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On the structure of solutions sets of differential and integral equations in Banach spaces

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Abstract. This paper gives some Kneser-type theorems for differential equations and Volterra integral equations in Banach spaces.

In this paper we investigate some topological properties of solutions sets of ordinary differential equations and Volterra integral equations in Banach spaces. More precisely, we prove that under some very general assumptions the following theorem is true: If all solutions of equation (1.1) or (2.1) exist on a compact interval J, then the set S of these solutions, considered as a subset of the Banach space of continuous functions, is a compact R_δ, i.e., S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. As every compact R_δ is a continuum, our results generalize the well-known Kneser–Kamke–Hukuhara–Sato theorem ([8], [9], [11], [13], [18] and [19]) for the finite dimensional case (see also Kelley [12]).

1. Kneser's theorem for ordinary differential equations. Assume that $N = \{1, 2, \ldots\}$, $R = (-\infty, \infty)$, E is a Banach space, W is an open subset of $R\times E$ and $(t_0, x_0) \in W$. We consider the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

where f is a continuous function from W into E.

For each compact interval $J = [t_0, a]$ and each bounded closed subset V of W we introduce the following notations:

$C(J, E)$ is the space of continuous functions $u\colon J\to E$ with the norm $\|u\|_c = \sup\{\|u(t)\|\colon t \in J\}$;

$C_1(J, E)$ is the space of continuously differentiable functions $u\colon J\to E$ with the norm $\|u\|_1 = \|u\|_c+\|u'\|_c$;

$S_n(J, V) = \{u \in C_1(J, E)\colon u(t_0) = x_0, \ \|u'(t)-f(t, u(t))\| \leqslant 1/n$ and $(t, u(t)) \in V$ for every $t \in J\}$;

α denotes the measure of non-compactness in $C(J, E)$ (cf. [14]).

We make the following assumptions on f:

(D_1) f is bounded on every bounded closed subset V of W;

(D_2) $\lim_{n\to\infty} \alpha(S_n(J, V)) = 0$ for each $J = [t_0, a]$ and each bounded closed subset V of W.

Obviously,

(1.2) $\lim_{n\to\infty} \alpha(S_n(J, V)) = 0 \Leftrightarrow$ *for every sequence* (u_n), $u_n \in S_n(J, V)$ *for* $n = 1, 2, \ldots$, *there exists a subsequence* (u_{n_k}) *uniformly convergent on* J.

Remark. It can be shown that condition (D_2) is fulfilled whenever f satisfies the assumptions of any one of the known existence theorems for (1.1) (cf. [3], [4], [6], [10], [16], [23] and [26]).

We say that a function g: $W \to E$ is *locally Lipschitzean* if for each $p \in W$ there is an open set U_p with $p \in U_p \subset W$ and a $k_p > 0$ such that $\|g(t, x) - g(t, y)\| \leqslant k_p \|x - y\|$ for all (t, x), $(t, y) \in U_p$.

Using standard techniques of finite dimensional theory the following lemma can be proved (cf. [2], [7], [15]).

LEMMA 1. *Let* g: $W \to E$ *be a continuous locally Lipschitzean function, and let* g *be bounded on every bounded closed subset* V *of* W.

1° *If* V *is a bounded closed subset of* W *such that the distance* $d(V, \partial W) > 0$, *then there exists a number* $h > 0$, *dependent only on* W, V, *and* $M = \sup\{\|g(t, x)\|: (t, x) \in V\}$, *such that for every* $(a, y) \in V$ *there exists a unique solution* x *of the initial value problem*

$$x' = g(t, x), \quad x(a) = y,$$

defined on $[a-h, a+h]$.

2° *Every solution* x *of the initial value problem*

$$x' = g(t, x), \quad x(t_0) = x_0,$$

can be extended to the maximal interval of existence (q_-, q_+).

LEMMA 2. *Suppose that* f *satisfies* (D_1), (D_2). *Let* V *be a bounded closed subset of* W *such that* $d(V, \partial W) > 0$. *Then there exists a positive number* h, *dependent only on* W, V, *and* $M = \sup\{\|f(t, x)\|: (t, x) \in V\}$, *which has the following property: if* u *is a solution of* (1.1) *on an interval* $[t_0, a]$, *and* $u(t) \in V$ *for* $t_0 \leqslant t \leqslant a$, *then* u *may be continued to the interval* $[t_0, a+h]$.

