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Random differential inclusions with convex right hand sides

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Abstract. The main result of the present paper deals with the existence of solutions of random functional-differential inclusions of the form

$$\dot{x}(t, \omega) \in G(t, \omega, x(\cdot, \omega), \dot{x}(\cdot, \omega))$$

with G taking as its values nonempty compact and convex subsets of n-dimensional Euclidean space $\mathbf{R}^n$.

1. Notations and definitions. Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space, where $\mathscr{F}$ is a σ-field of subsets of Ω and μ a probability measure defined on $\mathscr{F}$. Let $I = [\sigma, \sigma+a]$ be a closed bounded interval and denote by $\mathscr{L}(I)$ the Lebesgue σ-field on I. By $C(I, \mathbf{R}^n)$ we denote the Banach space of all continuous functions $x\colon I \to \mathbf{R}^n$ with the supremum norm $|\cdot|_C$ and $L(I, \mathbf{R}^n)$ stands for the space of all L-integrable functions $u\colon I \to \mathbf{R}^n$ endowed with the norm $|u| = \int_I |u(t)|\,dt$, where $|\cdot|$ is a norm of $\mathbf{R}^n$. By $AC(I, \mathbf{R}^n)$ we denote the space of all absolutely continuous functions $x\colon I \to \mathbf{R}^n$ with the norm defined by

$$\|x\| = |x(\sigma)| + \int_{\sigma}^{\sigma+a} |\dot{x}(t)|\,dt.$$

The symbols $\mathscr{B}(C)$, $\mathscr{B}(L)$ and $\mathscr{B}(AC)$ will denote the Borel σ-fields of $C(I, \mathbf{R}^n)$, $L(I, \mathbf{R}^n)$ and $AC(I, \mathbf{R}^n)$, respectively.

We shall also consider the metric space $(\mathrm{Conv}(\mathbf{R}^n), h)$ of all nonempty compact convex subsets of $\mathbf{R}^n$ with the Hausdorff metric h defined by $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$, for $A, B \in \mathrm{Conv}(\mathbf{R}^n)$, where $\bar{h}(B, A) = \max_{b\in B} \inf_{a\in A} |b-a|$.

Let $(T, \mathscr{F})$ be a measurable space, X a separable metric space and $F\colon T \to \mathscr{P}(X)$ a set-valued function, where $\mathscr{P}(X)$ is the space of all nonempty subsets of X.

F is said to be *measurable* (*weakly measurable*) if $F^-(E) = \{t \in T\colon F(t) \cap E \neq \emptyset\} \in \mathscr{F}$ for every closed (open) set $E \subset X$.

Let X and Y be topological Hausdorff spaces. We will say that a multifunction $F\colon X \to \mathscr{P}(Y)$ is *upper semicontinuous* (*u.s.c.*) at $\bar{x} \in X$ if for every neighbourhood U of $F(\bar{x})$ there exists a neighbourhood V of $\bar{x}$ such that $F(x) \subset U$ for every $x \in V$.

F is called *u.s.c. on* X if it is u.s.c. at every $x \in X$.

F is said to be *lower semicontinuous* (*l.s.c.*) at $\bar{x} \in X$ if for every open set U in Y with $F(\bar{x}) \cap U \neq \emptyset$ there exists a neighbourhood V of $\bar{x}$ such that $F(x) \cap U \neq \emptyset$ for every $x \in V$.

F is called *l.s.c. on* X if it is l.s.c. at every $x \in X$.

If F is simultaneously u.s.c. and l.s.c. we call it *continuous*.

THEOREM 1.1 ([5], Th. 2.2, p. 64). *Let X and Y be metric spaces. A set-valued function $F\colon X \to \mathrm{Comp}(Y)$ is u.s.c. on X if and only if for every $x \in X$ and every sequence (x_n) in X converging to x and every sequence (y_n) in Y with $y_n \in F(x_n)$ there is a convergent subsequence of (y_n) whose limit belongs to $F(x)$.*

Assume now X and Y are normed linear spaces. They will be considered as locally convex topological Hausdorff spaces with their weak topologies.

