1, 2, 3.$$

Hence

$$(4.10) \quad \alpha \geq \frac{\ln(e\alpha + \varepsilon)^{-1}}{\ln \frac{d}{(\lambda_n^{(j)})^{1/n}}} + \frac{\ln n}{\ln \frac{d}{(\lambda_n^{(j)})^{1/n}}}, \quad n \geq N, \quad j = 1, 2, 3.$$

Since (4.10) holds for all $\alpha > \varrho$, we have $\varrho \geq \gamma_j$ ($j = 1, 2, 3$). This is obviously true if $\varrho = \infty$. Since the opposite inequality holds, we conclude that $\varrho = \gamma_j$ ($j = 1, 2, 3$) as asserted.

Let f be a function defined and bounded on a compact set E with the positive transfinite diameter d .

Denote by

$$\Gamma(f, E) = \{ \mu > 0 : \limsup_{n \rightarrow \infty} n^{1/\mu} (\mathcal{E}_n(f, E))^{1/n} < \infty \}.$$

THEOREM 4.3. *If $\Gamma(f, E) \neq \emptyset$, then the function*

$$\tilde{f}(z) = \pi_1(z) + \sum_{n=2}^{\infty} (\pi_n(z) - \pi_{n-1}(z)), \quad z \in C$$

is entire and

- 1° $\tilde{f}(z) = f(z)$ for $z \in E$,
- 2° order $\tilde{f} = \varrho$, where $\varrho = \inf \Gamma(f, E)$,
- 3° if $0 < \varrho < \infty$, then

$$\text{type } \tilde{f} = \begin{cases} \frac{\beta^\varrho}{d^\varrho e \varrho}, & \text{when } \beta < \infty, \\ \infty, & \text{when } \beta = \infty, \end{cases}$$

where

$$\beta = \limsup_{n \rightarrow \infty} n^{1/\varrho} (\mathcal{E}_n(f, E))^{1/n}.$$

Proof. Since $\Gamma(f, E) \neq \emptyset$; then $\lim_{n \rightarrow \infty} (\mathcal{E}_n(f, E))^{1/n} = 0$. Hence and by the Bernstein-Walsh inequality the function \tilde{f} is entire. Moreover, by Lemma 3.3 for any fixed $\gamma \in \Gamma(f, E)$ and $\varepsilon > 0$ there exists an r_0 such that

$$\tilde{M}(r) \leq e^{r^{\gamma+\varepsilon}} \quad \text{for } r \geq r_0.$$

Therefore \tilde{f} is an entire function of order $\varrho' \leq \varrho$. By Theorem 4.2 we have

$$\varrho' = \limsup_{n \rightarrow \infty} \frac{\ln n}{\frac{d}{(\mathcal{E}_n(f, E))^{1/n}}}.$$

Then for every fixed $\mu > \varrho'$ there is a finite N_μ such that

$$n^{1/\mu} (\mathcal{E}_n(f, E))^{1/n} < d.$$

Hence and by the definition of the set $\Gamma(f, E)$ we see that $\mu \in \Gamma(f, E)$. Therefore $\varrho' \geq \varrho$. Since the opposite inequality holds, we conclude that $\varrho' = \varrho$ as asserted.

Let $0 < \varrho < \infty$, and $\beta < \infty$. Write β in the form

$$\beta = d(e\sigma\varrho)^{1/\varrho}, \quad \text{where } \sigma = \frac{\beta^\varrho}{d^\varrho e \varrho}.$$

Since

$$\|\pi_n - \pi_{n-1}\| \leq \|f - \pi_n\| + \|f - \pi_{n-1}\| \leq 2 \mathcal{E}_{n-1}(f, E),$$

so by Lemma 3.3 the type σ' of the function \tilde{f} is finite and $\sigma' \leq \sigma$. Moreover, by Theorem 4.1 we get

$$\limsup_{n \rightarrow \infty} n^{1/\varrho} (\mathcal{E}_n(f, E))^{1/n} = d(e\sigma'\varrho)^{1/\varrho}.$$

So finally $\sigma = \sigma'$. The proof is completed.