Proof. Let r be a positive number such that $d(V, \partial W) \geqslant 2r$. Then the set $B(V, r) = \{p \in R \times E: d(p, V) \leqslant r\}$ is bounded, closed and contained in W. Let $M = \sup\{\|f(t, x)\|: (t, x) \in B(V, r)\}$, $h = \min(r, r/(M+1))$, $I = [a, a+h]$ and $B = \{x \in E: \|x - u(a)\| \leqslant r\}$. Obviously, $I \times B \subset B(V, r)$. By [15]; Lemma 1; for each $n \in N$ there is a continuous locally Lipschitzean function g_n: $W \to E$ such that $g_n(a, u(a)) = f(a, u(a))$ and $\|g_n(t, x) - f(t, x)\| \leqslant 1/n$ for $(t, x) \in W$. Because $\|g_n(t, x)\| \leqslant M + 1/n \leqslant M+1$ for $(t, x) \in W$, $n \in N$, it follows from Lemma 1 that there exists a solution $y_n: I \to B$ of the problem $y' = g_n(t, y)$, $y(a) = u(a)$. Put $J = [t_0, a+h]$ and

$$v_n(t) = \begin{cases} u(t) & \text{for } t_0 \leqslant t \leqslant a, \\ y_n(t) & \text{for } a \leqslant t \leqslant a+h. \end{cases}$$

Since $y_n'(a) = g_n(a, y_n(a)) = g_n(a, u(a)) = f(a, u(a)) = u'(a)$ and $\|y_n'(t) - f(t, y_n(t))\| = \|g_n(t, y_n(t)) - f(t, y_n(t))\| \leqslant 1/n$ for $t \in I$, the function v_n belongs to $S_n(J, B(V, r))$. By (D_2) and (1.2), there exists a subsequence (v_{n_k}) of (v_n) converging uniformly on J to a function v. It is easy to verify that v satisfies (1.1) on J. Moreover, $v|[t_0, a] = u$. This completes the proof.

Now consider the sequence of sets W_n defined by

$$W_n = \{(t, x) \in W: |t| < n, \|x\| < n, d((t, x), \partial W) > 1/n\}.$$

We observe that all W_n are bounded open subsets of W, $W = \bigcup_{n=1}^{\infty} W_n$, $\overline{W}_n \subset W_{n+1}$ and $d(\overline{W}_n, \partial W) \geqslant 1/n$. Using the same argument as in the proofs of Theorems II. 3.1, II. 3.2 in [7], by Lemmas 1, 2 and (1.2), we obtain the following theorems:

THEOREM 1. *If f satisfies* (D_1), (D_2), *then every solution x of* (1.1) *may be extended to the right maximal interval of existence* $[t_0, q)$.

THEOREM 2. *For $n = 1, 2, \ldots$, consider the equations*

$$(D_n) \qquad x' = g_n(t, x), \quad x(t_0) = x_0,$$

where the functions $g_n: W \to E$ are locally Lipschitzean or satisfy (D_1), (D_2). *Suppose that f satisfies* (D_1), (D_2), *and* $\lim_{n\to\infty} g_n(t, x) = f(t, x)$ *uniformly on W. Let u_n be a solution of* (D_n) *with right maximal interval of existence* $[t_0, q_n)$ *for $n = 1, 2, \ldots$ Then there is a solution u of* (1.1) *with right maximal interval* $[t_0, q)$ *and a subsequence* (u_{n_k}) *such that for any $d \in (t_0, q)$ we have* $[t_0, d] \subset [t_0, q_{n_k})$ *for k sufficiently large and* $\lim_{k\to\infty} u_{n_k}(t) = u(t)$ *uniformly on* $[t_0, d]$.

Now we shall prove the following generalized Kneser theorem for differential equations in Banach spaces:

THEOREM 3. *If f satisfies* (D_1), (D_2) *and all solutions of* (1.1) *exist on an interval $J = [t_0, d]$, then the set S of all solutions of* (1.1) *defined on J is a compact R_δ in $C(J, E)$.*

Proof. Let (u_n) be a sequence of solutions of (1.1) with right maximal intervals of existence $[t_0, q_n)$. Applying Theorem 2 to the case $g_n = f$, $n = 1, 2, \ldots$, we see that there exist a solution u of (1.1) with right maximal interval of existence $[t_0, q)$ and a subsequence (u_{n_k}) of (u_n) such that

$\lim_{k\to\infty} u_{n_k}(t) = u(t)$ uniformly on compact subintervals of $[t_0, q)$. Note that $q_n > d$ and $q > d$, since all solutions of (1.1) exist on J. Consequently $\lim_{n\to\infty} u_{n_k}(t) = u(t)$ uniformly on J. This shows that the set S is compact in $C(J, E)$. Hence it follows that the set $H = \{(t, x(t)): x \in S, t \in J\}$ is compact in $R \times E$. Obviously $H \subset W$. Let r be a positive number such that $d(H, \partial W) \geqslant 2r$. Put $V = \{(t, x) \in R \times E: d((t, x), H) < r\}$. Then V is a bounded open subset of W and $\bar{V} \subset W$. Let $M = \sup\{\|f(t, x)\|: (t, x) \in \bar{V}\}$. By the Dugundji extension theorem there is a continuous function $g: R \times E \to E$ such that $g|\bar{V} = f|\bar{V}$ and $\|g(t, x)\| \leqslant M$ for $(t, x) \in R \times E$. From the definition of V it follows that S contains all solutions of the equation

$$x' = g(t, x), \quad x(t_0) = x_0, \tag{1.3}$$

defined on J. Obviously, if $u \in S$, then u is a solution of (1.3) on J.