We will say that a set-valued function $F\colon X \to \mathscr{P}(Y)$ is *weakly-weakly upper semicontinuous* (*w.-w.u.s.c.*) on X if for every weakly closed set $M \subset Y$ the set $F^-(M) = \{x \in X\colon F(x) \cap M \neq \emptyset\}$ is sequentially weakly closed in X.

We say that $F\colon X \to \mathscr{P}(Y)$ is *weakly-strongly upper semicontinuous* (*w.-s.u.s.c.*) on X if for every weakly closed set $M \subset Y$ the set $F^-(M)$ is closed (in the norm topology) in X.

Similarly we define *s.-w.u.s.c.* and *s.-s.u.s.c.* mappings on X. Weak forms of lower semicontinuity are obtained upon replacing $F^-(M)$ by $F_-(M) = \{x \in X\colon F(x) \subset M\}$.

Denote by $\mathscr{T}$ and $\mathscr{D}$ the mappings defined on $L(I, \mathbf{R}^n)$ and $AC(I, \mathbf{R}^n)$ respectively by

$$(\mathscr{T}u)(t) = \int_{\sigma}^{t} u(\tau)\,d\tau \quad \text{for } u \in L(I, \mathbf{R}^n),\ t \in I, \tag{1.1}$$

$$(\mathscr{D}x)(t) = \dot{x}(t) \quad \text{for } u \in AC(I, \mathbf{R}^n) \text{ and a.e. } t \in I. \tag{1.2}$$

THEOREM 1.2 ([4], Prop. 2.1, p. 12). *The mapping $\mathscr{T}$ defined by* (1.1) *has the following properties*:

(i) *$\mathscr{T}$ is a linear isometry of $L(I, \mathbf{R}^n)$ onto $AC(I, \mathbf{R}^n)$*,

(ii) *the restriction of $\mathscr{T}$ to each weakly compact set $A \subset L(I, \mathbf{R}^n)$ is strongly-weakly sequentially continuous as a mapping of A into $C(I, \mathbf{R}^n)$, i.e. for every $u \in A$ and every sequence (u_n) in A weakly converging to u we have* $|\mathscr{T}u_n - \mathscr{T}u|_C \to 0$.

COROLLARY 1.1. ([4], Cor. 2.1, p. 13). *The mapping $\mathscr{T}$ is weakly-weakly continuous as a mapping of $L(I, \mathbf{R}^n)$ to $AC(I, \mathbf{R}^n)$, i.e. it is continuous as a mapping between the spaces $L(I, \mathbf{R}^n)$ and $AC(I, \mathbf{R}^n)$ with their weak topologies.*

COROLLARY 1.2 ([4], Cor. 2.2, p. 14). *For every weakly compact set $\Lambda \subset L(I, \mathbf{R}^n)$ the set $K = \mathcal{T}\Lambda$ is a compact subset of $C(I, \mathbf{R}^n)$ and a weakly compact subset of $AC(I, \mathbf{R}^n)$. Furthermore, if Λ is convex, K is also convex.*

Let $\Lambda = \{u \in L(I, \mathbf{R}^n): |u(t)| \leqslant m(t)$ for a.e. $t \in I\}$, where m is an L-integrable function. It is not difficult to see that Λ is a uniformly integrable, bounded and convex subset of $L(I, \mathbf{R}^n)$. Hence, by the Dunford Theorem (see [1]) it is relatively sequentially weakly compact and by the Eberlein–Shmul'yan Theorem (see [1]) it is relatively weakly compact in $L(I, \mathbf{R}^n)$. Since Λ is convex and closed it is also weakly closed in $L(I, \mathbf{R}^n)$, and therefore Λ is weakly compact. Hence and by Corollary 1.2, $K = \mathcal{T}\Lambda$ is a compact convex subset of $C(I, \mathbf{R}^n)$.

