ANNALES
POLONICI MATHEMATICI
XXXIV (1977)

The problem of the number of switches in parabolic equations with control

by ANDRZEJ KARAFIAT (Kraków)

Abstract. The main theorem of the paper says that the optimal control in a process described by a parabolic equation with a boundary control has a finite number of switches. The estimation of that number is given. Other theorems concern some fundamental properties of the optimal control and describe its changes by moving of the final moment.

Introduction. We will consider in this paper the optimal control problem in a process described by one parabolic equation of one space variable with a control contained in a boundary condition of mixed type. The norm of the solution at the final moment in the space L^∞ is taken as a cost functional.

The main purpose of this paper is to prove a theorem about the finite number of switches of an optimal control. Theorem 2.11 estimates that number. We also prove theorems concerning the existence and uniqueness of the optimal control (2.1 and 2.9), the bang-bang principle (2.8) and also a theorem giving an inequality for the number of zeros of a solution of a parabolic equation (1.11). The third section presents the changes in optimal control, and especially in the number of switches with the movement of the final moment.

This paper was inspired by the paper of Ju. W. Egorov [2] and the books of J. L. Lions [4] and A. G. Butkowskiĭ [1].

The author wishes to express his gratefulness to Professor Andrzej Pliś for his helpful advice and assistance.

1. Let us consider a process described by a partial differential equation of parabolic type with a one-dimensional space variable

$$(1)\quad a(t,x)\frac{\partial^2 y(t,x)}{\partial x^2}+b(t,x)\frac{\partial y(t,x)}{\partial x}+c(t,x)y(t,x)-\frac{\partial y(t,x)}{\partial t}=0$$

in the domain $P:=(0,T]\times(-1,1)$ with the initial condition

$$(2)\qquad y(0,x)=g(x),\quad x\in[-1,1],$$

and boundary conditions

$$(3)\qquad \begin{aligned} -\frac{\partial y}{\partial x}\Big|_{x=-1} + \beta(t)y(t,\,-1) &= u(t),\\ \frac{\partial y}{\partial x}\Big|_{x=1} + \gamma(t)y(t,1) &= v(t), \end{aligned}\qquad t\in(0,T].$$

We shall make the following assumptions concerning this problem:

The functions $a(t,x)$, $b(t,x)$, $c(t,x)$ *are defined and continuous on* $\bar{P}$ *(where* $\bar{P}$ *is the closure of* P*)*;

There exist constants L_1, $L_2>0$ *such that for any* $(t,x)\in P$

$$L_1\leqslant a(t,x)\leqslant L_2;$$

There exist constants A, $\alpha>0$ *such that for any* (t_1,x_1), $(t_2,x_2)\in P$:

$$(4)\qquad \begin{aligned} |a(t_1,x_1)-a(t_2,x_2)| &\leqslant A(|x_1-x_2|^\alpha+|t_1-t_2|^\alpha),\\ |b(t_1,x_1)-b(t_1,x_2)| &\leqslant A|x_1-x_2|^\alpha,\\ |c(t_1,x_1)-c(t_1,x_2)| &\leqslant A|x_1-x_2|^\alpha; \end{aligned}$$

$c(t,x)\leqslant 0$ *for each* $(t,x)\in P$;

The functions $\beta(t)$, $\gamma(t)$ *are non-negative and continuous on* $(0,T]$;

The function g *is bounded, piecewise continuous, continuous at* ±1 *and continuous from the left on* $[-1,1]$;

The functions $u(t)$, $v(t)$ *are measurable and bounded on* $(0,T]$.

In order to describe the dependence of the solution y of (1)–(3) on g, u, v, we can use the formula ([3], p. 144)

$$y(t,x)=\int_0^t[\Gamma(t,x,\tau,1)-\Gamma(t,x,\tau,\,-1)\varphi(\tau;\,1)]d\tau+ \\ +\int_{-1}^{1}\Gamma(t,x,0,\,\xi)g(\xi)d\xi,$$

which we can write (changing the order of integration) as

$$(5)\qquad y(t,x)=\int_0^t[K_0(t,x,\tau,\,-1)u(\tau)+K_0(t,x,\tau,1)v(\tau)]d\tau+ \\ +\int_{-1}^{1}K_1(t,x,0,\,\xi)g(\xi)d\xi,$$

where $K_0(t,x,\cdot,\pm1)$, $K_1(t,x,0,\cdot)$ are the elements of the spaces $L^2(0,T)$, $L^2(-1,1)$ ([3]) for each fixed $(t,x)\in P$. Therefore, for each fixed $(t,x)\in P$

the functional

$$L^2\big((0, T]\big) \times L^2\big((0, T]\big) \times L^2([-1, 1]) \ni (u, v, g) \to y(t, x) \in R$$

is linear and continuous.

At this point let us state precisely what we mean by the solution of problem (1)–(3) with a piecewise continuous initial condition and measurable boundary conditions. To do this let us denote by y_k the solutions of equation (1) with the following initial and boundary conditions:

$$(2_k) \qquad y_k(0, x) = g_k(x), \qquad x \in [-1, 1],$$

$$(3_k) \qquad \begin{aligned} -\frac{\partial y_k}{\partial x}\bigg|_{x=-1} + \beta(t) y_k(t, -1) &= u_k(t), \\ \frac{\partial y_k}{\partial x}\bigg|_{x=1} + \gamma(t) y_k(t, 1) &= v_k(t), \end{aligned} \qquad t \in (0, T],$$

where

$$\frac{\partial y_k(t, x)}{\partial x}\bigg|_{x=\pm 1} := \lim_{\xi \to \pm 1} \frac{\partial y_k(t, \xi)}{\partial x}$$

and g_k, u_k, v_k, are continuous functions with $\operatorname{supp} g_k$ contained in the segment $(-1, 1)$. In this case there exists a solution of problem (1)–(3) in the ordinary sense. Let us choose the sequences $u_k \to u$, $v_k \to v$, $g_p \to g$ in $L^2\big((0, T]\big)$, $L^2([-1, 1])$, respectively, which fulfil the above conditions. Since the functions g_k, u_k, v_k, can be chosen in a such manner that

$$(6) \qquad \begin{aligned} \sup_{x \in [-1,1]} |g_k(x)| &\leqslant \sup_{x \in [-1,1]} |g(x)|, \\ \sup_{t \in (0,T)} |u_k(t)| &\leqslant \operatorname*{ess\,sup}_{t \in (0,T)} |u(t)|, \\ \sup_{t \in (0,T)} |v_k(t)| &\leqslant \operatorname*{ess\,sup}_{t \in (0,T)} |v(t)|, \end{aligned}$$

the functions $y_k(t, x)$ are bounded by the same constant ([3]). Hence there is a subsequence $\{k_l\}$ such that the limit

$$y(t, x) = \lim_{l \to \infty} y_{k_l}(t, x)$$

exists for any $(t, x) \in P$ and the derivatives

$$\frac{\partial y}{\partial x}, \ \frac{\partial^2 y}{\partial x^2}, \ \frac{\partial y}{\partial t}$$

are limits of suitable derivatives of the functions y_{k_l}.

The convergence $y_k \to y$ in P (because $K_0, K_1 \in L^2$) implies that these limits are unique. Function y defined in this manner satisfies equation (1) in the domain P ([3]). y will be called the solution of problem (1)–(3). It agrees with the generalized solution of this problem ([6]).

Let us write

(7) $\quad P := (0, T] \times (-1, 1); \qquad L := \{T\} \times (-1, 1), \qquad S := \partial P \setminus L,$

where "$:=$" means "equal by definition".

We fix the initial function $g(x)$ and denote by $y(t, x; u, v)$ the solution of problem (1)–(3) with the functions u, v in (3). When there is no doubt as to u, v we will write simply $y(t, x)$.

1.1. Proposition. *We can apply the theorems about weak differential inequalities of parabolic type* ([7]) *to the solution defined above because those inequalities for the approximating solution y_k remain true for y. We can also apply the strong maximum principle* ([3]), *because the solution $y(t, x)$ is continuous for $t \in (0, T]$, $x = \pm 1$ and for $t = 0$, $x \in [-1, 1]$ can be approximated by a solution of problem* (1), (2_k), (3_k) *with conditions* (6) *fulfilled.*

1.2. Lemma. *Let $y(t, x)$ be a solution of equation* (1) *in the domain P. Using notation* (7), *for any $(T, x_0) \in L$, $h > 0$, we get:*

1° *If $y(T, x_0) \geqslant 0$, then there is a component (a connected subset) A_0 of the set*

$$A := \{(t, x) \in \overline{P}:\ y(t, x) > y(T, x_0) - h\}$$

and a point $(t', x') \in S$ such that (T, x_0), $(t', x') \in A_0$ and $y(t', x') > y(T, x_0)$ if $y(t, x) \not\equiv y(T, x_0)$ on P;

2° *If $y(T, x_0) \leqslant 0$, then there is a component B_0 of the set*

$$B := \{(t, x) \in \overline{P}:\ y(t, x) < y(T, x_0) + h\}$$

and the point $(t', x') \in S$ such that (t', x'), $(T, x_0) \in B_0$ and $y(t', x') < y(T, x_0)$ if $y(t', x') \not\equiv y(T, x_0)$ on P.

Proof. 1° The existence of the component A_0 containing the point (T, x_0) is obvious. The strong maximum principle implies that either $y(t, x) = y(T, x_0)$ on A_0 and thus on P or there is a $(t', x') \in \partial A_0 \cap S$ such that $y(t', x') > y(T, x_0)$, because $y(t, x) = y(T, x_0) - h$ on $\partial A_0 \cap \operatorname{int} P$. Thus 1° is proved. 2° follows from 1° by a substitution of $(-y)$ into 1°.