Put $X = \{u \in C_1(J, E): u(t_0) = x_0\}$ and $Y = \{u \in C_1(J, E): u(t_0) = 0\}$. We will regard X as a complete metric subspace of $C_1(J, E)$, and Y as a Banach subspace of $C_1(J, E)$. Define a continuous (cf. [21]) mapping $G: C(J, E) \to X$ by the formula

$$G(x)(t) = x_0 + \int_{t_0}^{t} g(s, x(s))\,ds \quad \text{for } x \in C(J, E) \text{ and } t \in J.$$

Let $T = I - G$, where I denotes the identity mapping on X; T is a continuous mapping $X \to Y$.

Since all solutions of (1.1) exist on J, and their graphs are contained in V, Theorem 2 proves that there is a number $p > 0$ such that

(1.4) *For each continuous locally Lipschitzean function $h: V \to E$, which satisfies the inequality $\|h(t, x) - f(t, x)\| \leqslant p$ for $(t, x) \in V$, any solution x of the initial value problem $x' = h(t, x)$, $x(t_0) = x_0$, can be continued to J.*

Put $S_n = \{u \in X: \|u - G(u)\|_1 \leqslant 1/n\}$. Choose $m \in N$ and $\varepsilon > 0$ such that $1/m + 2\varepsilon \leqslant p$. By [15]; Lemma 1; there exists a continuous locally Lipschitzean function $h: R \times E \to E$ such that

$$\|h(t, x) - g(t, x)\| \leqslant \varepsilon \quad \text{for } (t, x) \in R \times E.$$

For any $u \in S_n$, $n \geqslant m$, put

$$s_u(t) = \begin{cases} u'(t_0) - h(t_0, u(t_0)) & \text{for } t \leqslant t_0, \\ u'(t) - h(t, u(t)) & \text{for } t \in J, \\ u'(d) - h(d, u(d)) & \text{for } t \geqslant d, \end{cases}$$

and $h_u(t, x) = h(t, x) + s_u(t)$ for $(t, x) \in R \times E$. Then h_u is a continuous locally Lipschitzean function such that $\|h_u(t, x) - g(t, x)\| \leqslant p$ for (t, x)

$\in R\times E$, and u satisfies on J the equation $x' = h_u(t, x)$, $x(t_0) = x_0$. By (1.4), it hence follows that the graphs of all $u \in S_n$, $n \geqslant m$, are contained in V, so that $S_n \subset S_n(J, \bar{V})$ for $n \geqslant m$. Therefore $\lim_{n\to\infty} a(S_n) \leqslant \lim_{n\to\infty} a(S_n(J, \bar{V})) = 0$.

Now, repeating the same argument as in [21], we conclude that the set $S = T^{-1}(0)$ is a compact R_δ in $C(J, E)$.

2. Kneser's theorem for Volterra integral equations. Now we shall consider the Volterra integral equation

$$x(t) = p(t) + \int_0^t f(t, s, x(s))\,ds, \tag{2.1}$$

where $\int$ denotes the Bochner integral.

Let $I = [0, c)$ be an interval in R, and let W be an open subset of a Banach space E. Assume that

(I_1) $p\colon I \to W$ is a continuous function;

(I_2) $(t, s, x) \to f(t, s, x)$ is a function of the set $\{0 \leqslant s \leqslant t < c,\ x \in W\}$ into E, which satisfies the following conditions:

1° for each fixed $x \in W$ and $t \in I$ the function $s \to f(t, s, x)$ is strongly L-measurable on $[0, t]$;

2° for each fixed t, s, $0 \leqslant s \leqslant t < c$, the function $x \to f(t, s, x)$ is continuous on W;

3° for each compact interval $J = [0, a] \subset I$ and each bounded closed set $K \subset W$ there exist real-valued functions $(t, s) \to \mu(t, s)$ and $(\tau, t, s) \to \psi(\tau, t, s)$ $(0 \leqslant s \leqslant t \leqslant \tau \leqslant a)$ such that

(i) for each fixed t, τ the functions $s \to \psi(\tau, t, s)$ and $s \to \mu(t, s)$ are L-integrable on $[0, t]$;

(ii) $\sup\{\|f(\tau, s, x) - f(t, s, x)\|\colon x \in K\} \leqslant \psi(\tau, t, s)$, $\sup\{\|f(t, s, x)\|\colon x \in K\} \leqslant \mu(t, s)$;

(iii) $l(h, J, K) = \sup\{\int_t^\tau \mu(\tau, s)\,ds + \int_0^t \psi(\tau, t, s)\,ds\colon t, \tau \in J,\ 0 \leqslant \tau - t \leqslant h\} \to 0$ when $h \to 0$.

