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# On a new characterization of the exponential functions

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**§ 1.** In the present paper we give a characterization of the exponential functions by some simultaneous functional equations and inequalities involving only one variable. Another such characterization has been given in [1].

We do not assume previous knowledge of the exponential functions. We start at the point where the operation of exponentiation is defined for natural exponents only. However, we assume the knowledge of the basic facts about the limit processes.

The characterization in question is contained in the following

THEOREM. *For every real number $a$ there exists exactly one function $\varphi_a(x)$ (real-valued and of a real variable) fulfilling for every $x \in (-\infty, \infty)$ the following three conditions*:

(1) $$\varphi_a(2x) = [\varphi_a(x)]^2 ,$$

(2) $$\varphi_a(-x) = \frac{1}{\varphi_a(x)} ,$$

(3) $$\varphi_a(x) \geqslant 1 + ax .$$

Proof. At first we shall establish some properties of the sequence

(4) $$\left(1+\frac{x}{n}\right)^n .$$

We shall prove that sequence (4) is increasing for $n > -x$. We have

$$\left(1+\frac{x}{n+1}\right)^{n+1} : \left(1+\frac{x}{n}\right)^n = \frac{n^n(n+x+1)^{n+1}}{(n+1)^{n+1}(n+x)^n}$$
$$= \frac{n+x}{n}\left(\frac{n(n+x+1)}{(n+1)(n+x)}\right)^{n+1}$$
$$= \frac{n+x}{n}\left(1-\frac{x}{(n+1)(n+x)}\right)^{n+1} .$$

For $n > -x$ we have

$$-\frac{x}{(n+1)(n+x)} > -1 ,$$

and hence, by the Bernoulli inequality,

$$\left(1-\frac{x}{(n+1)(n+x)}\right)^{n+1} \geqslant 1-\frac{x}{n+x} = \frac{n}{n+x} .$$

Consequently, for $n > -x$,

$$\left(1+\frac{x}{n+1}\right)^{n+1} : \left(1+\frac{x}{n}\right)^{n} \geqslant 1 ,$$

which proves that sequence (4) is increasing. It is evidently also positive:

$$\left(1+\frac{x}{n}\right)^{n} > 0 \quad \text{for} \quad n > -x .$$

Next we prove that sequence (4) is bounded above. Let $N_x$ be the least integer exceeding $|x|$. Then, on account of what has just been proved, we have for $n > N_x$

$$\left(1-\frac{x}{n}\right)^{n} = \left(1+\frac{(-x)}{n}\right)^{n} \geqslant \left(1-\frac{x}{N_x}\right)^{N_x} > 0 .$$

Hence, for $n > N_x$

(5) $$\left(1+\frac{x}{n}\right)^{n} = \frac{\left(1-\frac{x^2}{n^2}\right)^{n}}{\left(1-\frac{x}{n}\right)^{n}} \leqslant \frac{1}{\left(1-\frac{x}{N_x}\right)^{N_x}} .$$

Thus sequence (4), being increasing and bounded above, converges for all real $x$. We may write

(6) $$\eta(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} .$$

Evidently

(7) $$\eta(0) = 1 .$$

Moreover, since sequence (4) is increasing and positive (at least from a certain term on),

(8) $$\eta(x) > 0 \quad \text{for} \quad x \in (-\infty, \infty) .$$

Let $\delta_n$ be a sequence such that $\lim_{n\to\infty}\delta_n = 0$. We shall prove that

(9) $$\lim_{n\to\infty}\left\{\left(1+\frac{x+\delta_n}{n}\right)^{n} - \left(1+\frac{x}{n}\right)^{n}\right\} = 0 .$$

The sequence $\delta_n$ is bounded; consequently, there is a $b$ such that $|x| < b$ and $|x+\delta_n| < b$, $n = 1, 2, 3, \ldots$ Hence for $n > b$ we have

$$0 < \left(1+\frac{x}{n}\right)^n < \left(1+\frac{b}{n}\right)^n, \qquad 0 < \left(1+\frac{x+\delta_n}{n}\right)^n < \left(1+\frac{b}{n}\right)^n,$$

and

$$\left|\left(1+\frac{x+\delta_n}{n}\right)^n - \left(1+\frac{x}{n}\right)^n\right| = \frac{|\delta_n|}{n}\sum_{i=0}^{n-1}\left(1+\frac{x+\delta_n}{n}\right)^i\left(1+\frac{x}{n}\right)^{n-1-i}$$

$$< \frac{|\delta_n|}{n} n\left(1+\frac{b}{n}\right)^{n-1} = \frac{|\delta_n|\left(1+\frac{b}{n}\right)^n}{1+\frac{b}{n}} \leqslant \frac{|\delta_n|\eta(b)}{\left(1+\frac{b}{n}\right)}.$$

This proves relation (9).