1.3. Corollary. *We assume that for the fixed solution y of equation* (1) *there are numbers $-1 < x_1 < x_2 < x_3 < 1$ such that $y(T, x_1)$, $y(T, x_3) > 0$, $y(T, x_2) \leqslant 0$. Then:*

1° *there is a component B_0 of the set*

$$B := \left\{(t, x) \in \overline{P}:\ y(t, x) < \tfrac{1}{2}\min\{y(T, x_1), y(T, x_3)\}\right\}$$

and the point $(t'', x'') \in S$ such that (T, x_2), $(t'', x'') \in \partial B_0$ and $y(t'', x'') < 0$;

2° *if* A_1; A_2 *are the components of the set*

$$A := \{(t, x) \in \bar{P}:\ y(t, x) > \tfrac{1}{2}\min(y(T, x_1), y(T, x_3))\}$$

such that $(T, x_1) \in A_1$, $(T, x_3) \in A_2$, *then for any* $(t'; x') \in S \cap A_1$, $(t''', x''') \in S \cap A_2$ *the order of the points* $(t'; x')$, (t'', x''), (t''', x''') *on* S *is the same as the order of* (T, x_1), (T, x_2), (T, x_3) *on* L, *i.e.,* $x' < x'' < x'''$ *if* $|x''| < 1$ *and* $t' \leqslant t'' < t'''$ *if* $x'' = 1$, $t' > t'' \geqslant t'''$ *if* $x'' = -1$.

Proof. 1° is a simple consequence of Lemma 1.2. Since $B_0 \cap L \neq \emptyset$ and $B_0 \cap S \neq \emptyset$, the set $\bar{P} \setminus B_0$ is not connected (because $L \setminus B_0$ is not connected). Therefore the points (T, x_1), (T, x_3) belong to its different components. Since $A_1 \cup A_2 \subset \bar{P} \setminus B_0$, $A_1 \cap A_2 = \emptyset$. Hence on the boundary S the point (t'', x'') divides the sets $A_1 \cap S$, $A_2 \cap S$, which was to be proved.

1.4. COROLLARY. *Substituting the function* $(-y)$ *into Corollary* 1.3, *we obtain the following statement*:

If $y(T, x_1)$, $y(T, x_3) < 0$, $y(T, x_2) \geqslant 0$, *then there is a point* $(t'', x'') \in S$ *such that for any* (t', x'), $(t''', x''') \in S$ *"joined by components" in the sense of Corollary* 1.3 *with* (T, x_1), (T, x_3) *inequalities of* 2° *hold and* $y(t'', x'') > 0$.

1.5. DEFINITION. Let g: $(a, b) \to R^2$ be a fixed continuous function, let $K := \{g(t):\ t \in (a, b)\}$ be a Jordan arc and let f: $K \to R$ be piecewise continuous, continuous from the left.

1° We say that the *function f changes its sign n times on the arc K* iff

$$n = \sup\{m:\ \exists a < x_1 < x_2 < \ldots < x_{m+1} < b:\ f(g(x_i)) \cdot f(g(x_{i+1})) < 0,\quad i = 1, \ldots, m\}.$$

2° Let the function f be continuous on K. We say that this *function has n zeros at which it does not change sign* iff

$$n = \sup\{m:\ \exists a < x_1 < x_2 < \ldots < x_{3m} < b:\ \text{for } i = 3k-1,\quad k = 1, \ldots, m,\ f(g(x_i)) = 0,\ f(g(x_{i-1}))f(g(x_{i+1})) > 0 \quad \text{and } f(g(x'))f(g(x'')) \geqslant 0,\ \forall x';\ x'' \in (x_{i-1}, x_{i+1})\}.$$

1.6. LEMMA. *Let* $y(t, x)$ *be a solution of equation* (1) *in the domain* P. *If the function* $y(T, x)$ *of the variable* x *changes sign* m_1 *times on* $(-1, 1)$ *and has there* m_2 *zeros at which it does not change sign, then the function* $y(t, x)|_{(t,x) \in S}$ *changes sign on the boundary* S *at least* $m_1 + 2m_2$ *times.*

Proof. Since the number of sign changes of the function $y(T, x)$ is finite, there are $a, h_1, h_2, > 0$ such that y has a constant sign on $T \times (-1, h_1 - 1) \cup (T - s, T] \times \{h_1 - 1\}$ and on $\{T\} \times (1 - h_2, 1) \cup (T - s, T] \times \{1 - h_2\}$.

Remark. Under the assumptions of Lemma 1.6 for a positive s small enough the function $y(T-s, x)$ changes sign on the interval $(h_1-1, 1-h_2)$ at least m_1+2m_2 times.

Indeed, the interval $(h_1-1, 1-h_2)$ contains m_1+2m_2+1 subintervals such that on each of them the function $y(T, x)$ has a constant sign and is not equal to zero. Applying to each two or three abutting subintervals Corollaries 1.3 and 1.4, we can note that the interval $(h_1-1, 1-h_2)$ has at least m_1+2m_2+1 subintervals on which the function $y(T-s, x)$ has a constant sign and is not equal to zero. Moreover, Corollaries 1.3 and 1.4 imply that the function $y(T-s, x)$ changes sign on $(h_1-1, 1-h_2)$ at least m_1+2m_2 times, which completes the proof of this remark.

This remark implies the existence of at least m_1+2m_2+1 components of the sets

$$C\cap\{(t, x)\colon\ t\leqslant T-s\},\ D\cap\{(t, x)\colon\ t\leqslant T-s\},$$

where $C\colon = \bar{P}\cap y^{-1}\big((-\infty, 0)\big)$, $D\colon = \bar{P}\cap y^{-1}\big((0, +\infty)\big)$, because the common part of each component and the interval $\{T-s\}\times(-1, 1)$ is connected (Corollaries 1.3 and 1.4). The components of C and D alternate, and thus their traces on S alternate too. Hence $y(t, x)$ as the function on S changes sign at least m_1+2m_2 times on $S\cap\{(t, x)\colon\ t\leqslant T-s\}$. This means that the Lemma 1.6 is true.

1.7. Definition. The integrable function $f\colon (a, b)\to R$ is said *to have n switches on the interval (a, b)* iff n is the smallest number such that there are points $a = a_0 < a_1 < a_2 < \ldots < a_n < a_{n+1} = b$ for which the function f has a constant sign almost everywhere on each interval (a_i, a_{i+1}), $i = 0, \ldots, n$, i.e., for almost all x', $x''\in(a_i, a_{i+1})$, $f(x')\cdot f(x'')\geqslant 0$. The points fulfilling this definition $(a_1, \ldots, a_n)$ are said to be the *switching points* of the function f.

1.8. Lemma. *Let $y(t, x)$ be the solution of problem* (1)–(3), *let A be the component of the set D defined above and let $A\cap L\neq\emptyset$, $A\cap(\{0\}\times\times[-1, 1]) = \emptyset$.*

Write

$$t_1 = \inf\{t\colon\ (t, -1)\in A\};\qquad t_2 = \inf\{t\colon\ (t, 1)\in A\}.$$

If the functions u, v have a constant sign almost everywhere on the intervals (t_1, t_3) and (t_2, t_4), respectively, then $u(t)\geqslant 0$ on (t_1, t_3) or $v(t)\geqslant 0$ on (t_2, t_4) a.e. and, moreover, $u(t) > 0$ or $v(t) > 0$ on the set of positive measure contained in (t_1, t_3) or (t_2, t_4).

1.9. Remark. Let us suppose in Lemma 1.8 that A is a component of C defined above, then, the other assumptions being unaltered, $u(t)\leqslant 0$ on (t_1, t_3) or $v(t)\leqslant 0$ on (t_2, t_4) a.e. and $u(t) < 0$ or $v(t) < 0$ on the set of positive measure contained in a suitable interval.

The proof is an immediate consequence of Lemma 1.8 applied to the function $(-y)$.

1.10. Remark. If the component A in Lemma 1.8 has no common points with the boundary $(0, T]\times\{-1\}$ (or $(0, T]\times\{1\}$), then this lemma gives the inequalities for the function v (or u).

The same refers to Remark 1.9.

Proof of Lemma 1.8. Let

$$A_0 := \{(t, x)\in A: t\leqslant t_5 := \min(t_3, t_4)\}.$$

Consider the solution y on A_0. Since $y(t, x) = 0$ for $(t, x)\in(\operatorname{int}P)\cap\partial A_0$ so if $u(t)\leqslant 0$ for $t\in(t_1, t_5)$ and $v(t)\leqslant 0$ for $t\in(t_2, t_5)$ a.e., we have, according to the theorem on differential inequalities ([7]), $y(t, x)\leqslant 0$ on A_0. Hence the proof is completed, because this contradicts the definition of A_0.

1.11. THEOREM. *Let us consider the solution $y(t, x)$ of problem* (1)–(3). *Suppose that the functions u, v have p and r switches on $(0, T)$, respectively, and the function g (condition* (2)) *changes sign n times on $[-1, 1]$. If the function $y(T, x)$ of the variable x changes sign on $(-1, 1)$, m_1 times and has m_2 zeros at which it does not change sign, then*

1° $m_1+2m_2\leqslant n+p+r+2$,

2° *if* $g(x)\equiv 0$, *then* $m_1+2m_2\leqslant p+r+1$,

3° *if* $g(x) = 0$, $a_1, \ldots, a_p$ *are the switching points of* u, $b_1, \ldots, b_r$ *are the switching points of* v *and*

$$\int_0^{a_1} u(t)\,dt\int_0^{b_1} v(t)\,dt > 0,\ \textit{then}\ m_1+2m_2\leqslant p+r.$$

Proof. Let us assume s, h_1, h_2 as in the proof of Lemma 1.6. We can apply Lemma 1.8 to the domain $P' := (0, T-s]\times(-1, 1)$. It follows that with every component of the sets C or D having common points with $L' := \{T-s\}\times(-1, 1)$ and disjoint with $\{0\}\times(-1, 1)$ we can associate the segment contained in the intersection of that component and $(0, T)\times(\{-1\}\cup\{1\})$, where the sign of the function u (or v) is the same as the sign of y. Alternation of components implies alternation of those segments (because at most one of the components can have common points with $(0, T)\times\{-1\}$ and $(0, T)\times\{1\}$). Now the number of those segments cannot be higher than $p+r+2$; therefore the number of components of C and D disjoint with $\{0\}\times(-1, 1)$, cannot be higher than $p+r+2$, either. Since at most $n+1$ of the components can have common points with $\{0\}\times(-1, 1)$, then there are at most $n+p+r+3$ of them. There are m_1+2m_2+1 components at least having common points with L' (see the remark in the proof of Lemma 1.6); hence $m_1+2m_2\leqslant n+p+r+2$, which proves 1°.