(I_3) For each compact interval $J = [0, a] \subset I$ and each bounded closed set $K \subset W$ let $S_n(J, K)$ denote the set of all continuous functions $u\colon J \to K$ which satisfy the inequality

$$\Big\| u(t) - p(t) - \int_0^t f(t, s, u(s))\,ds \Big\| \leqslant 1/n \quad \text{for } t \in J.$$

Then $\lim_{n\to\infty} a(S_n(J, K)) = 0$.

Remark. In particular, hypotheses $(I_1)-(I_3)$ are fulfilled whenever p, f satisfy the assumptions of Theorem 1 of Theorem 2 of [22].

The main result of this section is the following generalized Kneser theorem for equation (2.1):

THEOREM 4. *If all solutions of* (2.1) *exist on a subinterval* $J = [0, d]$ *of* I, *then the set* S *of all solutions of* (2.1) *defined on* J *is a compact* R_δ *in* $C(J, E)$.

Before proving Theorem 4, we shall prove six lemmas which will simplify the proof of this theorem.

Let $J_0 = [0, b]$, where $d < b < c$. Denote by Z the set all continuous functions $z\colon J \to W$ such that $\|z(0)\| < d(p(0), \partial W)/2$. Put $w(h, z) = \sup\{\|z(t)-z(s)\|\colon t, s \in J_0, |t-s| \leqslant h\}$ for any $z \in Z$. Let us fix $z \in Z$. Choose $h_x > 0$ in such a way that

$$w(h_z, p) + w(h_z, z) < d(p(0), \partial W)/2.$$

For any ε, $0 < \varepsilon < h_z$, we consider the equation

$$x(t) = \begin{cases} p(t) + z(t) & \text{for } 0 \leqslant t \leqslant \varepsilon, \\ p(t) + z(t) + \int\limits_0^{t-z} f(t-\varepsilon, s, x(s))\, ds & \text{for } \varepsilon \leqslant t \leqslant b. \end{cases} \tag{2.2}$$

We can easily prove the following

LEMMA 3. 1° *Let* V *be a bounded closed subset of* W *such that* $d(V, \partial W) \geqslant 2r > 0$, *and let* h *be a positive number whech satisfies the inequality* $w(p, h) + w(h, z) + l(h, J_0, B(V, r)) \leqslant r$. *If* u *is a solution of* (2.2) *on an interval* $[0, a]$, *and* $u(t) \in V$ *for* $t \in [0, a]$, *then* u *may be continued to the interval* $[0, a_h]$, *where* $a_h = \min(b, a+h)$.

2° *In the domain* $[0, b) \times W$ *every solution* x *of* (2.2) *may be continued to the maximal interval of existence* $[0, q]$.).

LEMMA 4. *Let* V *be a bounded closed subset of* W.

1° *If* u *is a solution of* (2.2) *defined on a compact interval* $[0, a] \subset J_0$ *with values in* V, *then*

$$\Big\| u(t) - p(t) - \int\limits_0^t f(t, s, u(s))\, ds \Big\| \leqslant \|z(t)\| + l(\varepsilon, J_0, V) \tag{2.3}$$

fort $\in [0, a]$.

2° *If* (u_n) *is a sequence of functions* $u_n\colon [0, a] \to V$ *such that* $\lim\limits_{n\to\infty} \Big\|u_n(t) - p(t) - \int\limits_0^t f(t, s, u_n(s))\, ds\Big\| = 0$ *uniformly on* $[0, a]$, *then there exists a subsequence* (u_{n_k}) *of* (u_n) *which converges uniformly on* $[0, a]$ *to a solution* u *of equation* (2.1).

Proof. Let μ, ψ be functions defined in (I_2) corresponding to the pair J_0, V.