Let $G: I \times \Omega \times K \times \Lambda \to \mathrm{Conv}(\mathbf{R}^n)$ be given and for each $\omega \in \Omega$ and $(x, z) \in K \times \Lambda$, denote by $\mathcal{F}(G)(\omega, x, z)$ the collection of all L-integrable functions $u: I \to \mathbf{R}^n$ having the property that $u(t) \in G(t, \omega, x, z)$ a.e. in I. The set $\mathcal{F}(G)(\omega, x, z)$ is called the *subtrajectory integrals of the set-valued function* $G(\cdot, \omega, x, z)$.

By $\mathcal{T}\mathcal{F}(G)(\omega, x, z)$ we denote the image of $\mathcal{F}(G)(\omega, x, z)$ by $\mathcal{T}$. It will be called the *trajectory integrals of* $G(\cdot, \omega, x, z)$.

If $\mathcal{F}(G)(\omega, x, z) \neq \emptyset$ we say $G(\cdot, \omega, x, z)$ is *Aumann integrable.* Its Aumann integral over the measurable subset $U \subset I$ is denoted by $\int_U G(t, \omega, x, z)\, dt$, i.e.

$$\int_U G(t, \omega, x, z)\, dt = \Big\{\int_U f(t)\, dt \colon f(t) \in \mathcal{F}(G)(\omega, x, z)\Big\}.$$

The family $\{G(\cdot, \omega, x, z)\}_{(\omega,x,z) \in \Omega \times K \times \Lambda}$ will be assumed uniformly integrable, i.e. such that for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\left\|\int_U G(t, \omega, x, z)\, dt\right\| \leqslant \varepsilon$ for every $\omega \in \Omega$, $(x, z) \in K \times \Lambda$ and all measurable sets $U \subset I$ with $\mu(U) \leqslant \delta$, where for $A \in \mathrm{Comp}(\mathbf{R}^n)$ we set $\|A\| = \sup_{a \in A} |a|$. It is known that the family $\{G(\cdot, \omega, x, z)\}_{(\omega,x,z) \in \Omega \times K \times \Lambda}$ is uniformly integrable if and only if the collection of all selectors of all its members is uniformly integrable.

THEOREM 1.3 ([4], Lemma 2.4, p. 16). *Let $G: I \times \Omega \times K \times \Lambda \to \mathrm{Comp}(\mathbf{R}^n)$ be such that $G(\cdot, \omega, x, z)$ is Aumann integrable for each fixed $\omega \in \Omega$ and $(x, z) \in K \times \Lambda$. Then*

(i) *$\mathcal{T}\mathcal{F}(G)(\omega, x, z)$ is a nonempty, closed and bounded subset of $AC(I, \mathbf{R}^n)$,*

(ii) *$\mathcal{T}\,\overline{[\mathcal{F}(G)(\omega, x, z)]}{}_L^w$ is compact in $C(I, \mathbf{R}^n)$ and weakly compact in $AC(I, \mathbf{R}^n)$.*

THEOREM 1.4 ([4], Lemma 2.5, p. 17). *Under the assumptions of Theorem 1.3,*

(i) $\mathcal{T}\,\overline{[\mathcal{F}(G)(\omega, x, z)]}{}_L^w = \overline{[\mathcal{T}\mathcal{F}(G)(\omega, x, z)]}{}_C$,

(ii) $\mathcal{T}\,\overline{[\mathcal{F}(G)(\omega, x, z)]}{}_L^w = \overline{[\mathcal{T}\mathcal{F}(G)(\omega, x, z)]}{}_{AC}^w$.

THEOREM 1.5 ([4], Cor. 2.4, p. 19). *Under the assumptions of Theorem* 1.3, $\mathcal{T}\mathcal{F}(\mathrm{co}\,G)$ *is a nonempty convex compact subset of* $C(I, \mathbf{R}^n)$. *Moreover,*

(i) $\mathcal{T}\mathcal{F}(\mathrm{co}\,G) = \overline{[\mathcal{T}\mathcal{F}(G)]}_C$,

(ii) $\mathcal{T}\mathcal{F}(\mathrm{co}\,G) = \overline{[\mathcal{T}\mathcal{F}(G)]}^w_{AC}$.