2° If $g(x) \equiv 0$ on $[-1, 1]$, then no component of C and D has common points with $\{0\} \times (-1, 1)$. Thus, there are at most $p+r+2$ of those components, which proves 2°.

3° If $g(x) \equiv 0$ and $\int\limits_0^{a_1} u(t)\,dt \int\limits_0^{b_1} v(t)\,dt > 0$, then the segments $(0, a_1) \times \times \{-1\}$, $(0, b_1) \times \{1\}$ cannot be contained in two different components fulfilling the assumptions of Lemma 1.8. It follows that the number of those components is not higher than $p+r+1$, which, similarly to 2°, proves 3° and hence Theorem 1.11.

Remark. If in case 1° $m_1+2m_2 = n+p+r+2$ or in case 2° $m_1+ +2m_2 = p+r+1$ or in case 3° $m_1+2m_2 = p+r$, then there are $m_1+ +2m_2+1$ components of C and D.

1.12. Corollary. *With the notation and assumptions of Theorem* 1.11, *under the condition that the assumption of the foregoing remark is true, there is a* $h > 0$ *such that for* a_p, b_r *as the last switching points of* u *and* v *the following inequalities hold*:

$$y(T, x) \int\limits_{b_r}^{T} v(t)\,dt > 0 \qquad \text{for } x \in (1-h, 1),$$

$$y(T, x) \int\limits_{a_p}^{T} u(t)\,dt > 0 \qquad \text{for } x \in (-1, -1+h).$$

Proof. The foregoing remark says that there are m_1+2m_2+1 components of C and D in P. Hence the components containing the points (T, h_1-1) and $(T, 1-h_2)$ defined in the proof of Lemma 1.6 have common parts with the boundary S contained in the segments $(a_p, T) \times \{-1\}$: $(b_r, T) \times \{1\}$. Lemma 1.8 says that in those common parts the following inequalities hold:

$$\operatorname{sign} y(t, x) = \operatorname{sign} \int\limits_{a_p}^{T} u(t)\,dt = \operatorname{sign} y(T, h_1-1)$$

and

$$\operatorname{sign} y(t, x) = \operatorname{sign} \int\limits_{b_r}^{T} v(t)\,dt = \operatorname{sign} y(T, 1-h_2),$$

respectively. Consequently, the strong maximum principle and the definition of a_p, b_r imply that $y(T, x) \not\equiv 0$ for $x \in (-1, h_1-1) \cup (1-h_2, 1)$ and so the proof is completed.

2. We will now state the optimal control problem for equation (1) with conditions (2) and (3). We adopt all the assumptions of Section 1 about problem (1)–(3).

We will try to find such a pair of functions $(u_0, v_0) \in U \times U$, where

(8) $$U: = \{u \text{ measurable: } |u(t)| \leqslant K \text{ for } t \in (0, T]\},$$

that

(9) $$J(u_0, v_0) = \inf\{J(u, v): u, v \in U\},$$

where

(10) $$J(u, v): = \sup\{|y(T, x; u, v)|: |x| < 1\}.$$

The pair of functions $(u, v) \in U \times U$ will be called an *admissible control* and the solution (u_0, v_0) of problem (8)–(10) — an *optimal control.*

Similarly to J. L. Lions ([4]) we prove

2.1. THEOREM. *There exists an optimal solution of problem* (1)–(3), (8)–(10).

Proof. Let (u_n, v_n) be a sequence of admissible controls which minimizes $J(u, v)$. $U \times U$ is a convex, closed and bounded subset of $L^2\big((0, T]\big) \times L^2\big((0, T]\big)$, and so there is a subsequence (u_{n_k}, v_{n_k}) weakly convergent to $(u_0, v_0) \in U \times U$. Let us suppose, for the sake of simplicity, that $(u_{n_k}, v_{n_k}) = (u_n, v_n)$. For each fixed $(t, x) \in P$ the functions from (5) $K_0(t, x, \cdot, -1)$, $K_0(t, x; \cdot, 1)$ belong to $L^2\big((0, T]\big)$; hence, for each $(t, x) \in P$, $y(t, x; u_n, v_n) \to y(t, x; u_0, v_0)$. On the other hand, for each $x \in (-1, 1)$

$$|y(T; x; u_n, v_n)| \leqslant \inf J(u, v) + \varepsilon_n,$$

where $\varepsilon_n \to 0$; therefore

$$|y(T, x; u_0, v_0)| \leqslant \inf J(u, v),$$

and so

$$J(u_0, v_0) \leqslant \inf J(u, v),$$

which means that the proof is completed, because

$$J(u_0, v_0) = \inf J(u, v).$$

Before the main theorems we shall introduce auxiliary lemmas about optimal control defined in the above way.

2.2. LEMMA. *Consider any functions* $f_1, \ldots, f_n: (-1, 1) \to R$ *and numbers* $x_1, \ldots, x_{n-1} \in (-1, 1)$. *There are numbers* $\lambda_1, \ldots, \lambda_n$ *such that* $|\lambda_1| + \ldots + |\lambda_n| > 0$ *and*

(11) $$\sum_{i=1}^{n} \lambda_i f_i(x_j) = 0, \quad j = 1, \ldots, n-1.$$

Proof. Write $a_{ij}: = f_i(x_j)$. Condition (11) can be written as follows:

$$\begin{gathered} a_{11}\lambda_1 + \ldots + a_{n1}\lambda_n = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ a_{1,n-1}\lambda_1 + \ldots + a_{n,n-1}\lambda_n = 0. \end{gathered}$$

It is a linear homogeneous system of $n-1$ equations of n variables. It posesses a non-zero solution which fulfils the lemma.

ASSUMPTION A. We say that problem (1)–(3) fulfils Assumption A iff for any functions $(u, v) \in U \times U$ not equal to zero a.e. simultaneously and having a finite number of switches on $(0, T)$, the solution of (1)–(3) with $g(x) = 0$ is not equal to zero at the moment T on any interval contained in $(-1, 1)$.

This assumption is true, for example, for the heat equation (Egorov, [2]). It is rather artificial, but it allows us to extend this theory to all problems fulfilling this assumption. The author's attempts to find any closer characterization of these problems were unsuccessful.

2.3. LEMMA. *Suppose there are given measurable sets $A_1, \ldots, A_n$, $B_1, \ldots, B_m \subset (0, T)$ of positive measure such that for $i < j$ for any $t_i \in A_i$, $t_j \in A_j$ ($t_i \in B_i$, $t_j \in B_j$): $t_i < t_j$. Let the numbers $-1 < x_1 < x_2 < \ldots < x_{n+m-1} < 1$ be given. Then there are positive coefficients $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_m$ such that if $y(t, x)$ is the solution of problem* (1)–(3) *fulfilling Assumption* A, $g(x) \equiv 0$ *and*

$$u(t) := \begin{cases} \lambda_i(-1)^i, & t \in A_i, \\ 0, & t \notin \bigcup_{i=1}^{n} A_i, \end{cases}$$

$$v(t) := \begin{cases} \mu_i(-1)^{i+1}, & t \in B_i, \\ 0, & t \notin \bigcup_{i=1}^{m} B_i \end{cases}$$

then $y(T, x) = 0$ iff $x \in \{x_1, \ldots, x_{n+m-1}\}$ and $y(T, x)$ changes sign on $(-1, 1)$ $(n+m-1)$ times.

Proof. Using (5), we obtain

$$y(T, x) = \sum_{i=1}^{n} \lambda_i(-1)^i \int_{A_i} K_0(t, x, \tau, -1)\, d\tau + \sum_{i=1}^{m} \mu_i(-1)^{i+1} \int_{B_i} K_0(t, x, \tau, 1)\, d\tau.$$

Lemma 2.2 allows us to choose the coefficients $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_m$ not all equal to zero and such that $y(T, x_i) = 0$, $i = 1, \ldots, n+m-1$. Assumption A implies that $y(T, x)$ cannot equal zero on any interval (x_i, x_{i+1}). Let m_1, m_2, p, r be the same as in Theorem 1.11. Now $m_1 + m_2 \geq n+m-1$. Since the definition of u, v implies that $p \leq n-1$, $r \leq m-1$, Theorem 1.11, 2° and the above inequalities imply $n+m-1 \leq m_1 + m_2 \leq m_1 + 2m_2 \leq p + r + 1 \leq n+m-1$; therefore $p = n-1$, $r = m-1$, $m_2 = 0$, $m_1 = n+m-1$. This means that:

1° the function $y(T, x)$ changes sign $n+m-1$ times and $y(T, x_i) = 0$, $i = 1, \ldots, n+m-1$;

2° there are no other zeros of $y(T, x)$ on $(-1, 1)$;

3° the functions u, v have $n-1$ and $m-1$ switches on $(0, T)$, respectively;

4°

$$\int_{A_1} u(t)\,dt \int_{B_1} v(t)\,dt = \lambda_1(-1)\mu_1 < 0$$

because the inequality of Theorem 1.11, 3° is false. 3° implies that the λ_i are different from zero and have the same sign, because only in this case the function u defined in this way has $n-1$ switches on $(0, T)$. 3° and 4° imply that μ_i also have the same sign. If necessary, we multiply λ_i and μ_i by (-1) to receive positive coefficients. This does not change the proof which has just been completed.

ASSUMPTION B. Equation (1) is said to fulfil Assumption B iff the coefficients $a(t, x)$, $b(t, x)$, $c(t, x)$ of that equation are defined and differentiable with respect to x in some neighbourhood of the set $\bar{P}$ and if the derivatives

$$\frac{\partial a}{\partial x}, \quad \frac{\partial b}{\partial x}, \quad \frac{\partial c}{\partial x}$$

are continuous in $\bar{P}$ and fulfil the Hölder condition (4) respectively, like a, b, c.