1° Suppose that a function $u\colon [0, a]\to V$ is a solution of (2.2). Then

$$\Big\|u(t)-p(t)-\int_0^t f\big(t, s, u(s)\big)ds\Big\| = \Big\|z(t)-\int_0^t f\big(t, s, u(s)\big)ds\Big\|$$
$$\leqslant \|z(t)\|+\int_0^t \mu(t, s)ds \quad \text{for } 0\leqslant t\leqslant \varepsilon,$$

and

$$\Big\|u(t)-p(t)-\int_0^t f\big(t, s, u(s)\big)ds\Big\|$$
$$= \Big\|z(t)+\int_0^{t-\varepsilon} f\big(t-\varepsilon, s, u(s)\big)ds-\int_0^t f\big(t, s, u(s)\big)ds\Big\|$$
$$\leqslant \|z(t)\|+\int_0^{t-\varepsilon} \Big\|f\big(t, s, u(s)\big)-f\big(t-\varepsilon, s, u(s)\big)\Big\|ds+\int_{t-\varepsilon}^t \Big\|f\big(t, s, u(s)\big)\Big\|ds$$
$$\leqslant \|z(t)\|+\int_0^{t-\varepsilon} \psi(t, t-\varepsilon, s)ds+\int_{t-\varepsilon}^t \mu(t, s)ds$$

for $\varepsilon\leqslant t\leqslant a$, which proves (2.3).

2° If (u_n) is a sequence of functions $u_n\colon [0, a]\to V$ such that

$$(*)\qquad \lim_{n\to\infty}\Big\|u_n(t)-p(t)-\int_0^t f\big(t, s, u_n(s)\big)ds\Big\| = 0 \quad \text{uniformly on } [0, a],$$

then, by (I_3) and (1.2), there exists a subsequence (u_{n_j}) of (u_n) which converges uniformly on $[0, a]$ to a function $u\colon [0, a]\to V$. Since $\big\|f\big(t, s, u_{n_j}(s)\big)-f\big(t, s, u(s)\big)\big\|\leqslant 2\mu(t, s)$ and $\lim_{j\to\infty} f\big(t, s, u_{n_j}(s)\big) = f\big(t, s, u(s)\big)$ for each $0\leqslant s\leqslant t\leqslant a$, the Lebesgue dominated convergence theorem proves that

$$\lim_{j\to\infty}\int_0^t \Big\|f\big(t, s, u_{n_j}(s)\big)-f\big(t, s, u(s)\big)\Big\|ds = 0 \quad \text{for } t\in[0, a].$$

By $(*)$, this implies $u(t) = p(t)+\int_0^t f\big(t, s, u(s)\big)ds$ for $0\leqslant t\leqslant a$, which ends the proof.

LEMMA 5. *Let V be a bounded closed subset of W such that $d(V, \partial W)\geqslant 2r > 0$, and let h be a positive number which satisfies the inequality $w(h, p)+l\big(h, J_0, B(V, r)\big)\leqslant r$. If u is a solution of (2.1) on an interval $[0, a]$, and $u(t)\in V$ for each $0\leqslant t\leqslant a$, then u may be continued to the interval $[0, a_h]$, where $a_h = \min(b, a+h)$.*

Proof. For a small $\varepsilon > 0$ we define a function v_ε by the formula

$$v_\varepsilon(t) = \begin{cases} u(t) & \text{for } 0 \leqslant t \leqslant a, \\ u(a) & \text{for } a \leqslant t \leqslant a+\varepsilon, \\ p(t-\varepsilon) + \int\limits_0^{t-\varepsilon} f\big(t-\varepsilon, s, v_\varepsilon(s)\big)ds & \text{for } a+\varepsilon \leqslant t \leqslant a_h. \end{cases}$$

Obviously, v_ε is continuous on $[0, a_h]$, and $v_\varepsilon(t) \in B(V, r)$ for $t \in [0, a_h]$, because

$$\begin{aligned} &\|v_\varepsilon(t) - u(a)\| \\ &= \Big\|p(t-\varepsilon) + \int\limits_0^{t-\varepsilon} f\big(t-\varepsilon, s, v_\varepsilon(s)\big)ds - p(a) - \int\limits_0^a f\big(a, s, v_\varepsilon(s)\big)ds\Big\| \\ &\leqslant \|p(t-\varepsilon) - p(a)\| + \int\limits_a^{t-\varepsilon} \big\|f\big(t-\varepsilon, s, v_\varepsilon(s)\big)\big\|ds + \int\limits_0^a \big\|f\big(t-\varepsilon, s, v_\varepsilon(s)\big) - \\ &\qquad\qquad - f\big(a, s, v_\varepsilon(s)\big)\big\|ds \\ &\leqslant w(h, p) + l\big(h, J_0, B(V, r)\big) \end{aligned}$$

for $a+\varepsilon \leqslant t \leqslant a_h$.

