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APPLICATIONES MATHEMATICAE
20, 1 (1988), pp. 23–27

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GRAPHO-ANALYTIC METHOD OF SOLVING $2 \times n$ BI-MATRIX GAMES

In this paper we give a method to calculate the Nash equilibriums for bi-matrix games $\Gamma_{A,B}$, where A and B are $(2 \times n)$-matrices. The reasonings are based on grapho-analytic considerations. We obtain effective formulae. In particular, we apply these formulae to show that in a $2 \times n$ non-degenerate bi-matrix game the number of Nash equilibriums is finite and odd.

Let $A = (a^{ij})$ and $B = (b^{ij})$, where $i = 1, 2$ and $j = 1, \ldots, n$. In the sequel we use the convention that if the same index appears as a subscript and a superscript in a multiplication, a summation with respect to this index is taken.

The sets of pure strategies of two players in the considered game are: $I = \{1, 2\}$ – the strategy set for the first player, and $J = \{1, \ldots, n\}$ – the strategy set for the second player.

We denote by X and Y the sets of mixed strategies for the first and the second players, respectively. The elements of X and Y are probabilistic measures on I and J, respectively.

If the first player uses a strategy $x = (x_i)_{i=1}^2 \in X$ and the second a strategy $y = (y_j)_{j=1}^n \in Y$, then their pay-offs are

$$H_A(x, y) = a^{ij} x_i y_j \quad \text{and} \quad H_B(x, y) = b^{ij} x_i y_j, \tag{1}$$

respectively.

A pair of strategies $(x, y) \in X \times Y$ is called a *situation of the game*. A situation (x^0, y^0) is said to be a *Nash equilibrium* or a *solution of the game* if for each $x \in X$ and $y \in Y$ the inequalities

$$H_A(x, y^0) \leqslant H_A(x^0, y^0), \quad H_B(x^0, y) \leqslant H_B(x^0, y^0) \tag{2}$$

are satisfied.

Each person dealing with game theory is familiar with the concept of Nash equilibrium. Nevertheless, we give some historical notes.

The concept of Nash equilibrium was introduced by Nash [4] who carried over non-zero sum games in this way the classical concept of a saddle point initiated by Borel [1]. The existence of Nash equilibrium is usually proved in a non-constructive manner using fixed point theorems. For a constructive proof see [6]. The calculation of the equilibriums for zero sum games is usually made with the use of linear programming methods [2]. The over-carriage of the calculation methods for non-zero games is not immediate. An algorithm for non-degenerate bi-matrix games was suggested by Lemke and Howson [3]. A further improvement of their algorithm is due to Shapley [5] and other authors. Let us mention that those algorithms are not "effective" if one wants to show the dependence of the solutions on the elements of the matrices A and B. For simple cases, so-called grapho-analytic methods are more useful. Such an approach for zero sum games is shown in Vorob'ev's monograph [7]. Here we discuss such a method for a $2\times n$ bi-matrix game.

We look for an equilibrium (x^0, y^0), where

$$x^0 = (x_1^0, x_2^0) \in X \quad \text{and} \quad y^0 = (y_1^0, y_2^0, \ldots, y_n^0) \in Y.$$

Conditions (2) are obviously equivalent to

$$a^{i'j} y_j^0 \leqslant a^{ij} x_i^0 y_j^0, \quad i' = 1, 2, \tag{3}$$

$$b^{ij'} x_i^0 \leqslant b^{ij} x_i^0 y_j^0, \quad j' = 1, 2, \ldots, n. \tag{4}$$

LEMMA. *The situation (x^0, y^0) is an equilibrium for the bi-matrix game $\Gamma_{A,B}$ if and only if each inequality*

$$a^{i'j} y_j^0 < \max_i a^{ij} y_j^0$$

implies

$$x_{i'}^0 = 0,$$

and each inequality

$$b^{ij'} x_i^0 < \max_j b^{ij} x_i^0$$

implies

$$y_{j'}^0 = 0.$$

Proof. The assertion is obviously a consequence of inequalities (3) and (4) and the properties $\sum_i x_i^0 = 1$, $x_i^0 \geqslant 0$, $\sum_j y_j^0 = 1$, $y_j^0 \geqslant 0$.

As a direct consequence one has

COROLLARY 1. *Suppose that the $2\times n$ bi-matrix game $\Gamma_{A,B}$ has an equilibrium (x^0, y^0) with essentially mixed strategy $x^0 = (x_1^0, x_2^0)$. Then $a^{1j} y_j^0 = a^{2j} y_j^0$.*

Proof. Otherwise one has either $x_1^0 = 0$ or $x_2^0 = 0$.

Let $b_k = (b^{1k}, b^{2k}) \in \boldsymbol{R}^2$, $k = 1, 2, \ldots, n$, be points in the plane having as a convex hull

$$B^* = \operatorname{conv}\{b_1, b_2, \ldots, b_n\}.$$

The strategy $x^0 = (x_1^0, x_2^0)$ is considered as a vector in $\boldsymbol{R}^2$.