Now we shall prove the fundamental property of $\eta(x)$. We have

$$\eta(x)\eta(y) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n.$$

In virtue of (9) with $\delta_n = xy/n$

$$\lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = \eta(x+y),$$

whence

$$\eta(x+y) = \eta(x)\eta(y). \tag{10}$$

Now, it follows from (10) and (7) that $\eta(x)$ fulfils conditions (1) and (2). The inequality

$$\eta(x) \geqslant 1+x \tag{11}$$

is evident for $x \leqslant -1$ in view of (8). For $x > -1$ we have $n > -x$, $n = 1, 2, 3, \ldots$, and sequence (4) is increasing from the first term on. Hence (11) follows, since $1+x$ is the first term of sequence (4).

Te function $\eta(x)$ fulfils the conditions of the theorem with $a = 1$. One can now easily check that the functions

$$\varphi_a(x) = \eta(ax) = \lim_{n\to\infty}\left(1+\frac{ax}{n}\right)^n \tag{12}$$

fulfil conditions (1), (2), (3). It remains to prove the uniqueness.

Let us fix an $a$ and let a function $\varphi(x)$ fulfil conditions (1), (2), (3). It follows from (1) by induction that

$$\varphi(2^n t) = [\varphi(t)]^{2^n},$$

whence, putting $t = x/2^n$, we have

$$\varphi(x) = \left[\varphi\left(\frac{x}{2^n}\right)\right]^{2^n}. \tag{13}$$

It follows from (13) and (3) for large $n$ (so that $1+ax/2^n > 0$)

$$\varphi(x) \geqslant \left(1+\frac{ax}{2^n}\right)^{2^n}. \tag{14}$$

Similarly, for large $n$,

$$\varphi(-x) \geqslant \left(1-\frac{ax}{2^n}\right)^{2^n} > 0 .$$

Hence by (2)

$$\varphi(x) = \frac{1}{\varphi(-x)} \leqslant \frac{1}{\left(1-\frac{ax}{2^n}\right)^{2^n}} = \frac{\left(1+\frac{ax}{2^n}\right)^{2^n}}{\left(1-\frac{a^2x^2}{4^n}\right)^{2^n}}. \tag{15}$$

The sequence $\left(1+\frac{ax}{2^n}\right)^{2^n}$ tends, in view of (6), to $\eta(ax)$. The sequence $\left(1-\frac{a^2x^2}{4^n}\right)^{2^n}$ is a subsequence of $\left(1-\frac{a^2x^2}{n^2}\right)^n$ and tends, in view of (9), to 1. Thus inequalities (14) and (15) yield

$$\varphi(x) = \eta(ax) ,$$

which proves the uniqueness and completes the proof of the theorem.

**§ 2.** (12) and (10) imply

$$\varphi_a(x+y) = \varphi_a(x)\varphi_a(y) \tag{16}$$

for all real $x, y$. Writing $\varphi_a(1) = p$ we get from (16) by induction

$$\varphi_a(k) = p^k \quad \text{for} \quad k = 1, 2, 3, \ldots \tag{17}$$

Thus the function $\varphi_a(x)$ may be regarded as a generalization of the operation of exponentiation to the case of arbitrary real exponents $x$. We may write

$$\varphi_a(x) = p^x , \quad p = \varphi_a(1) , \tag{18}$$

the symbol $p^x$ being defined just by relation (18). The functions $\varphi_a(x)$ will be called *exponential functions*, and the number $p = \varphi_a(1)$ will be called the *base* of the function $\varphi_a(x)$. In particular, the base of the function $\eta(x) = \varphi_1(x)$ will be denoted by $e$:

$$\eta(x) = e^x .$$

Formula (6) gives immediately

$$(19)\qquad e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The exponential functions form a one-parameter family. They may all be expressed by the function $\eta(x)$ (formula (12)), therefore their properties follow easily from those of $\eta(x)$. In the sequel we shall establish the most important properties of the latter.

**§ 3.** By (11) $\eta(x)>1$ for $x>0$. Hence we get for $h>0$

$$\eta(x+h)=\eta(x)\eta(h)>\eta(x)$$

in virtue of (10) and (8). Thus the *function* $\eta(x)$ *is strictly increasing.*

Next, the evident inequality

$$\left[\eta\left(\frac{x}{2}\right)-\eta\left(\frac{y}{2}\right)\right]^2\geqslant 0$$

yields the relation

$$2\eta\left(\frac{x}{2}\right)\eta\left(\frac{y}{2}\right)\leqslant\left[\eta\left(\frac{x}{2}\right)\right]^2+\left[\eta\left(\frac{y}{2}\right)\right]^2,$$

whence we get by (10) and (1)

$$(20)\qquad \eta\left(\frac{x+y}{2}\right)\leqslant\frac{\eta(x)+\eta(y)}{2}.$$

Thus the *function* $\eta(x)$ *is convex* (1).