2.4. LEMMA. *Let equation* (1) *fulfil Assumption* B *and let the function* g *in condition* (2) *be fixed. Then there are constants* M_1, M_2 *such that, for every* $K > 0$ *and for any* u, v *occurring in* (3) *and fulfilling the inequalities*

$$|u(t)| \leqslant K, \qquad |v(t)| \leqslant K$$

for a.e. t, *the variation* $W_{-1}^{1} y(T, x;\ u, v)$ *of the solution* y *at the moment* T *equals* $M_1 + M_2 K$ *at most.*

Proof. Write $y(t, x;\ u, v) = y_1(t, x) + y_2(t, x)$, where y_1 is the solution of (1) with the initial condition (2) and the boundary condition

$$-\frac{\partial y_1}{\partial x}\bigg|_{x=-1} + \beta(t)y(t, -1) = \frac{\partial y_1}{\partial x}\bigg|_{x=1} + \gamma(t)y(t, 1) = 0 \tag{12}$$

and $y_2(t, x)$ is the solution of (1) with (3) and the initial condition

$$y_2(0, x) \equiv 0, \qquad x \in [-1, 1].$$

For $t_0 > 0$ the function $y_1(t_0, x)$ is continuous with respect to x; thus, in the domain $(t_0, T] \times (-1, 1)$, y_1 is the solution in the ordinary sense.

The derivative $\partial y_1/\partial x$ exists and is continuous in the interior of P and by (12) is bounded there. Let

$$M_1 := 2\sup\left\{\left|\frac{\partial y_1}{\partial x}(T, x)\right|,\ x \in (-1, 1)\right\}. \tag{13}$$

Then $W^1_{-1} y_1(T, x) \leqslant M_1$.

Assumption B allows us to use the formula

$$a\frac{\partial^3 y_2}{\partial x^3} + \left(\frac{\partial a}{\partial x} + b\right)\frac{\partial^2 y_2}{\partial x^2} + \left(\frac{\partial b}{\partial x} + c\right)\frac{\partial y_2}{\partial x} + \frac{\partial c}{\partial x} y_2 - \frac{\partial^2 y_2}{\partial x \partial t} = 0.$$

Changing the order of differentiation and writing

$$f(t, x) := \frac{\partial c(t, x)}{\partial x} y_2(t, x),$$

we obtain the equation for the derivative $\partial y_2/\partial x$:

$$a\frac{\partial^2}{\partial x^2}\left(\frac{\partial y_2}{\partial x}\right) + \left(\frac{\partial a}{\partial x} + b\right)\frac{\partial}{\partial x}\left(\frac{\partial y_2}{\partial x}\right) + \left(\frac{\partial b}{\partial x} + c\right)\left(\frac{\partial y_2}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial y_2}{\partial x}\right) = -f$$

with the initial condition $\dfrac{\partial y_2}{\partial x}(0, x) \equiv 0$, $x \in [-1, 1]$, and the boundary condition

$$-\frac{\partial y_2}{\partial x}\bigg|_{x=-1} = u(t) - \beta(t) y_2(t, -1),$$

$$\frac{\partial y_2}{\partial x}\bigg|_{x=1} = v(t) - \gamma(t) y_2(t, 1).$$

There is a constant L_1 such that $|y_2(t, x)| \leqslant L_1 K$ for $(t, x) \in P$ if $|u(t)| \leqslant K$, $|v(t)| \leqslant K$ for $t \in (0, T]$ ([3]). Therefore, if we write:

$$L_2 := \sup\{\sup(|\beta(t)|, |\gamma(t)|),\ t \in (0, T]\},$$

$$L_3 := \sup\left\{\left|\frac{\partial c}{\partial x}(t, x)\right|,\ (t, x) \in P\right\},$$

$$L_4 := \sup\left\{\left|\frac{\partial b}{\partial x}(t, x) + c(t, x)\right|,\ (t, x) \in P\right\},$$

then

$$\sup\left\{\left|\frac{\partial y_2}{\partial x}(t, x)\right|,\ (t, x) \in P\right\} \leqslant e^{L_4 T}[K + L_1 L_2 K + (e^{2\lambda} - 1) L_3 L_1 K]$$

for λ sufficiently large and depending only on the coefficients of equation (1) ([3]). If we define

$$M_2 = 2e^{L_4 T}[1 + L_1 L_3 (e^{2\lambda} - 1) + L_1 L_2],$$

the inequality

$$\left|\frac{\partial y_2}{\partial x}(t, x)\right| \leqslant \tfrac{1}{2} M_2 K \quad \text{for } (t, x) \in P$$

is true and implies

$$W^1_{-1} y_2(T, x) \leqslant M_2 K \tag{14}$$

which, together with (13), was to be proved.

2.5. DEFINITION. Let f be a real continuous function not equal to zero identically on $[a, b]$. We say that this function has N *essential extremes* on $[a, b]$ iff N is the largest number n such that there is a sequence $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$ for which

$$|f(x_1)| = |f(x_2)| = \ldots = |f(x_n)| = \sup\{|f(x)| \, | \, x \in [a, b]\}$$

and $f(x_i) \cdot f(x_{i+1}) < 0$, $i = 1, \ldots, n-1$.

According to this definition, the sequence $(x_1, \ldots, x_N)$ is called the *sequence of essential extremes* of the function f on $[a, b]$.

2.6. COROLLARY. *Let $y(t, x)$ be a solution of* (1)–(3) *satisfying Assumption* B *and $J(u, v) \geqslant r > 0$ for each admissible control (u, v). Then the number of essential extremes of the function $y(T, x)$ on $[-1, 1]$ is equal to $(2r)^{-1}(M_1 + M_2 K) + 1$ at most, where M_1, M_2, K are constants from Lemma* 2.4.

Proof. Let N be the number of essential extremes of the function $y(T, x)$. Lemma 2.4 and the definition of an admissible control imply that

$$M_1 + M_2 K \geqslant W^1_{-1} y(T, x) \geqslant \sum_{i=1}^{N-1} |f(x_{i+1}) - f(x_i)| \geqslant 2r(N-1),$$

where $(x_1, \ldots, x_N)$ are essential extremes of the function $y(T, x)$. Hence $N \leqslant (M_1 + M_2 K)(2r)^{-1} + 1$ and the proof is completed.

2.7. LEMMA. *Consider the solution $y(t, x)$ of problem* (1)–(3), (u, v) *being an admissible control. Suppose that Assumptions* A *and* B *are true, $y(T, x)$ has N essential extremes on $[-1, 1]$ and is not equal to zero. If the measurable sets $A_1, \ldots, A_k$, $B_1, \ldots, B_{N-k} \subset (0, T)$ fulfil the assumptions of Lemma* 2.3 *and a constant $\varepsilon \neq 0$ can be defined so that the functions*

$$u'(t) := \begin{cases} u(t) + \varepsilon(-1)^i, & t \in A_i, \\ u(t), & t \notin \bigcup_{i=1}^{k} A_i, \end{cases}$$

$$v'(t) := \begin{cases} v(t) + \varepsilon(-1)^{i+1}, & t \in B_i, \\ v(t), & t \notin \bigcup_{i=1}^{N-k} B_i, \end{cases}$$

belong to U and

$$(-1)^k y(T, x_1) < 0, \tag{15}$$

where x_1 is the first element of the sequence of essential extremes of $y(T, x)$, then there are positive coefficients $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_{N-k}$, so that, adopting the notation

$$\begin{aligned} u''(t) &:= \begin{cases} u(t) + \lambda_i \varepsilon(-1)^i, & t \in A_i, \\ u(t), & t \notin \bigcup_{i=1}^{k} A_i, \end{cases} \\ v''(t) &:= \begin{cases} v(t) + \mu_i \varepsilon(-1)^{i+1}, & t \in B_i, \\ v(t), & t \notin \bigcup_{i=1}^{N-k} B_i, \end{cases} \end{aligned} \tag{16}$$

we obtain $u'', v'' \in U$ and $J(u'', v'') < J(u, v)$.

Proof. Let $x_1 < x_2 < \ldots < x_N$ be a sequence of essential extremes of $y(T, x)$. Let us define

$$\begin{gathered} x_i' := \sup\{x \in (x_i, x_{i+1}):\ y(T, x) = 0 \text{ and for } \bar{x} \in (x_i, x) \\ y(T, \bar{x})\,[y(T, x_i)]^{-1} > -1\}, \quad i = 1, \ldots, N-1, \\ x_0' := -1, \quad x_N' := 1. \end{gathered}$$

The definition of x_i' and the assumption about the number of essential extremes imply that $y(T, x) \neq -y(T, x_{i+1})$ on the interval (x_i', x_{i+1}'), $i = 0, \ldots, N-1$.

The continuity of $y(T, x)$ gives us the inequality

$$d := \min_i \inf\{(y(T, x)\,[y(T, x_{i+1})]^{-1} + 1):\ x \in (x_i', x_{i+1}')\} > 0. \tag{17}$$

Now we introduce functions $y_i(t, x)$ as solutions of (1) with the initial conditions $y_i(0, x) \equiv 0$ and boundary conditions

$$-\frac{\partial y_i}{\partial x}\bigg|_{x=-1} + \beta(t) y_i(t, -1) = \begin{cases} \varepsilon(-1)^i, & t \in A_i, \\ 0, & t \notin A_i, \\ 0, & i = k+1, \ldots, N, \end{cases}$$

$$i = 1, \ldots, k,$$

$$\frac{\partial y_i}{\partial x}\bigg|_{x=1} + \gamma(t) y_i(t, 1) = \begin{cases} 0, & i = 1, \ldots, k, \\ \varepsilon(-1)^{i+1-k}, & t \in B_{i-k}, \\ 0, & t \notin B_{i-k}, \end{cases} \qquad i = k+1, \ldots, N,$$

where ε is taken from the definition of u' and v'.

We choose coefficients

$$\lambda_1', \ldots, \lambda_k', \mu_1', \ldots, \mu_{N-k}' > 0 \tag{18}$$

so that

$$\sum_{i=1}^{k} \lambda_i' y_i(T, x_j') + \sum_{i=k+1}^{N} \mu_{i-k}' y_i(T, x_j') = 0, \quad j = 1, \ldots, N-1.$$

This is possible by Lemma 2.3.