On the other hand,

$$\begin{aligned} &\Big\|v_\varepsilon(t) - p(t) - \int\limits_0^t f\big(t, s, v_\varepsilon(s)\big)ds\Big\| \\ &= \Big\|u(a) - p(t) - \int\limits_0^t f\big(t, s, v_\varepsilon(s)ds\Big\| \\ &= \Big\|p(a) - p(t) + \int\limits_0^a f\big(a, s, v_\varepsilon(s)\big)ds - \int\limits_0^t f\big(t, s, v_\varepsilon(s)\big)ds\Big\| \\ &\leqslant \|p(t) - p(a)\| + \int\limits_0^a \big\|f\big(t, s, v_\varepsilon(s)\big) - f\big(a, s, v_\varepsilon(s)\big)\big\|ds + \int\limits_a^t \big\|f\big(t, s, v_\varepsilon(s)\big)\big\|ds \\ &\leqslant w(\varepsilon, p) + l\big(\varepsilon, J_0, B(V, r)\big) \quad \text{for } a \leqslant t \leqslant a+\varepsilon, \end{aligned}$$

and

$$\begin{aligned} &\Big\|v_\varepsilon(t) - p(t) - \int\limits_0^t f\big(t, s, v_\varepsilon(s)\big)ds\Big\| \\ &\quad= \Big\|p(t-\varepsilon) - p(t) + \int\limits_0^{t-\varepsilon} f\big(t-\varepsilon, s, v_\varepsilon(s)\big)ds - \int\limits_0^t f\big(t, s, v_\varepsilon(s)\big)ds\Big\| \\ &\quad\leqslant \|p(t) - p(t-\varepsilon)\| + \int\limits_0^{t-\varepsilon} \big\|f\big(t, s, v_\varepsilon(s)\big) - f\big(t-\varepsilon, s, v_\varepsilon(s)\big)\big\|ds + \\ &\qquad\qquad + \int\limits_{t-\varepsilon}^t \big\|f\big(t, s, v_\varepsilon(s)\big)\big\|ds \\ &\quad\leqslant w(\varepsilon, p) + l\big(\varepsilon, J_0, B(V, r)\big) \quad \text{for } a+\varepsilon \leqslant t \leqslant a_h. \end{aligned}$$

This shows that

$$\lim_{\varepsilon\to 0}\left\|v_\varepsilon(t)-p(t)-\int_0^t f\big(t,s,v_\varepsilon(s)\big)ds\right\| = 0 \quad \text{uniformly on } [0,a_h].$$

Let (ε_n) be a sequence of positive real numbers tending to 0 as $n\to\infty$, and let $u_n = v_{\varepsilon_n}$. By Lemma 4 there exists a subsequence (u_{n_j}) of (u_n) which converges uniformly on $[0,a_h]$ to a solution v of equation (2.1). Moreover, $v|[0,a] = u$, because $u_n|[0,a] = u$ for each n, so the proof is complete.

Now, using the same argument as in the proofs of Theorems II. 3.1, II. 3.2. in [7], by Lemmas 3–5, we obtain the following

LEMMA 6. *In the domain* $[0,b)\times W$ *every solution* x *of* (2.1) *may be continued to the maximal interval of existence* $[0,q)$;

and

LEMMA 7. *In the domain* $[0,b)\times W$ *we consider equation* (2.1) *and the equations*

$$(\mathrm{I}_n)\qquad x(t) = \begin{cases} p(t)+z_n(t) & \text{for } 0\leqslant t\leqslant \varepsilon_n,\\ p(t)+z_n(t)+\displaystyle\int_0^{t-\varepsilon_n} f\big(t-\varepsilon_n,s,x(s)\big)ds & \text{for } \varepsilon_n\leqslant t\leqslant b,\end{cases}$$

where $z_n\in Z$, $0<\varepsilon_n<h_{z_n}$ *and* $n=1,2,\ldots$ *Suppose that* $\lim_{n\to\infty}\varepsilon_n = 0$ *and* $\lim_{n\to\infty} z_n(t) = 0$ *uniformly on* J_0. *Let* u_n *be a solution of* (I_n) *or* (2.1) *with maximal interval of existence* $[0,q_n)$ *for* $n=1,2,\ldots$ *Then there is a solution* u *of* (2.1) *with maximal interval of existence* $[0,q)$ *and a subsequence* (u_{n_j}) *such that for any* $a\in(0,q)$ *we have* $[0,a]\subset[0,q_{n_j})$ *for* j *sufficiently large and* $\lim_{j\to\infty}u_{n_j}(t) = u(t)$ *uniformly on* $[0,a]$.

The next lemma is a modification of the Dugundji extension theorem.