COROLLARY 2. *Let $(x^0, y^0) \in X \times Y$ be an equilibrium and let l be a support line for B^* in the direction x^0. Then, whenever $b_{j'}$ does not belong to l, the equality $y_{j'}^0 = 0$ is satisfied.*

Proof. The linear form $F(x_1, x_2) = x_1 x_1^0 + x_2 x_2^0$ attains the maximum on the set $\{b_1, b_2, \ldots, b_n\}$ only on $b_j \in l$. If $b_{j'} \notin l$, then according to the Lemma $y_{j'}^0 = 0$. The geometric illustration is given in Fig. 1.

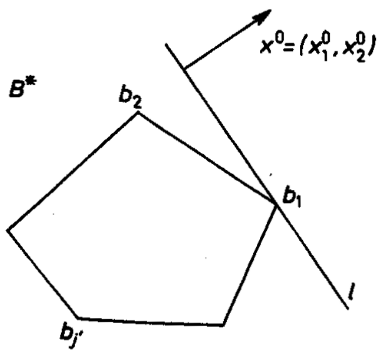

Fig. 1

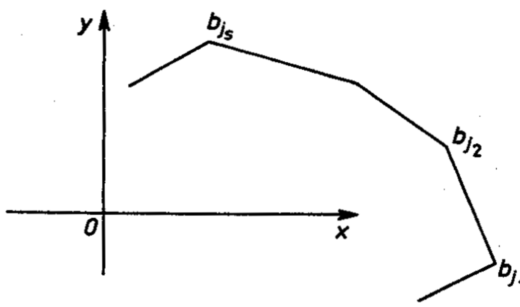

Fig. 2

DEFINITION. The considered $2\times n$ bi-matrix game is called *non-degenerate* if

1° each side of B^* contains exactly two of the points $b_1, b_2, \ldots, b_n$;
2° each two of the points $b_1, b_2, \ldots, b_n$ are different;
3° the vectors $(1, 0)$ and $(0, 1)$ are not support directions for sides of B^*;
4° $a^{1j} \neq a^{2j}$ for $j = 1, 2, \ldots, n$.

The above concept of non-degeneracy does not coincide thoroughly with the corresponding concept of non-degeneracy in [7]. The above definition is accepted for simplicity of the reasonings.

THEOREM 1. *For a non-degenerate bi-matrix game with $2\times n$ matrices A and B under the above notation let $b_{j_1}, b_{j_2}, \ldots, b_{j_s}$ be the consistent vertices of B^* counted counter-clockwise from the vertex b_{j_1} with support direction $(1, 0)$ to the vertex b_{j_s} with support direction $(0, 1)$ (Fig. 2). Then all the equilibriums in mixed strategies*

$$(x^0, y^0) \in X \times Y, \quad x^0 = (x_1^0, x_2^0), \quad y^0 = (y_1^0, y_2^0, \ldots, y_n^0)$$

are given with

$$x^0 = (1, 0), \quad y^0 = (0, \ldots, 0, \underset{j_1}{1}, 0, \ldots, 0) \quad \textit{if } a^{2j_1} < a^{1j_1}, \tag{5}$$

$$x^0 = (0, 1), \quad y^0 = (0, \ldots, 0, \underset{j_s}{1}, 0, \ldots, 0) \quad \textit{if } a^{1j_s} < a^{2j_s}, \tag{6}$$

$$x^0=(x_1^0,\ x_2^0),\qquad y^0=(0,\ \ldots,\ y_k^0,\ y_{k+1}^0,\ 0,\ \ldots,\ 0), \tag{7}$$

where

$$x_1^0=\frac{b^{2j_{k+1}}-b^{2j_k}}{b^{1j_k}+b^{2j_{k+1}}-b^{1j_{k+1}}-b^{2j_k}},\qquad x_2^0=\frac{b^{1j_k}-b^{1j_{k+1}}}{b^{1j_k}+b^{2j_{k+1}}-b^{1j_{k+1}}-b^{2j_k}},$$

$$y_k^0=\frac{a^{2j_{k+1}}-a^{1j_{k+1}}}{a^{1j_k}+a^{2j_{k+1}}-a^{1j_{k+1}}-a^{2j_k}},\qquad y_{k+1}^0=\frac{a^{1j_k}-a^{2j_k}}{a^{1j_k}+a^{2j_{k+1}}-a^{1j_{k+1}}-a^{2j_k}}$$

if $\operatorname{sgn}(a^{1j_k}-a^{2j_k})\neq\operatorname{sgn}(a^{1j_{k+1}}-a^{2j_{k+1}})$, $k=1,\ 2,\ \ldots,\ s-1$.

The breaking of any of the conditions for non-degenerate games means that the corresponding couple $(x^0,\ y^0)$ does not give an equilibrium situation in mixed strategies.