Next we take such $\varepsilon' \neq 0$ that for the function

$$F(x) := \sum_{i=1}^{k} \lambda_i' y_i(T, x) + \sum_{i=k+1}^{N} \mu_{i-k}' y_i(T, x)$$

the following inequality holds:

$$|\varepsilon' F(x)| < d\,|y(T, x_1)| \quad \text{for any } x \in [-1, 1], \tag{19}$$

where d is defined by (17) and

$$\begin{aligned} &\sup\{|\varepsilon' \lambda_i'|,\ i = 1, \ldots, k\} \leqslant 1, \\ &\sup\{|\varepsilon' \mu_i'|,\ i = 1, \ldots, N-k\} \leqslant 1; \end{aligned} \tag{20}$$

moreover,

$$\varepsilon' F(x_1) y(T, x_1) < 0. \tag{21}$$

The function $F(x)$ is equal to zero at the points $x_1', \ldots, x_{N-1}'$ only, where it changes sign (Lemma 2.3), and that is why formula (21) is true and implies

$$\varepsilon' F(x_i) y(T, x_i) < 0, \quad i = 1, \ldots, N. \tag{22}$$

Inequality (21) and the definition F and y_i imply that

$$\varepsilon' \cdot \varepsilon \cdot \lambda_k' (-1)^k y(T, x_1) < 0$$

(Corollary 1.12). This, together with (20), (18) and (15), means:

$$\begin{aligned} &\varepsilon' \lambda_i' \in (0, 1], \quad i = 1, \ldots, k, \\ &\varepsilon' \mu_i' \in (0, 1], \quad i = 1, \ldots, N-k. \end{aligned}$$

Thus, if we write $\lambda_i := \varepsilon' \lambda_i'$, $\mu_i := \varepsilon' \mu_i'$ then, according to the assumption about u', v' and the convexity of U, the functions u'' and v'' defined by (16) belong to U. Next, inequalities (22) and (19) imply that $J(u'', v'') < J(u, v)$, which was to be proved.

2.8. Theorem. *If problem* (1)–(3), (8)–(10) *fulfils Assumptions* A *and* B, *if* (u, v) *is an optimal control and if* $J(u, v) > 0$, *then* $|u(t)| \equiv |v(t)| \equiv K$ *a.e. in* $(0, T]$.

Proof. Suppose that the theorem is false. Then there are measurable subsets $A, B \subset (0, T]$ of positive measure of $A \cup B$ and the number $\varepsilon_1 > 0$ such that, for all $t \in A$, $s \in B$, $|u(t)| < K - \varepsilon_1$, $|v(s)| < K - \varepsilon_1$. Since the variation $W^1_{-1}y(T, x)$ is bounded by a constant independent of the choice of admissible control (Lemma 2.4) and $J(u, v) > 0$, the function $y(T, x)$ has a finite number of essential extremes (Corollary 2.6). Let that number equals N. Then we can divide the sets A and B into N subsets of positive measure fulfilling the assumptions of Lemma 2.3 and take ε equal to $\pm\varepsilon_1$, choosing the sign of ε so that inequality (15) holds. Lemma 2.7 implies that the control (u, v) is not optimal, thus we have arrived at a contradiction of the assumption of optimality, which completes the proof of the theorem.

2.9. THEOREM. *If problem* (1)–(3), (8)–(10) *fulfils Assumptions* A *and* B *and if* $\inf\{J(u, v): (u, v) \in U \times U\} > 0$, *then two optimal controls can differ only on the set of measure zero.*

Proof. Let (u_1, v_1), (u_2, v_2) be two optimal solutions. The map

$$y: \; U \times U \ni (u, v) \to y(T, \cdot; \; u, v) \in L^\infty[-1, 1]$$

is affine and the sets $U \times U$ and

$$Y: = \{y(T, \cdot; \; u, v) \in L^\infty[-1, 1]: \; \|y(T, \cdot; \; u, v)\|_{L^\infty} \leqslant J(u_1, v_1)\}$$

are convex; so

$$\big(\tfrac{1}{2}(u_1+u_2), \; \tfrac{1}{2}(v_1+v_2)\big) \in U \times U \quad \text{and} \quad y\big(T, \cdot; \; \tfrac{1}{2}(u_1+u_2), \; \tfrac{1}{2}(v_1+v_2)\big) \in Y.$$

It follows that the control $\big(\frac{1}{2}(u_1+u_2), \; \frac{1}{2}(v_1+v_2)\big)$ is optimal too.

Using Theorem 2.8 we have

$$\tfrac{1}{2}|(u_1+u_2)(t)| = \tfrac{1}{2}|(v_1+v_2)(t)| = K$$

a.e. in $(0, T]$; thus $u_1(t) = u_2(t)$, $v_1(t) = v_2(t)$ a.e. in $(0, T]$ and the proof is completed.

2.10. DEFINITION. Consider the admissible control (u, v). If $t_1 < \ldots < t_k$ are switching points of u and $s_1 < \ldots < s_l$ are switching points of v, $k+l = n$, then for

$$\text{(a)} \qquad \int_0^{t_1} u(t)\,dt \int_0^{s_1} v(t)\,dt > 0,$$

we say that the control (u, v) *has n switches* on the interval $(0, T]$, and for

$$\text{(b)} \qquad \int_0^{t_1} u(t)\,dt \int_0^{s_1} v(t)\,dt < 0,$$

we say that the control (u, v) *has $n+1$ switches* on $(0, T]$.

2.11. THEOREM. *Consider problem* (1)–(3), (8)–(10) *fulfilling Assumptions* A *and* B, *the optimal solution* (u, v) *such that* $J(u, v) = r > 0$, *and* $W^1_{-1}y(T, x; \; u, v) = M$. *In this case the control* (u, v) *has at most* $M(2r)^{-1}$ *switches.*

2.12. Remark. In fact, we will show that if N is the number of essential extremes of $y(T, x)$, then the number of switches of the optimal control is $N-1$ at most.

Proof. As in Corollary 2.6,

$$2r(N-1) \leqslant W^1_{-1} y(T, x;\ u, v) = M; \tag{23}$$

therefore $N-1 \leqslant M(2r)^{-1}$ and the theorem follows from the remark.

Proof of the remark. Suppose that the number of switches is higher than $N-1$. Denote the sets on which $u(t)$, $v(t)$ are constant a.e. (Theorem 2.8) by $A_1, \ldots, A_{k+1}$, $B_1, \ldots, B_{l+1}$, respectively, so that those sets fulfil the assumptions of Lemma 2.3. If sign $u(t) = \operatorname{sign} v(t)$ in a neighbourhood of 0, then we denote the set between the first and the second switching point of v by B_1, so that

$$\int_{A_1} u(t)\,dt \int_{B_1} v(t)\,dt < 0.$$

Thus, supposing that the remark is false, we have, according to Definition 2.10,

$$k+l > N-2. \tag{24}$$

Note that, for ε defined as

$$\varepsilon := \operatorname{sign} \int_{A_1} u(t)\,dt,$$

either $\varepsilon(-1)^k y(T, x_1) < 0$, or $\varepsilon(-1)^{k+1} y(T, x_1) < 0$, where

$$x_1 := \inf\big\{x \in (-1, 1):\ |y(T, x)| = \sup\{|y(T, z)| : z \in (-1, 1)\}\big\}.$$

Hence either the sequence $A_1, \ldots, A_k$, $B_1, \ldots, B_{l+1}$, or $A_1, \ldots, A_{k+1}$, $B_1, \ldots, B_l$, composed of at least N sets (cf. (24)), fulfils with this ε the assumptions of Lemma 2.7. This lemma implies that the control (u, v) is not optimal, and this contradicts the assumptions and proves Remark 2.12.

2.13. Complementary remarks. There are some examples which realize equality in Remark 2.12. They can be chosen so that Entier $M(2r)^{-1} = N-1$. Thus the estimation given by Theorem 2.11 and Remark 2.12 is the best possible.

The question arises as to whether Theorems 2.1–2.11 are true when (2) is replaced by the Dirichlet condition

$$y(t, -1) = u(t), \quad y(t, 1) = v(t).$$

Some of these theorems cannot be true, because otherwise for the optimal control (u_0, v_0) $|u_0(t)| = |v_0(t)| = K$ for $T-h < t < T$ and $\sup\{|y(T, x)|,$

$x \in [-1, 1]\} \geqslant K$ even if $\sup|g(x)| < K$. But in this case, for $u \equiv v \equiv 0$,

$$\sup\{|y(T, x)|,\ x \in [-1, 1]\} < \sup|g(x)| < K$$

and (u_0, v_0) is not an optimal control.

Another question connected with this problem concerns the possibility of the generalization of Theorem 2.11 to the case where the cost functional is

$$J(u, v) = \int_{-1}^{1} [y(T, x;\ u, v)]^2 dx.$$

3. The purpose of this section is to describe changes of the optimal control of problem (1)–(3), (8)–(10) when the final moment T is being changed.

The domain $P = (0, T] \times (-1, 1)$ will not be fixed now, but it will depend on T. We adopt all the assumptions of Section 1 about problem (1)–(3). The optimal control problem remains the same.

3.1. Lemma. *If* $\{(u_n, v_n)\} \subset U \times U \subset L^2\big((0, T]\big) \times L^2\big((0, T]\big)$ *is a sequence of controls weakly convergent to* $(u_0, v_0) \in U \times U$, *then*

$$y(t, x;\ u_n, v_n) \to y(t, x;\ u_0, v_0)$$

uniformly on $\overline{P}$.

Proof. In accordance with (5):

$$y(t, x;\ u_n, v_n) = \int_0^t [K_0(t, x, \tau, -1)u_n(\tau) + K_0(t, x, \tau, 1)v_n(\tau)]d\tau + \\ + \int_{-1}^{1} K_1(t, x, 0, \xi)g(\xi)d\xi.$$

The second term of the right-hand side does not depend on (u, v). Thus, it is enough to prove the uniform convergence of the first term. To do this we shall use the following theorem ([5]):

If E is a compact space, $\{f_n\}$ is a sequence of the equicontinuous functions on E and $f_n(x) \to f(x)$ on the set Z dense in E, then the sequence f_n is uniformly convergent on E.