Remark. In Theorem 4.3 $\Gamma(f, E)$ may be replaced by

$$\Gamma_j(f, E) = \{\mu > 0: \limsup_{n \rightarrow \infty} n^{1/\mu} (\lambda_n^{(j)})^{1/n} < \infty\} \quad \text{for } j = 1, 2, 3.$$

5. Coefficients of the Newton series of an entire function. Let $\{z_n\}$ be a bounded sequence of points of C . Put

$$(5.0) \quad \begin{aligned} w_n(z) &= (z-z_0) \dots (z-z_n) && \text{for } n = 0, 1, \dots, \\ w_{-1}(z) &= 1 && \text{for } z \in C. \end{aligned}$$

LEMMA 5.1. *If $f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$ is of order ρ , $0 < \rho < \infty$, and of type σ , then*

$$\limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} \leq (e\sigma\rho)^{1/\rho}.$$

Proof. Let $L = \sup_{n \geq 0} |z_n|$. Since

$$\left| \frac{w_n(z)}{z^{n+1}} \right| = \left| \left(1 - \frac{z_0}{z}\right) \dots \left(1 - \frac{z_n}{z}\right) \right| \geq \left(1 - \frac{L}{z}\right)^{n+1} \quad \text{for } n = 0, 1, \dots; |z| > L,$$

we have

$$\inf_{|z|=r} |w_n(z)| \geq \left(1 - \frac{L}{r}\right)^{n+1} \quad \text{for } n = 0, 1, \dots; r > L.$$

Applying these inequalities and the formula

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{w_n(z)} dz \quad (r > M),$$

we get

$$(5.1) \quad |a_n| \leq \left(\frac{r}{r-L}\right)^{n+1} \frac{M(r)}{r^n} \quad \text{for } n \geq 1, r > L,$$

where $M(r)$ is defined by (3.1). By the definition of the type σ for any fixed $K > \sigma$ there exists a finite r_K such that

$$M(r) \leq e^{Kr^\rho} \quad \text{for all } r \geq r_K.$$

Let n_0 be such that $\left(\frac{n}{K\rho}\right)^{1/\rho} > L$ for all $n \geq n_0$. Now, for any fixed n the minimum value $\left(\frac{eK\rho}{n}\right)^{n/\rho}$ of the expression $r^{-n} \exp(Kr^\rho)$ is reached for $r = \left(\frac{n}{K\rho}\right)^{1/\rho}$. It follows that

$$|a_n| \leq \left(\frac{n^{1/\rho}}{n^{1/\rho} - (K\rho)^{1/\rho} L}\right)^{n+1} \left(\frac{eK\rho}{n}\right)^{1/\rho} \quad \text{for } n \geq n_0.$$

Hence

$$\limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} \leq (eK\rho)^{1/\rho}.$$

Thus

$$\limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} \leq (e\sigma\rho)^{1/\rho}.$$

The proof is completed.

LEMMA 5.2. If $\limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} \leq (e\sigma\rho)^{1/\rho}$, then

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$$

is an entire function and for every $\varepsilon > 0$ there exists an R_ε such that

$$M(r) \leq e^{(\sigma+\varepsilon)r^\rho} \quad \text{for all } r > R_\varepsilon.$$

Proof. Let $\varepsilon > 0$ and $K = \sigma + \varepsilon/2$. Taking N_ε and $R > 1$ sufficiently large, we have

$$|a_n| \leq \left(\frac{eK\rho}{n}\right)^{n/\rho} \quad \text{for } n \geq N_\varepsilon,$$

and

$$\left|1 - \frac{z_j}{z}\right| \leq e^\varepsilon \quad \text{for } |z| \geq R, j = 1, 2, \dots$$

Hence

$$M(r) \leq \sum_{n=0}^{N_\varepsilon} |a_n| e^{n\varepsilon} r^n + \sum_{n=N_\varepsilon+1}^{\infty} e^{n\varepsilon} \left(\frac{eK\rho}{n}\right)^{n/\rho} r^n \quad \text{for } r > R.$$

Therefore, repeating the proof of Lemma 3.3, we get the proof of Lemma 5.2.

If c_n ($n = 0, 1, \dots$) is the n -th coefficient of the Taylor series of an entire function f of order ρ , $0 < \rho < \infty$, and type σ , then [4]

$$\limsup_{n \rightarrow \infty} n^{1/\rho} |c_n|^{1/n} = (e\sigma\rho)^{1/\rho}.$$

This formula may be generalized as follows.

THEOREM 5.1. Let $\{z_n\}$ be a bounded sequence of complex numbers.