Proof. If $x^0=(1,\ 0)$, then

$$y^0=(0,\ \ldots,\ 0,\ \underset{j_1}{1},\ 0,\ \ldots,\ 0)$$

according to Corollary 1 and, by the Lemma, $a^{2j_1}<a^{1j_1}$ (remind that the case $a^{2j_1}=a^{1j_1}$ is not considered). On the contrary, if $a^{2j_1}<a^{1j_1}$, then, obviously,

$$x^0=(1,\ 0),\qquad y^0=(0,\ \ldots,\ 0,\ \underset{j_1}{1},\ 0,\ \ldots,\ 0)$$

is an equilibrium situation. Hence (5) is proved. In quite a similar way (6) can be obtained.

Let $\vec{n}_k=(\alpha_k,\ \beta_k)$ be a unit normal vector to the side $b_{j_k}b_{j_{k+1}}$ of B^*, $k=1,\ 2,\ \ldots,\ s-1$. Then α_k and β_k are the numbers x_1^0 and x_2^0, respectively, from (7). Since $\vec{n}_k$ is a support direction for $b_{j_k}b_{j_{k+1}}$, it follows from the Lemma and Corollaries 1 and 2 that $(x^0,\ y^0)$ with $x_1^0=\alpha_k$, $x_2^0=\beta_k$ is an equilibrium if and only if

$$y^0=(0,\ \ldots,\ 0,\ y_k^0,\ y_{k+1}^0,\ 0,\ \ldots,\ 0),$$
$$y_k^0\geqslant 0,\qquad y_{k+1}^0\geqslant 0,\qquad y_k^0+y_{k+1}^0=1$$

and

$$a^{1j_k}y_k^0+a^{1j_{k+1}}y_{k+1}^0=a^{2j_k}y_k^0+a^{2j_{k+1}}y_{k+1}^0,$$

whence one gets (7).

Another possibility for an equilibrium situation is $(x^0,\ y^0)$ with $\beta_{k-1}<x_2^0<\beta_k$, $k=1,\ 2,\ \ldots,\ s$. In this case we have

$$y^0=(0,\ \ldots,\ 0,\ \underset{j_k}{1},\ 0,\ \ldots,\ 0)$$

and, by Corollary 1, $a^{1j_k}=a^{2j_k}$, which is an excluded case. Hence no more equilibriums are obtained.

Theorem 1 gives the following particular case of the Lemke–Howson Theorem:

THEOREM 2. *For a non-degenerate* $2 \times n$ *bi-matrix game the number of the equilibriums* $(x^0, y^0) \in X \times Y$ *is finite and odd.*

Proof. Theorem 1 shows that the considered number of equilibriums, denote it by q, is finite. Accounting the conditions in which formulae (5)–(7) give equilibriums one sees that this number is given by the number of the changes of the sign in the sequence

$$-1, \Delta_1, \Delta_2, \ldots, \Delta_s, 1,$$

where $\Delta_k = a^{1j_k} - a^{2j_k}$. Obviously,

$$\begin{aligned}(-1)^q &= \operatorname{sgn}(-\Delta_1 \Delta_1 \Delta_2 \ldots \Delta_{s-1} \Delta_s \Delta_s) \\ &= \operatorname{sgn}(-\Delta_1^2 \Delta_2^2 \ldots \Delta_s^2) = -1,\end{aligned}$$

whence q is odd.

The Lemke–Howson Theorem [3] in its general case is about $m \times n$ bi-matrix games. The idea used here is different from its authors' proof. From Theorem 2 we obtain directly the existence theorem:

COROLLARY 3. *Each non-degenerate* $2 \times n$ *bi-matrix game has an equilibrium in mixed strategies.*

Proof. The number of equilibriums is odd, and hence not zero.

References

[1] E. Borel, *La théorie du jeu et les équations intégrales à noyau symétrique*, C. R. Acad. Sci. 173 (1921), pp. 1304–1308.

[2] G. B. Dantzig, *A proof of the equivalence of the programming problem and the game problem*, in: *Activity Analysis of Production and Allocation*, Cowles Commission Monograph No. 13, New York 1951.

[3] C. E. Lemke and J. T. Howson, *Equilibrium points of bi-matrix games*, SIAM J. 12 (1964), pp. 413–423.

[4] J. F. Nash, *Non-cooperative games*, Ann. of Math. 54 (1951), pp. 286–295.

[5] L. S. Shapley, *A note on the Lemke–Howson algorithm*, pp. 175–189 in: *Mathematical Programming Study*, Amsterdam 1974.

[6] H. Skarf, *The approximation of fixed points of a continuous mapping*, SIAM J. Appl. Math. 15 (1967), pp. 1328–1343.

[7] Н. Н. Воробьев, *Основы теории игр — бескоалиционные игры*, Наука, Москва 1984.

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Received on 1985.06.19;
revised version on 1986.12.30