This theorem implies that if f_n, f are equicontinuous on E and $f_n \to f$ on Z, then $f_n \to f$ uniformly.

Let us write $\overline{P} = E$, $P = Z$,

$$f_n(t, x) := \int_0^t [K_0(t, x, \tau, -1)u_n(\tau) + K_0(t, x, \tau, 1)v_n(\tau)]d\tau. \tag{25}$$

The mapping $(u_n, v_n) \to f_n(t, x)$ is a linear continuous functional for each fixed $(t, x) \in P$; therefore $f_n(t, x) \to f_0(t, x)$ on P. We have to prove that $\{f_n\}_{n=0}^{\infty}$ is a family of equicontinuous functions.

Lemma 2.4 implies that the derivative $\partial f/\partial x$ is bounded in P by a constant independent of an admissible control. Hence, for each (t, x_1), $(t, x_2) \in \bar{P}$,

$$|f_n(t, x_1) - f_n(t, x_2)| \leqslant c|x_1 - x_2|.$$

We have to prove only that for any $x \in [-1, 1]$

$$|f_n(t_1, x) - f_n(t_2, x)| \leqslant h(t_1 - t_2),$$

where h does not depend on x, n and $\lim\limits_{t\to 0} h(t) = 0$. To prove this, let us write, according to [3],

$$Z(t, x, \tau, \xi) := 2^{-1}\pi^{-1/2}a^{-1/2}(\tau, \xi)(t-\tau)^{-1/2}\cdot\exp\{a^{-1}(\tau, \xi)(x-\xi)^2 \times \\ \times [4(t-\tau)]^{-1}\},$$

$$\Phi(t, x, \tau, \xi) := \sum_{n=1}^{\infty}(LZ)_n(t, x, \tau, \xi),$$

where $(LZ)_1 = LZ$ and L is the differential operator which associates the left-hand side of (1) with y,

$$(LZ)_{n+1}(t, x, \tau, \xi) := \int_{\tau}^{t}\int_{-1}^{1} LZ(t, x, \vartheta, \zeta)\cdot(LZ)_n(\vartheta, \zeta, \tau, \xi)\,d\zeta\,d\vartheta,$$

$$\Gamma(t, x, \tau, \xi) := Z(t, x, \tau, \xi) + \int_{\tau}^{t}\int_{-1}^{1} Z(t, x, \vartheta, \zeta)\Phi(\vartheta, \zeta, \tau, \xi)\,d\zeta\,d\vartheta.$$

We shall define on the boundary $[0, T]\times(\{-1\}\cup\{1\})$

$$F(t, x) := \int_{-1}^{1}\frac{\partial\Gamma(t, x, 0, \xi)}{\partial v(t, x)}g(\xi)\,d\xi + p(t, x)\int_{-1}^{1}\Gamma(t, x, 0, \xi)g(\xi)\,d\xi - \\ - w(t, x),$$

where

$$\frac{\partial f(t, x)}{\partial v(t, x)} := \lim_{\substack{z\to x\\ |z|<1}} a(t, z)\frac{\partial f(t, z)}{\partial x}$$

and

$$p(t, 1) := a(t, 1)\cdot\gamma(t), \quad p(t, -1) := a(t, -1)\cdot\beta(t),$$

$$w(t, 1) := a(t, 1)\cdot v(t), \quad w(t, -1) := a(t, -1)\cdot u(t).$$

Let us write, moreover,

$$M_1(t, x, \tau, \xi) := \frac{2\partial \Gamma(t, x, \tau, \xi)}{\partial v(t, x)} + 2p(t, x)\Gamma(t, x, \tau, \xi),$$

$$M_{n+1}(t, x, \tau, \xi) := \int_0^t [M_1(t, x, \sigma, 1)\ M_n(\sigma, 1, \tau, \xi) - $$
$$- M_1(t, x, \sigma, -1)M_n(\sigma, -1, \tau, \xi)]d\sigma,$$

$$\varphi(t, x) := 2F(t, x) + 2\int_0^t \Big[\sum_{n=1}^{\infty} M_n(t, x, \tau, 1)F(\tau, 1) - $$
$$- M_n(t, x, \tau, -1)F(\tau, -1)\Big]d\tau.$$

This explains the symbols used in the formula, which implies (5) (Section 1).

Applying that notation we get

$$|f_n(t_1, x) - f_n(t_2, x)| = \Big|\int_0^{t_1} [\Gamma(t_1, x, \tau, 1)\varphi(\tau, 1;\ u_n, v_n) - $$
$$- \Gamma(t_1, x, \tau, -1)\varphi(\tau, -1;\ u_n, v_n)]d\tau - $$
$$- \int_0^{t_2} [\Gamma(t_2, x, \tau, 1)\varphi(\tau, 1;\ u_n, v_n) - \Gamma(t_2, x, \tau, -1)\varphi(\tau, -1; u_n, v_n)]d\tau\Big|$$
$$\leqslant W_1 + W_2 + W_3,$$

where, if we assume that $t_2 - h < t_1 < t_2$,

$$W_1 := \Big|\int_0^{t_2-h} \{[\Gamma(t_1, x, \tau, 1) - \Gamma(t_2, x, \tau, 1)]\cdot\varphi(\tau, 1;\ u_n, v_n) - $$
$$- [\Gamma(t_1, x, \tau, -1) - \Gamma(t_2, x, \tau, -1)]\varphi(\tau, -1;\ u_n, v_n)\}d\tau\Big|,$$

$$\text{(26)} \qquad W_2 := \Big|\int_{t_2-h}^{t_1} \{[\Gamma(t_1, x, \tau, 1) - \Gamma(t_2, x, \tau, 1)]\cdot\varphi(\tau, 1;\ u_n, v_n) - $$
$$- [\Gamma(t_1, x, \tau, -1) - \Gamma(t_2, x, \tau, -1)]\varphi(\tau, -1;\ u_n, v_n)\}d\tau\Big|,$$

$$W_3 := \Big|\int_{t_1}^{t_2} [\Gamma(t_1, x, \tau, 1)\varphi(\tau, 1;\ u_n, v_n) - $$
$$- \Gamma(t_2, x, \tau, -1)\varphi(\tau, -1; u_n, v_n)]\, d\tau\Big|,$$

we get

$$\text{(27)} \qquad |Z(t_1, x, \tau, \xi) - Z(t_2, x, \tau, \xi)|$$
$$= \big(4\pi a(\tau, \xi)\big)^{-1/2} |(t_1 - \tau)^{-1/2} \exp[-(x-\xi)^2 a^{-1}(\tau, \xi)\cdot 4^{-1}(t_1-\tau)^{-1}] - $$
$$- (t_2 - \tau)^{-1/2} \exp[-a^{-1}(\tau, \xi)\ (x-\xi)^2 4^{-1}(t_2-\tau)^{-1}]|$$

$$= \big(4\pi a(\tau,\xi)\big)^{-1/2}|[(t_1-\tau)^{-1/2}-(t_2-\tau)^{-1/2}]\exp[-a^{-1}(\tau,\xi)\times$$
$$\times(x-\xi)^2 4^{-1}(t_1-\tau)^{-1}]+$$
$$+(t_2-\tau)^{-1/2}[\exp\{-a^{-1}(\tau,\xi)(x-\xi)^2 4^{-1}(t_1-\tau)^{-1}\}-$$
$$-\exp\{-a^{-1}(\tau,\xi)(x-\xi)^2 4^{-1}(t_2-\tau)^{-1}\}]|$$
$$\leqslant \big(4\pi a(\tau,\xi)\big)^{-1/2}|[(t_2-\tau)^{1/2}-(t_1-\tau)^{1/2}]\cdot[(t_1-\tau)(t_2-\tau)]^{-1/2}+$$
$$+(t_2-\tau)^{-1/2}\exp[-a^{-1}(\tau,\xi)(x-\xi)^2 4^{-1}(t_2-\tau)^{-1}]\times$$
$$\times\{\exp[-a^{-1}(\tau,\xi)(x-\xi)^2 4^{-1}(t_1-\tau)^{-1}+$$
$$+a^{-1}(\tau,\xi)(x-\xi)^2 4^{-1}(t_2-\tau)^{-1}]-1\}|$$
$$\text{for } \tau\leqslant t_2-h<t_1<t_2,\ x,\xi\in(-1,1)$$
$$\leqslant \mathrm{const}|(t_2-t_1)\big[(t_1+h-t_2)^{1/2}h^{1/2}\big((t_1+h-t_2)^{1/2}+(h)^{1/2}\big)\big]^{-1}+$$
$$+h^{-1/2}\{\exp[-\mathrm{const}(t_2-t_1)^{-1}4^{-1}(t_1+h-t_2)^{-1}h^{-1}]-1\}|\to 0,$$

having fixed h and $t_1\to t_2$, which, together with formula (cf. [3])

$$|\Phi(t,x,\tau,\xi)|\leqslant\frac{\mathrm{const}}{(t-\tau)^{\mu}|x-\xi|^{3-2\mu-\alpha}}$$

for any $\mu\in(1-\frac{1}{2}\alpha,1)$, where α is from (4), implies

$$(28)\qquad \Big|\int_{\tau}^{t_2}\int_{-1}^{1}Z(t_2,x,\sigma,\zeta)\Phi(\sigma,\zeta,\tau,\xi)d\zeta d\sigma+$$
$$+\Big|\int_{\tau}^{t_1}\int_{-1}^{1}[Z(t_1,x,\sigma,\zeta)-Z(t_2,x,\sigma,\zeta)]\Phi(\sigma,\zeta,\tau,\xi)d\zeta d\sigma\Big|\to 0.$$

Omitting terms independent of (u,v) which do not belong to the first term of the right-hand side of (5), we can write

$$\varphi(t,1;\,u,v)=2v(t)+2\sum_{n=1}^{\infty}\int_0^t[M_n(t,1,\tau,1)v(\tau)-M_n(t,1,\tau,-1)\times$$
$$\times u(\tau)]d\tau.$$

Therefore, for $(u,v)\in U\times U$,

$$|\varphi(t,x;\,u,v)|\leqslant 2K+2K\sum_{n=1}^{\infty}\int_0^t[|M_n(t,1,\tau,1)|+|M_n(t,1,\tau,-1)|]d\tau.$$

This series is bounded and absolutely convergent (cf. [3]) independently of $(u,v)\in U\times U$. Hence, (26)–(28) and the formula for $\Gamma(t,x,\tau,\xi)$ imply that for any $\varepsilon>0$ there is a $\delta>0$ such that for each $t_1\in(t_2-\delta,t_2)$ $W_1\leqslant\varepsilon/3$. The summability of Γ as a function of τ on $(0,t)$ implies that there is an $h>0$ such that $W_2\leqslant\varepsilon/3$, $W_3\leqslant\varepsilon/3$ for $t_1\in(t_2-h,t_2)$. So we have completed the proof of the equicontinuity of $\{f_n\}$, and thus the proof of the Lemma as well.