In the above theorems, $\overline{[A]}^w_L$, $\overline{[A]}^w_{AC}$ denote the closure of A in the weak topology of $L(I, \mathbf{R}^n)$ and $AC(I, \mathbf{R}^n)$ respectively whereas $\overline{[A]}_C$ is the closure of A in the norm topology of $C(I, \mathbf{R}^n)$.

Finally, we recall the following definitions (see [5], pp. 150-151).

A set-valued function $G\colon I \times \Omega \times K \times \Lambda \to \mathrm{Comp}(\mathbf{R}^n)$ is said to be *weakly-weakly upper semicontinuous* (*w.-w.u.s.c.*) [*weakly-weakly lower semicontinuous* (*w.-w.-l.s.c.*)] *in its last two variables* if for every $(x, z) \in K \times \Lambda$ and every sequence $\{(x_n, z_n)\}$ in $K \times \Lambda$ such that $|x_n - x|_C \to 0$ and $z_n \rightharpoonup z$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \bar{h}\Big(\int_U G(t, \omega, x_n, z_n)\,dt,\ \int_U G(t, \omega, x, z)\,dt\Big) = 0 \tag{1.3}$$

$$\Big[\lim_{n\to\infty} \bar{h}\Big(\int_U G(t, \omega, x, z)\,dt,\ \int_U G(t, \omega, x_n, z_n)\,dt\Big) = 0\Big] \tag{1.4}$$

for every measurable set $U \subset I$ and for a.e. $\omega \in \Omega$, where $\rightharpoonup$ denotes weak convergence in $L(I, \mathbf{R}^n)$.

If $G\colon I \times \Omega \times K \times \Lambda \to \mathrm{Comp}(\mathbf{R}^n)$ is simultaneously w.-w.u.s.c. and w.-w.l.s.c. in its last two variables we call it *weakly-weakly continuous* (*w.-w.c.*).

Replacing in the above definition $z_n \rightharpoonup z$ by $|z_n - z| \to 0$ we obtain the respective "weak-strong" notions. Finally, replacing (1.3) [(1.4)] by

$$\lim_{n\to\infty} \int_{\sigma}^{\sigma+a} \bar{h}\big(G(t, \omega, x_n, z_n), G(t, \omega, x, z)\big)\,dt = 0 \tag{1.5}$$

$$\Big[\lim_{n\to\infty} \int_{\sigma}^{\sigma+a} \bar{h}\big(G(t, \omega, x, z), G(t, \omega, x_n, z_n)\big)\,dt = 0\Big] \tag{1.6}$$

gives the strong-weak counterparts.

THEOREM 1.6 ([4], Lemma 3.7, p. 31). *Suppose* $G\colon I \times \Omega \times K \times \Lambda \to \mathrm{Comp}(\mathbf{R}^n)$ *has convex values and is w.-w.l.s.c. in its last two variables. Then* $\mathcal{T}\mathcal{F}(G)(\omega, \cdot, \cdot)$ *is w.-s.l.s.c. on* $K \times \Lambda$.

Now we define a set-valued function $G \square \mathcal{D}$ on $I \times \Omega \times K$ by setting $(G \square \mathcal{D})(t, \omega, x) = G(t, \omega, x, \mathcal{D}x)$ for $t \in I$, $\omega \in \Omega$ and $x \in K$, where $\mathcal{D}$ is defined by (1.2).

By $\mathcal{F}(G \square \mathcal{D})(\omega, x)$ we denote the subtrajectory integrals of the function $(G \square \mathcal{D})(\cdot, \omega, x)$.

Observe that $\mathcal{F}(G \square \mathcal{D})(\omega, x) = \mathcal{F}(G)(\omega, x, \mathcal{D}x)$ for $\omega \in \Omega$ and $x \in K$. By $\mathcal{T}\mathcal{F}(G \square \mathcal{D})(\omega, x)$ we denote the trajectory integrals of $(G \square \mathcal{D})(\ , \omega, x)$.