If $f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$ is an entire function of order ρ ($0 < \rho < \infty$) and of type σ , then

$$\limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} = (e\sigma\rho)^{1/\rho}.$$

Proof. Let $\sigma < \infty$. By Lemma 5.1 we get

$$(5.2) \quad \limsup_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} = \beta \leq (e\sigma\rho)^{1/\rho}.$$

Write β in the form

$$\beta = (e\sigma'\rho)^{1/\rho}, \quad \text{where } \sigma' = \frac{\beta^\rho}{e\rho}.$$

Now, by Lemma 5.2, for any fixed $K > \sigma'$ there exists a finite $A = A(K)$ such that

$$M(r) \leq e^{Kr^e} \quad \text{for } r \geq A.$$

Thus, by the definition of type of f we have $\sigma' \geq \sigma$. Therefore we must have

$$\beta = (e\varrho\sigma')^{1/e} \geq (e\varrho\sigma)^{1/e}.$$

Since the opposite inequality holds, we conclude that $\sigma' = \sigma$, as asserted.

Let $\sigma = \infty$. If

$$\limsup_{n \rightarrow \infty} n^{1/e} |a_n|^{1/n} < \infty,$$

then Lemma 5.2 would imply that the type σ of the function f is finite. This contradiction concludes the proof.

Applying the method of proving Theorems 4.2 and 4.3, respectively, one can get the following two theorems.

THEOREM 5.2. *If f is of order ϱ , then*

$$\varrho = \limsup_{n \rightarrow \infty} \frac{\ln n}{\ln \frac{1}{|a_n|^{1/n}}}.$$

THEOREM 5.3. *If the set Γ defined by*

$$\Gamma = \{\mu > 0: \limsup_{n \rightarrow \infty} n^{1/\mu} |a_n|^{1/n} < \infty\}$$

is non-empty, then

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$$

is an entire function of order ϱ , where $\varrho = \inf \Gamma$, and if $0 < \varrho < \infty$ the type of f is given by

$$\sigma = \frac{1}{e\varrho} \limsup_{n \rightarrow \infty} n |a_n|^{e/n}.$$

Let $\{z_n\}$ be an extremal sequence of a compact set $E \subset C$ and let f be an entire function of order ϱ , $0 < \varrho < \infty$, and type σ , $\sigma < \infty$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z) \quad \text{for } z \in C,$$

where $w_n(z)$ is defined by (5.0).

Writing

$$p_n(z) = a_0 + a_1 w_0(z) + \dots + a_n w_{n-1}(z) \quad \text{for } n = 1, 2, \dots$$

we have

$$\mathcal{E}_n(f, E) \leq \|f - p_n\| \leq \sum_{\nu=n+1}^{\infty} |a_\nu| \|w_{\nu-1}(z_\nu)\| \quad \text{for } n = 1, 2, \dots$$

Hence by Theorem 5.1 and by (2.6) for every $K > \sigma$ and any fixed $\varepsilon > 0$ there exists an $n_0 = n_0(K, \varepsilon)$ such that

$$\mathcal{E}_n(f, E) \leq \sum_{\nu=n}^{\infty} \left(\frac{eK\rho}{\nu} \right)^{\nu/e} (d + \varepsilon)^\nu \quad \text{for } n \geq n_0,$$

where d is the transfinite diameter of the compact E . Therefore

$$\mathcal{E}_n(f, E) \leq \left(\frac{eK\rho}{n} \right)^{n/e} (d + \varepsilon)^n \left[1 + \sum_{\nu=1}^{\infty} \left(\frac{n}{n+\nu} \right)^{n/e} \left(\frac{eK\rho}{n+\nu} \right)^{\nu/e} (d + \varepsilon)^\nu \right] \\ \text{for } n \geq n_0.$$

It is clear that the series

$$\beta_n = \sum_{\nu=1}^{\infty} \left(\frac{n}{n+\nu} \right)^{n/e} \left(\frac{eK\rho}{n+\nu} \right)^{\nu/e} (d + \varepsilon)^\nu, \quad n = 1, 2, \dots$$

is convergent and there exists an $M < \infty$ such that $\beta_n < M$ for all n . Hence and by Lemma 3.2 we get

$$(5.3) \quad \limsup_{n \rightarrow \infty} n^{1/e} (\lambda_n^{(j)})^{1/n} \leq d(e\rho\sigma)^{1/e} \quad \text{for } j = 1, 2, 3,$$

where $\lambda_n^{(j)}$ is defined by (4.0).

We note that inequality (5.3) holds also in the case where the transfinite diameter of E is equal to zero. Thus, we can drop in both parts of Theorem 4.1 the assumption that $d(E) > 0$.

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