LEMMA 8. *Let* V *be a bounded closed subset of* W, *and let* μ, ψ *be functions defined in* (I_2) *corresponding to the pair* J_0, V. *Then there exists a function* $(t,s,x)\to g(t,s,x)$, *defined for* $0\leqslant s\leqslant t\leqslant b$ *and* $x\in E$, *such that*

1° $g(t,s,x) = f(t,s,x)$ *for each* $0\leqslant s\leqslant t\leqslant b$ *and* $x\in V$;

2° *for each fixed* $x\in E$ *and* $t\in J_0$ *the function* $s\to g(t,s,x)$ *is strongly* L*-measurable on* $[0,t]$;

3° *for each fixed* t, s, $0\leqslant s\leqslant t\leqslant b$, *the function* $x\to g(t,s,x)$ *is continuous on* E;

4° $\|g(\tau,s,x)-g(t,s,x)\|\leqslant\psi(\tau,t,s)$ *for* $0\leqslant s\leqslant t\leqslant\tau\leqslant b$, $x\in E$;

5° $\|g(t,s,x)\|\leqslant\mu(t,s)$ *for* $0\leqslant s\leqslant t\leqslant b$, $x\in E$.

Proof. For any $x \in E \setminus V$ define

$$Q(x) = \{y \in E:\ \|y-x\| < d(x, \partial V)/3\}.$$

Then $Q(x)$ is an open subset of E and $E\setminus V \subset \bigcup_{x\in E\setminus V} Q(x)$. Since any metric space is paracompact, there is an open locally finite refinement $\{P_a:\ a \in A\}$ of $\{Q(x):\ x \in E\setminus V\}$ which covers $E\setminus V$. Put $v_a(x) = d\big(x, E\setminus(V\cup P_a)\big)/ / \sum_{\beta\in A} d\big(x, E\setminus(V\cup P_\beta)\big)$ for each $x \in E$ and $a \in A$. It is easy to verify (cf. [1], p. 238) that v_a is continuous on E. For any $a \in A$ choose $x_a \in P_a$ and $y_a \in \partial V$ in such a way that $\|x_a - y_a\| < 2d(x_a, \partial V)$. Put

$$g(t, s, x) = \begin{cases} f(t, s, x) & \text{for } 0 \leqslant s \leqslant t \leqslant b,\ x \in V, \\ \sum_{a\in A} f(t, s, y_a) v_a(x) & \text{for } 0 \leqslant s \leqslant t \leqslant b,\ x \in E\setminus V. \end{cases}$$

In the same way as in [1], p. 239, we prove that for each fixed t, s the function $x \to g(t, s, x)$ is continuous on E.

On the other hand, for any $x \in E\setminus V$ there exists a finite set $B_x \subset A$ such that

$$g(t, s, x) = \sum_{a\in B_x} f(t, s, y_a) v_a(x). \tag{2.4}$$

This implies that for each fixed t, x the function $s \to g(t, s, x)$ is strongly L-measurable on $[0, t]$.

Moreover, from (2.4) it follows that

$$\begin{aligned}\|g(\tau, s, x) - g(t, s, x)\| &= \Big\| \sum_{a\in B_x} \big(f(\tau, s, y_a) - f(t, s, y_a)\big) v_a(x) \Big\| \\ &\leqslant \sum_{a\in B_x} v_a(x) \|f(\tau, s, y_a) - f(t, s, y_a)\| \\ &\leqslant \sum_{a\in B_x} v_a(x) \psi(\tau, t, s) \leqslant \psi(\tau, t, s)\end{aligned}$$

and

$$\begin{aligned}\|g(t, s, x)\| = \Big\| \sum_{a\in B_x} f(t, s, y_a) v_a(x) \Big\| &\leqslant \sum_{a\in B_x} v_a(x) \|f(t, s, y_a)\| \\ &\leqslant \sum_{a\in B_x} v_a(x) \mu(t, s) \leqslant \mu(t, s)\end{aligned}$$

for each $0 \leqslant s \leqslant t \leqslant \tau \leqslant b$ and $x \in E$. This concludes the proof of Lemma 8.

Proof of Theorem 4. Suppose that all solutions of (2.1) exist on $J = [0, d]$. Denote by S the set of all solutions of (2.1) defined on J. Similarly as in the proof of Theorem 3, from Lemma 7 we deduce that S is compact in $C(J, E)$, and therefore the set $H = \{x(t):\ x \in S,\ t \in J\}$ is compact in E. Let r be a positive number such that $d(H, \partial W) \geqslant 2r$, and let $V = \{x \in E: d(x, H) < r\}$. Then V is a bounded open subset of E and $\bar{V} \subset W$. Let g be the function defined in Lemma 8 corresponding to the pair J_0, $\bar{V}$. From the definition of V it follows that S contains all solutions of the equation

$$x(t) = p(t) + \int_0^t g\big(t, s, x(s)\big)\, ds \tag{2.5}$$

defined on J. Conversely, if $u \in S$, then u is a solution of (2.5).