2. Auxiliary results. The following lemma will play the crucial role in this paper.

LEMMA 2.1. *Assume* $G\colon I\times\Omega\times K\times\Lambda\to\mathrm{Conv}(\mathbf{R}^n)$ *is such that*

(i) $G(\cdot,\cdot,x,z)$ *is measurable (with respect to the product* σ*-field* $\mathcal{L}(I)\times\mathcal{F}$) *for each* $(x,z)\in K\times\Lambda$,

(ii) $\|G(t,\omega,x,z)\|\leqslant m(t)$ *for a.e.* $t\in I$, $\omega\in\Omega$ *and* $(x,z)\in K\times\Lambda$, *where* m *is L-integrable,*

(iii) G *is w.-w.c. in its last two variables.*

Then

(a) $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,\cdot)$ *is continuous for every* $\omega\in\Omega$,

(b) $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\cdot,x)$ *is measurable for* $x\in K$.

Proof. We have to show that $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,\cdot)$ is u.s.c. and l.s.c. for fixed $\omega\in\Omega$. Fix $\omega\in\Omega$ and $x\in K$ and let (x_n) be a sequence in K converging to x, i.e. $|x_n-x|_C\to 0$. Furthermore, let (y_n) be a sequence in $C(I,\mathbf{R}^n)$ such that $y_n\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,x_n)$ for $n=1,2,\ldots$ Then there exists a sequence (v_n) in Λ such that $v_n\in\mathcal{F}(G\Box\mathcal{D})(\omega,x_n)$ and $y_n=\mathcal{T}v_n$ for $n=1,2,\ldots$ The set $\{v_n\}_{n\geqslant 1}$ is relatively weakly compact and so there exists a subsequence (v_{n_k}) of (v_n) weakly converging to some $v\in\Lambda$. Then by Theorem 1.2(ii) we have $\mathcal{T}v_{n_k}\to\mathcal{T}v$. Since $y_{n_k}=\mathcal{T}v_{n_k}$ and $y=\mathcal{T}v$ it follows that $|y_{n_k}-y|_C\to 0$. We shall show that $y\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,x)$, i.e. that $v(t)\in G(t,\omega,x,\mathcal{D}x)$ for a.e. $t\in I$.

Indeed, for each measurable set $E\subset I$ we have

$$\begin{aligned}\mathrm{dist}\Big(\int_E v(t)\,dt,\int_E G(t,\omega,x,\mathcal{D}x)\,dt\Big)&\leqslant\Big|\int_E v(t)\,dt-\int_E v_{n_k}(t)\,dt\Big|\\&\quad+\mathrm{dist}\Big(\int_E v_{n_k}(t)\,dt,\int_E G(t,\omega,x_{n_k},\mathcal{D}x_{n_k})\,dt\Big)\\&\quad+\bar{h}\Big(\int_E G(t,\omega,x_{n_k},\mathcal{D}x_{n_k})\,dt,\int_E G(t,\omega,x,\mathcal{D}x)\,dt\Big).\end{aligned}$$

It is clear that $\big|\int_E v(t)\,dt-\int_E v_{n_k}(t)\,dt\big|\to 0$ for each measurable set $E\subset I$. Since $v_{n_k}\in\mathcal{F}(G\Box\mathcal{D})(\omega,x_{n_k})$ by the definition of Aumann's integral we also have

$$\int_E v_{n_k}(t)\,dt\in\int_E G(t,\omega,x_{n_k},\mathcal{D}x_{n_k})\,dt$$

for each mesurable $E\subset I$. Moreover, G is w.-w.u.s.c. in its last two variables. Then for every measurable set $E\subset I$ one has

$$\lim_{n\to\infty}\bar{h}\Big(\int_E G(t,\omega,x_{n_k},\mathcal{D}x_{n_k})\,dt,\int_E G(t,\omega,x,\mathcal{D}x)\,dt\Big)=0.$$

Therefore, finally we get $\int_E v(t)\,dt\in\int_E G(t,\omega,x,\mathcal{D}x)\,dt$ for each measurable set $E\subset I$. Hence $v(t)\in G(t,\omega,x,\mathcal{D}x)$ for a.e. $t\in I$, i.e. $v\in\mathcal{F}(G\Box\mathcal{D})(\omega,x)$. Since $y=\mathcal{T}v$, we have $y\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,x)$. Therefore, by Theorem 1.1, $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega,\cdot)$ is u.s.c.