3.2. THEOREM. *Let problem* (1)–(3) *fulfils Assumptions* A *and* B.

1° *If $T_n \to T_0$ is an increasing sequence, if (u_n, v_n) is the optimal control in problem* (1)–(3), (8)–(10) *in the domain $P_n = (0, T_n] \times (-1, 1)$ and if $J(u_n, v_n) > 0$ for $n = 0, 1, 2, \ldots$, then the sequence*

$$\big(u_n'(t), v_n'(t)\big) = \begin{cases} \big(u_n(t), v_n(t)\big), & t \leqslant T_n, \\ (0, 0), & t \in (T_n, T_0], \end{cases}$$

converges:

$$u_n'(t) \to u_0(t), \quad v_n'(t) \to v_0(t) \quad \text{a.e. on } (0, T_0].$$

2° *If $T_n \to T_0$ is a decreasing sequence, if (u_n, v_n) is the optimal control for P_n and if $J(u_n, v_n) > 0$ for $n = 0, 1, 2, \ldots$, then the sequence*

$$\big(u_n'(t), v_n'(t)\big) = \begin{cases} \big(u_n(t), v_n(t)\big), & t \leqslant T_n, \\ (0, 0), & t \in (T_n, T_1], \end{cases}$$

converges:

$$u_n'(t) \to u_0'(t), \quad v_n'(t) \to v_0'(t) \quad \text{a.e. on } (0, T_1].$$

In cases 1° *and* 2° *there is a* $\lim_{n\to\infty} J(u_n, v_n) = J(u_0, v_0)$ *and* $J(u_i, v_i) < J(u_j, v_j)$ *for* $T_i > T_j$.

Proof. 1° The sequence (u_n', v_n') is contained in the bounded, convex, closed set $U \times U \subset L^2 \times L^2$. Hence there are weakly convergent subsequences u_{n_k}', v_{n_k}' (convergent in measure); thus there are subsequences convergent a.e. to some functions u_0'', v_0''. It will be sufficient to prove that for any such sequences $u_0''(t) = u_0(t)$, $v_0''(t) = v_0(t)$ a.e. on $(0, T_0]$ and to prove the last sentence of the theorem.

Denote the chosen subsequence by (u_n'', v_n''). Let $u_n'' \to u_0''$, $v_n'' \to v_0''$ a.e. and let $y_n(t, x)$ be the solution of (1)–(3) in the domain $(0, T_0] \times (-1, 1)$ with $u = u_n''$, $v = v_n''$ in condition (3).

The maximum principle for problem (1)–(3) implies that, for each $(t, x) \in (T_n, T_0] \times (-1, 1)$,

$$|y_n(t, x)| \leqslant \sup\{|y_n(T, x)|: \ |x| < 1\} = J(u_n, v_n)$$

([7], §§64 and 54). Since for $m > n \geqslant 1$ the control (u_n', v_n') does not satisfy the bang-bang principle on $(0, T_m]$, we have

$$J(u_m, v_m) < \sup\{|y_n(T_m, x)|: \ |x| < 1\} \leqslant J(u_n, v_n). \tag{29}$$

This means that the minimal cost is a strongly decreasing function of the final moment. This means that for any $n > 1$ we also have

$$J(u_0, v_0) < J(u_n, v_n). \tag{30}$$

The consequence of the uniform continuity of $y_0(t, x)$ on $(\frac{1}{2}T_0, T_0) \times \times (-1, 1)$ (Lemma 3.1 and (5)) is that

$$(31) \qquad \begin{aligned} J(u_0, v_0) &= \sup\{|y(T_0, x;\ u_0, v_0)|:\ |x| < 1\} \\ &= \lim_{n\to\infty}\big(\sup\{|y(T_n, x;\ u_0, v_0)|:\ |x| < 1\}\big) \\ &\geqslant \limsup_{n\to\infty}\big(\sup|y_n(T_n, x)|\big) \\ &= \limsup_{n\to\infty} J(u_n, v_n). \end{aligned}$$

Since $u_n'' \to u_0''$, $v_n'' \to v_0''$ weakly, then, for any $x \in (-1, 1)$, $y(T_0, x;\ u_n'', v_n'') \to y(T_0, x;\ u_n'', v_n'')$, which means that

$$(32) \qquad |y(T_0, x;\ u_0'', v_0'')| \leqslant \liminf_{n\to\infty} J(u_n'', v_n'') \leqslant \liminf_{n\to\infty} J(u_n, v_n).$$

It follows from (30), (31) and (32) that

$$J(u_0'', v_0'') \leqslant \liminf_{n\to\infty} J(u_n, v_n) \leqslant \limsup_{n\to\infty} J(u_n, v_n) \leqslant J(u_0, v_0)$$

and from the optimality of (u_0, v_0) and the uniqueness theorem — that $u_0''(t) = u_0(t)$, $v_0''(t) = v_0(t)$ a.e. on $(0, T_0]$ and

$$J(u_0'', v_0'') = \lim_{n\to\infty} J(u_n, v_n) = J(u_0, v_0),$$

which was to be proved.

2° Suppose, as above, that (u_n'', v_n'') is a subsequence convergent a.e. to (u_0'', v_0''). Likewise we can prove that the minimal cost is the decreasing function of the final moment. It remains to prove that $u_0''(t) = u_0'(t)$, $v_0''(t) = v_0'(t)$ a.e. on $(0, T_1]$, and that

$$J(u_0, v_0) = \lim_{n\to\infty} J(u_n, v_n).$$

The function $y(t, x;\ u_0'', v_0'')$ is uniformly continuous on $(\frac{1}{2}T_0, T_1] \times \times [-1, 1]$; thus for each $\varepsilon > 0$ there is an N such that for $n > N$

$$(33) \quad J(u_n, v_n) = \sup\{|y_n(T_n, x)|:\ |x| < 1\} \geqslant |y_n(T_n, x_n)|,$$

where x_n is such a number that the maximum of $y(T_n, x;\ u_0'', v_0'')$ is attained at x_n,

$$\begin{aligned} &\geqslant |y(T_n, x_n;\ u_0'', v_0'')| - \varepsilon = \sup\{|y(T_n, x;\ u_0'', v_0'')|:\ |x| < 1\} - \varepsilon \\ &\geqslant \sup\{|y(T_0, x;\ u_0'', v_0'')|:\ |x| < 1\} - 2\varepsilon. \end{aligned}$$

These inequalities are true because of the uniform convergence $y_n \to y$ (Lemma 3.1).

The optimality of (u_0, v_0) implies that

$$(34) \qquad J(u_0'', v_0'') := \sup\{|y(T_0, x;\ u_0'', v_0'')|:\ |x| < 1\} \geqslant J(u_0, v_0),$$

and the decrease of the minimal cost implies that

$$\limsup_{n\to\infty} J(u_n, v_n) \leqslant J(u_0, v_0). \tag{35}$$

Using (33)–(35) we have, for any $\varepsilon > 0$,

$$J(u_0, v_0) \geqslant \limsup_{n\to\infty} J(u_n, v_n) \geqslant \liminf_{n\to\infty} J(u_n, v_n) \geqslant J(u_0'', v_0'') - 2\varepsilon$$
$$\geqslant J(u_0, v_0) - 2\varepsilon;$$

hence there is a $\lim\limits_{n\to\infty} J(u_n, v_n) = J(u_0'', v_0'') = J(u_0, v_0)$.

Theorem 2.9 implies that $u_0''(t) = u_0(t)$, $v_0''(t) = v_0(t)$ a.e. on $(0, T_0]$, and, moreover, the pointwise convergence $u_n''(t) \to u_0''(t)$, $v_n''(t) \to v_0''(t)$ implies that $u_0''(t) = v_n''(t) \equiv 0$ on $(T_0, T_1]$. Therefore $u_0'' = u_0'$, $v_0'' = v_0'$ a.e. on $(0, T_1]$, which was to be proved.

3.3. THEOREM. *Let problem* (1)–(3) *fulfil Assumptions* A *and* B. *Suppose that* $T_n \to T_0 > 0$, (u_n, v_n) *is an optimal control of problem* (1)–(3), (8)–(10) *in* P_n, *and* $J(u_n, v_n) > 0$, $n = 0, 1, 2, \ldots$ *Then in each neighbourhood of any switching point of* $u_0(v_0)$ *for* n *sufficiently large there is a switching point of* $u_n(v_n)$.

3.4. Remark. This theorem implies that for n sufficiently large the function u_n (v_n) has at least as many switches as u_0 (v_0).

Proof of Theorem 3.3. Let

$$0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T_0,$$

$$0 = s_0 < s_1 < \ldots < s_j < s_{j+1} = T_0,$$

be the switching points of u_0 and v_0, respectively. Define functions $u_n'(t)$, $v_n'(t)$ on the interval $(0, T_0]$ as follows

$$\big(u_n'(t), v_n'(t)\big) = \begin{cases} \big(u_n(t), v_n(t)\big), & t \in \big(0, \min(T_n, T_0)\big], \\ (0, 0), & t \in \big(\min(T_n, T_0), T_0\big]. \end{cases}$$

According to Theorem 3.2, $u_n' \to u_0$, $v_n' \to v_0$ weakly in $L^2\big((0, T_0]\big)$ Theorem 2.8 implies that $|u_n(t)| = |v_n(t)| = K$ a.e. on $(0, T_n]$.