Define a mapping $G: C(J, E) \to C(J, E)$ by the formula

$$G(x)(t) = p(t) + \int_0^t g\big(t, s, x(s)\big)\, ds \quad \text{for } x \in C(J, E) \text{ and } t \in J.$$

By (I_2) and Lemma 8, we have

$$\|G(x)(t) - G(x)(s)\| \leqslant \|p(t) - p(s)\| + l(|t-s|, J_0, \bar{V})$$

for each $x \in C(J, E)$ and $t, s \in J$, which shows that the set $G\big(C(J, E)\big)$ is equiuniformly continuous. Put

$$w(h) = \sup\big\{\|y(t) - y(s)\|: \ y \in G\big(C(J, E)\big),\ t, s \in J,\ |t-s| \leqslant h\big\}.$$

Then $\lim_{h \to 0+} w(h) = 0$.

Now assume that x_n, $x \in C(J, E)$ and $\lim_{n\to\infty} \|x - x_n\|_c = 0$. Because $\lim_{n\to\infty} g\big(t, s, x_n(s)\big) = g\big(t, s, x(s)\big)$ and $\big\|g\big(t, s, x_n(s)\big) - g\big(t, s, x(s)\big)\big\| \leqslant 2\mu(t, s)$ for $0 \leqslant s \leqslant t \leqslant d$, the Lebesgue theorem proves that $\lim_{n\to\infty} G(x_n)(t) = G(x)(t)$ for every $t \in J$, and hence, by the equicontinuity of $G\big(C(J, E)\big)$, $\lim_{n\to\infty} \|G(x_n) - G(x)\|_c = 0$. This proves the continuity of G.

Further, let $T = I - G$, where I denotes the identity mapping on $C(J, E)$. Obviously T is a continuous mapping $C(J, E) \to C(J, E)$.

We shall show that T is 0-closed (cf. [20]). Since all solution of (2.1) exist on J, and their values belong to V, a simple argument based on Lemma 7 shows that there is a number $q > 0$ such that

(2.6) *For each continuous function $z: J_0 \to V$ and each $\varepsilon > 0$ such that $\|z(t)\| + \varepsilon \leqslant q$ for all $t \in J_0$, in the domain $[0, b) \times V$ any solution x of equation (2.2) may be continued to J.*

Put $S_n = \{u \in C(J, E): \|u - G(u)\|_c \leqslant 1/n\}$. Choose $m \in N$ and $\varepsilon > 0$ such that $1/m + w(\varepsilon) + \varepsilon + \int_0^t \mu(t, s)\, ds \leqslant q$ for $0 \leqslant t \leqslant \varepsilon$. For any $u \in S_n$, $n \geqslant m$, let

$$y_u(t) = u(t) - p(t) - \int_0^t g\big(t, s, u(s)\big)\, ds \quad \text{for } t \in J,$$

and

$$v_u(t) = \begin{cases} \int_0^t g\big(t, s, u(s)\big) ds & \text{for } 0 \leqslant t \leqslant \varepsilon, \\ \int_0^t g\big(t, s, u(s)\big) ds - \int_0^{t-\varepsilon} g\big(t-\varepsilon, s, u(s)\big) ds & \text{for } \varepsilon \leqslant t \leqslant d. \end{cases}$$

Then $\|y_u(t)\| \leqslant 1/m$ for $t \in J$, $\|v_u(t)\| \leqslant \int_0^t \mu(t, s) ds$ for $0 \leqslant t \leqslant \varepsilon$, and $\|v_u(t)\| \leqslant w(\varepsilon)$ for $\varepsilon \leqslant t \leqslant d$. Putting $z_u(t) = y_u(t) + v_u(t)$ for $t \in J$, and $z_u(t) = z_u(d)$ for $t \geqslant d$, we see that $\|z_u(t)\| + \varepsilon \leqslant q$ for $t \in J_0$ and u satisfies on J equation (2.2) with $z = z_u$. By (2.6) hence we deduce that the values of all $u \in S_n$, $n \geqslant m$, belong to V, so that $S_n \subset S_n(J, \bar{V})$ for $n \geqslant m$. Consequently, $\lim_{n\to\infty} \alpha(S_n) \leqslant \lim_{n\to\infty} \alpha\big(S_n(J, \bar{V})\big) = 0$, and therefore $T = I - G$ is 0-closed.

Applying Theorem 5 of [20], we see that the set $S = T^{-1}(0)$ is a compact R_δ in $C(J, E)$, which was to be proved.

Remark. If $f(t, s, x) = f_1(s, x) + f_2(s, x)$, where f_1 is completely continuous and f_2 satisfies the assumptions from the papers of C. Olech [16] and T. Ważewski [26], then f satisfies (D_2) but (I_3) does not hold. This example proves that Theorem 4 does not imply Theorem 3.

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