On the other hand, G is w.-w.l.s.c. in its last two variables. Therefore by Theorem 1.6, $\mathcal{T}\mathcal{F}(G)(\omega,.,.)$ is w.-s.l.s.c. on $K\times\Lambda$. Then the superposition $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})$ is l.s.c.

Now observe that it follows immediately from a lemma of Nowak (see [7], Lemma, p. 490) that $\mathcal{F}(G\Box\mathcal{D})(\cdot, x)$ is measurable for $x \in K$.

Let U be a closed subset of $C(I, \mathbf{R}^n)$. It is clear that

$$\{\omega\colon \mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x)\cap U \neq \emptyset\} = \{\omega\colon \mathcal{F}(G\Box\mathcal{D})(\omega, x)\cap\mathcal{T}^{-1}(U) \neq \emptyset\}$$

for each $x\in K$. The right-hand side is in $\mathcal{F}$ since $\mathcal{F}(G\Box\mathcal{D})(\cdot, x)$ is measurable. Therefore we also have $\{\omega\colon \mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x)\cap U \neq \emptyset\}\in\mathcal{F}$ for every closed set $U\subset C(I, \mathbf{R}^n)$. Thus $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\cdot, x)$ is measurable for fixed $x\in K$. ∎

The following fixed point theorem is a consequence of ([5], Th. 3.7, p. 177).

LEMMA 2.2. *Let* $G\colon I\times\Omega\times K\times\Lambda\to\mathrm{Conv}(\mathbf{R}^n)$ *be w.-w.l.s.c. in its last two variables and such that* $\|G(t, \omega, x, z)\|\leqslant m(t)$ *for a.e.* $t\in[\sigma, \sigma+a]$, $(x, z)\in K\times\Lambda$, *and* $\omega\in\Omega$, *where* m *is L-integrable. Then for fixed* $\omega\in\Omega$ *there exists* $x_\omega\in K$ *such that* $x_\omega\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x_\omega)$.

Proof. By Theorem 3.7 of [5] and Theorem 1.6, $\mathcal{T}\mathcal{F}(G)$ is w.-s.l.s.c. on $K\times\Lambda$. Then the superposition $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})$ is l.s.c. on K for fixed $\omega\in\Omega$. Furthermore, by Theorems 1.3–1.5 it has compact convex values contained in K for fixed $\omega\in\Omega$. Thus by Michael's selection theorem there is a continuous function $f\colon K\to C(I, \mathbf{R}^n)$ such that $f(x_\omega)\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x_\omega)$ for $x\in K$ and $\omega\in\Omega$. Since K is a compact convex subset of $C(I, \mathbf{R}^n)$ such that $f(K)\subset K$, by Schauder–Tikhonov's fixed point theorem there is $x_\omega\in K$ such that $x_\omega = f(x_\omega)$ for each $\omega\in\Omega$. Since $f(x_\omega)\in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x_\omega)$, we have $x_\omega \in\mathcal{T}\mathcal{F}(G\Box\mathcal{D})(\omega, x_\omega)$ for $\omega\in\Omega$. ∎

3. Existence of random solutions. Consider a random functional-differential inclusion of the form

$$\dot{x}(t, \omega)\in G\big(t, \omega, x(\cdot, \omega), \dot{x}(\cdot, \omega)\big) \quad \text{for } t\in[\sigma, \sigma+a] \tag{3.1}$$

with the initial condition

$$x(\sigma, \omega) = 0 \quad \text{for } \omega\in\Omega \tag{3.2}$$

where $G\colon I\times\Omega\times K\times\Lambda\to\mathrm{Conv}(\mathbf{R}^n)$, K and Λ are as defined in Section 1.