Next we suppose the contrary theorem to 3.3. Hence there are a positive number δ and a sequence $n_l \to \infty$ such that the functions u_{n_l} (v_{n_l}) have a constant sign on the interval $(t_i - \delta, t_i + \delta)$ (or $(s_i - \delta, s_i + \delta)$).

Let the first case be true. Then

$$2K\delta \leqslant \Big| \int\limits_{t_i-\delta}^{t_i+\delta} [u_0(t) - u_{n_l}'(t)]\, dt \Big| = \Big| \int\limits_0^t \chi(t)\, [u_0(t) - u_{n_l}'(t)]\, dt \Big|$$

for $l = 1, 2, \ldots$, where χ is the characteristic function of the interval $(t_i - \delta, t_i + \delta)$. This contradicts the weak convergence $u_n' \to u_0$, because the above integral is a linear bounded functional on $L^2\big((0, T_0]\big)$.

In the second case we can prove a contradiction analogically. Hence the proof is completed.

3.5. THEOREM. *Let problem* (1)–(3) *fulfils Assumptions* A *and* B. *Suppose that the function g appearing in* (2) *changes sign m times on* $[-1, 1]$, *and $(u, v) \in U \times U$ is the optimal control of problem* (1)–(3), (8)–(10) *in the domain* $P = (0, T] \times (-1, 1)$. *Suppose that there is a number $d > 0$ such that for $T' \in (T-d, T+d)$*

$$\inf\{\sup_{|x|<1} |y(T', x;\ u, v)|:\ (u, v) \in U \times U\} > 0.$$

If n is the number of switches of the control (u, v) and n' is the number of switches of the optimal control (u', v') of problem (1)–(3), (8)–(10) *in the domain* $P' := (0, T'] \times (-1, 1)$, *then*

1° *there is an $h > 0$ such that, for $T' \in (T-h, T+h)$, $0 \leqslant n'-n \leqslant m+2$;*

2° *if $m = 0$, then for $T' \in (T-h, T+h)$ the difference between the numbers of switches of u and u' (v and v', respectively) is* 0 *or* 1.

Proof. 1° We have proved (Remark 3.4) that the functions u', v' have at least as many switches as u, v, respectively, for T' close to T. We have proved that $u'(t) = u(t)$, $v'(t) = v(t)$ for t small enough or u' (v') has at least one switch more than u (v). Hence

$$\liminf_{T' \to T} n' \geqslant n.$$

We now have to prove that

$$\limsup_{T' \to T} n' \leqslant n + m + 2.$$

Denote by N the number of the essential extremes of the function $y(T, x;\ u, v)$ on $[-1, 1]$. This function changes sign $N-1$ times at least; thus

$$n \geqslant N-1-m-2 = N-m-3 \tag{36}$$

(Theorem 1.11). Since $y(t, x;\ u', v') \to y(t, x;\ u, v)$ uniformly by $T' \to T$ (Theorem 3.2 and Lemma 3.1), therefore $y(T', x;\ u', v') \to y(T, x;\ u, v)$ uniformly by $T' \to T$. Since the derivative

$$\frac{\partial}{\partial x}\, y(t, x;\ u', v')$$

is bounded in the neighbourhood of T by a constant independent of t, x, u, v (Lemma 2.4), thus for $|T'-T|$ sufficiently small the number of essential extremes of $y(T', x;\ u', v')$ on $[-1, 1]$ is not higher than N. In this way we have proved the following

3.6. Remark. Under the assumptions of Theorem 3.5 and if the function $y(T, x;\ u, v)$ has N essential extremes, for T' close to T the function $y(T', x;\ u', v')$ has at most N essential extremes on $[-1, 1]$.

Remark 2.12 implies that

$$n' \leqslant N-1; \tag{37}$$

hence the consequence of (36) is $n'-n \leqslant m+2$, which completes the proof of 1°.

2° Remark 3.4 allows us to consider only the case where

$$n' = n+2. \tag{38}$$

Inequalities (36) and (37) imply that

$$n' = N-1. \tag{39}$$

Let $0 < t_1' < t_2' < \ldots < t_k' < T'$, $0 < s_1' < s_2' < \ldots < s_l' < T'$ be the switching points of u' and v', respectively. Using Definition 2.10, if

$$\int_0^{t_1'} u'(t)\,dt \int_0^{s_1'} v'(t)\,dt \begin{cases} < 0, & \text{then } k+l = N-2, \\ > 0, & \text{then } k+l = N-1. \end{cases}$$

If

$$\int_{t_k'}^{T'} u'(t)\,dt\, y(T', x_1';\ u', v') > 0,$$

where

$$x_1' := \inf\{x \in [-1, 1]:\ |y(T', x;\ u', v')| = \sup_{|\xi|<1} |y(T', \xi;\ u', v')|\},$$

then the sets

$$A_0 := (0, t_1'), \quad A_1 := (t_1', t_2'), \quad \ldots, \quad A_k := (t_k', T')$$

and

$$B_l := (s_l', T'), \quad B_{l-1} := (s_{l-1}', s_l'), \quad \ldots, \quad B_1 := (s_1', s_2')$$

and $B_0 := (0, s_1')$ if $k+l = N-2$ with

$$\varepsilon := (-1)^{k+1} \operatorname{sign} \int_{t_k'}^{T'} u'(t)\,dt$$

fulfil the assumptions of Lemma 2.7, which contradicts the optimality of (u', v'). Thus we can conclude that

$$y(T', x_1';\ u', v') \int_{t_k'}^{T'} u'(t)\,dt < 0. \tag{40}$$

If $0 < t_1 < \ldots < t_j < T$ are the switching points of u, then

$$\int_{t_j}^{T} u(t)\,dt\, y(T, x;\; u, v) > 0$$

for $x \in (-1, -1+h)$ (Corollary 1.12) because for $n' = n+2$, $n = N-3$; (36) thus, writing

$$x_1 := \inf\{x \in [-1, 1]\colon\; |y(T, x;\; u, v)| = \sup_{|\xi|<1} |y(T, \xi;\; u, v)|\},$$

we have

$$y(T, x_1, u, v) \int_{t_j}^{T} u(t)\,dt > 0, \tag{41}$$

since the function $y(T, x;\, u, v)$ which has N essential extremes on $[-1, 1]$ changes sign $(N-1)$ times because $n = N-3$ (Theorem 1.11).

Since there is exactly N essential extremes on $[-1, 1]$ (which follows from $n' = N-1$, Remarks 2.12 and 3.6), we have, just as in the proof of 1°, $x_1' \to x_1$. Hence (41) implies (for T' close to T) the following inequality:

$$y(T', x_1';\; u', v') \int_{t_j}^{T} u(t)\,dt > 0, \tag{42}$$

$u' \to u$, $v' \to v$ weakly in $L^2\big((0, T]\big)$; thus

$$m\{t\colon\; |u'(t) - u(t)| + |v'(t) - v(t)| > \varepsilon\} \to 0$$

for any $\varepsilon > 0$ when T' is close enough to T

$$\int_{0}^{t_1} u(t)\,dt \int_{0}^{t_1} u'(t)\,dt > 0, \qquad \int_{t_1}^{t_2} u(t)\,dt \int_{t_1}^{t_2} u'(t)\,dt > 0,$$

etc., and hence

$$\int_{t_j}^{T} u(t)\,dt \int_{t_j}^{T} u'(t)\,dt > 0. \tag{43}$$

Inequalities (38)–(43) imply

$$k \geqslant j+1. \tag{44}$$

An analogous proof for the function v with switching points $s_1 < s_2 < \ldots < s_i$ yields

$$\int_{s_i}^{T} v(t)\,dt\, y(T, x_N;\; u, v) > 0, \tag{45}$$

$$\int_{s}^{T} v'(t)\,dt\, y(T', x_N';\; u', v') > 0, \tag{46}$$

where

$$x_N := \sup\{x \in [-1, 1]\colon\ |y(T, x;\ u, v)| = \sup_{|\xi|<1} |y(T, \xi;\ u, v)|\},$$

$$x'_N := \sup\{x \in [-1, 1]\colon\ |y(T', x;\ u', v')| = \sup_{|\xi|<1} |y(T', \xi;\ u', v')|\}$$

and

$$l \geqslant i+1. \tag{47}$$

If the function $y(T, x;\ u, v)$ changes sign $(N-1)$ times on $(-1, 1)$, then

$$i+j \geqslant N-3 \tag{48}$$

(Theorem 1.11). Since $k+l \leqslant N-1$, inequalities (44), (47), (48) imply 2° and Theorem 3.5 is completely proved.

We have proved that in case 2° if $n' = n+2$ inequalities (40) and (41) hold. They imply (44). Inequalities (45) and (46) are also true and they imply (47). This means that $k = j+1$, $l = i+1$. When $n' = n+1$, inequalities (40), (41) or (45), (46) hold, which implies (44) or (47). In this way we have proved the following

3.7. COROLLARY. *Using all the notation and assumptions of Theorem* 3.5, 2°, *we have* $k = j+1$ *iff inequalities* (40) *and* (41) *hold, and* $l = i+1$ *iff* (45) *and* (46) *are true.*

Remark. To complete these investigations one could prove that the number of switches is an increasing function of the final moment T.

References

[1] A. G. Butkowskiĭ, *Theory of optimal control of systems with distributed parameters* (in Russian), "Science", Moscow 1965.

[2] Ю. В. Егоров, *Некоторые задачи теории оптимального управления*, Ж. Выч. Мат. и Мат.-Физ. 3 (1963), No. 5.

[3] A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, New York 1964.

[4] J. L. Lions, *Controle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Paris 1968.

[5] S. Łojasiewicz, *Wstęp do teorii funkcji rzeczywistych*, PWN, Warszawa 1973.

[6] H. Marcinkowska, *Wstęp do teorii równań różniczkowych cząstkowych*, PWN, Warszawa 1972.

[7] J. Szarski, *Differential inequalities*, PWN, Warszawa 1965.

INSTYTUT MECHANIKI BUDOWLI
POLITECHNIKA KRAKOWSKA

Reçu par la Rédaction le 1. 4. 1975