A function $x\colon I\times\Omega\to\mathbf{R}^n$ is called a *random solution* of (3.1)–(3.2) if it is measurable in ω, absolutely continuous in t and such that

$$\begin{cases} \dot{x}(t, \omega)\in G\big(t, \omega, x(t, \omega), \dot{x}(t, \omega)\big) & \text{for } t\in I \text{ and a.e. } \omega\in\Omega,\\ x(\sigma, \omega) = 0. \end{cases}$$

In what follows for a given G we shall denote by $S(\omega)$ the set of all fixed points of $\mathcal{T}\mathcal{F}(G\Box\mathcal{D})$ and by $\mathscr{C}(G)$ the set of all random solutions of (3.1)–(3.2).

Now we can prove the following theorem.

THEOREM 3.1. *Let* $G\colon I\times\Omega\times K\times\Lambda\to\mathrm{Conv}(\mathbf{R}^n)$ *be such that*

(i) $G(\cdot, \cdot, x, z)$ *is measurable (with respect to the product σ-field* $\mathcal{L}(I)\times\mathcal{F}$) *for* $(x, z)\in K\times\Lambda$,

(ii) $\|G(t, \omega, x, z)\| \leqslant m(t)$ *for a.e.* $t \in I$ *and for fixed* $\omega \in \Omega$, $(x, z) \in K \times \Lambda$, *where* m *is L-integrable,*

(iii) G *is w.-w.c. in its last two variables.*

Then the set $\mathscr{C}(G)$ *is nonempty.*

Proof. By Lemmas 2.1 and 2.2 the multifunction $\mathscr{T}\mathscr{F}(G \square \mathscr{D})$ is such that the conditions of Theorem 2(ii) of [8] are satisfied. Then there exists a measurable function z: $\Omega \to K$ such that $z(\omega) \in S(\omega)$ for each $\omega \in \Omega$. Therefore for a.e. $\omega \in \Omega$ we have $z(\omega) \in \mathscr{T}\mathscr{F}(G \square \mathscr{D})(\omega, z(\omega))$. Thus there exists $v(\omega) \in \mathscr{F}(G \square \mathscr{D})(\omega, z(\omega))$ such that $z(\omega) = \mathscr{T}v(\omega)$. We define $x(\cdot, \omega) = z(\omega)$ for $\omega \in \Omega$. For every $\omega \in \Omega$, $z(\omega)$ is an absolutely continuous function from I to $\mathbf{R}^n$ with values equal to $x(t, \omega)$.

For fixed $t \in I$, let $w(t)$: $\Omega \to \mathbf{R}^n$ be defined by $w(t)(\omega) = x(t, \omega)$ for $\omega \in \Omega$. Observe that $w(t) = \Pi_t(z)$, where Π_t: $K \to \mathbf{R}^n$ is defined by $\Pi_t(x) = x(t)$ for fixed $t \in I$ and $x \in K$. It is not difficult to see that Π_t is continuous and hence measurable. Thus, $w(t) = \Pi_t(z)$ is measurable as a superposition of measurable functions.

But $z(\omega) = \mathscr{T}v(\omega)$, $x(t, \omega) = z(\omega)(t)$ and $v(\omega)(t) \in G(t, \omega, z(\cdot, \omega), \mathscr{D}z(\cdot, \omega))$ for $t \in I$ and a.e. $\omega \in \Omega$. Therefore $\dot{x}(t, \omega) \in G(t, \omega, z(t, \omega), \mathscr{D}z(t, \omega))$ for a.e. $t \in I$ and a.e. $\omega \in \Omega$.

It is not difficult to verify that we also have $x(\sigma, \omega) = 0$ for a.e. $\omega \in \Omega$. Thus, $\mathscr{C}(G) \neq \emptyset$. ■

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