For $x\in(-1,1)$ we have $N_x=1$ ($N_x$ being the least integer exceeding $|x|$). Thus we have by (5) for $x\in(-1,1)$ and $n>1$

$$(21)\qquad \left(1+\frac{x}{n}\right)^n\leqslant\frac{1}{1-x}.$$

(21), (6) and (11) yield the estimation valid for $x\in(-1,1)$

$$1+x\leqslant\eta(x)\leqslant\frac{1}{1-x},$$

i.e.

$$(22)\qquad x\leqslant\eta(x)-1\leqslant\frac{x}{1-x}.$$

(1) This is a weaker form of convexity. The stronger form: $\eta[\lambda x+(1-\lambda)y]\leqslant\lambda\eta(x)+(1-\lambda)\eta(y)$, $x,y\in(-\infty,\infty)$, $\lambda\in(0,1)$, can be proved by the use of (20) and of the continuity of $\eta(x)$, or directly from the fact that $\eta''(x)=\eta(x)>0$. The necessary properties of the function $\eta(x)$ (continuity and differentiability) follow.

From (22) we obtain two important limit relations:

$$\lim_{x\to 0}\eta(x) = 1\,, \qquad \lim_{x\to 0}\frac{\eta(x)-1}{x} = 1\,.$$

Hence

$$\lim_{h\to 0}\eta(x+h) = \lim_{h\to 0}\eta(x)\eta(h) = \eta(x)\,,$$

i.e. the *function* $\eta(x)$ *is continuous in* $(-\infty, \infty)$. Further,

$$\lim_{h\to 0}\frac{\eta(x+h)-\eta(x)}{h} = \lim_{h\to 0}\eta(x)\frac{\eta(h)-1}{h} = \eta(x)\,,$$

i.e. the *function* $\eta(x)$ *is differentiable in* $(-\infty, \infty)$ and

$$\eta'(x) = \eta(x)\,.$$

Hence it follows that $\eta(x)$ *is of class* $C^\infty$ *in* $(-\infty, \infty)$ *and all its derivatives are equal* $\eta(x)$. This allows us to write the Taylor-Maclaurin formula

$$\eta(x) = 1+x+\frac{x^2}{2}+\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}\eta(\theta_n x)\,, \qquad 0<\theta_n<1\,,$$

and since the remainder $\frac{x^n}{n!}\eta(\theta_n x)$ tends to zero as $n\to\infty$, we have

(23) $$\eta(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,.$$

Consequently, the *function* $\eta(x)$ *is analytic in* $(-\infty, \infty)$.

The relations

(24) $$\lim_{x\to\infty}\frac{\eta(x)}{x^m} = \infty\,, \qquad \lim_{x\to-\infty}x^m\eta(x) = 0\,,$$

for every fixed positive integer $m$, can easily be deduced from (23) and (2) or obtained by the use of de l'Hospital's rule. They can also be obtained in an elementary way from the facts that $\eta(x)$ is monotonic and fulfils conditions (1) and (2), just as corresponding relations have been obtained in [2] for the logarithmic functions.

**§ 4.** There is still one important question left unsettled. It was pointed out in § 2 that functions (12) may be regarded as an extension of the exponential functions (17), defined for a natural argument only, to arbitrary real values of $x$. Now, the functions

(25) $$\varphi(k) = p^k\,, \qquad k = 1, 2, 3, \ldots,$$

are defined for arbitrary real $p$. The question to be settled is whether all functions (25) admit an extension onto the whole real axis [2]; in other words, whether for every real number $p$ there exists an exponential function (18). It is obvious from the condition $p = \varphi_a(1) = \eta(a)$ that the answer is no: Number $p$ must be positive. But there is no further restriction: for every positive $p$ there is an exponential function $\varphi_a(x)$ whose base is just $p$. In fact, on account of (24) we have

$$(26) \qquad \lim_{x\to\infty}\eta(x) = \infty\,, \qquad \lim_{x\to-\infty}\eta(x) = 0\,.$$

((26) may also be deduced from (2) and (3).) Since the function $\eta(x)$ is continuous and strictly increasing, it has a unique inverse $\eta^{-1}(x)$, which is also continuous and strictly increasing and maps $(0, \infty)$ onto $(-\infty, \infty)$. Consequently, for every $p > 0$ the equation $p = \eta(a)$ has the unique solution

$$(27) \qquad a = \eta^{-1}(p)\,.$$

The function $\eta^{-1}(x)$ is called the *natural logarithm of $x$* and is denoted by $\ln x$. Thus (27) becomes

$$(28) \qquad a = \ln p\,.$$

Relation (27) or (28) establishes a connection between the base of the exponential function $\varphi_a(x)$ and the number $a$ occurring in (3). It is important, since, when we are obtaining properties of the function $\varphi_a(x)$, the parameter $a$ will play an important role. (E.g. the formula of differentiation $\varphi_a'(x) = a\varphi_a(x)$ contains $a$.) It may be more convenient to have relations involving the base of the exponential function. Thus the formula of differentiation may be written in view of (18) and (28) as

$$(p^x)' = p^x \ln p\,.$$

**§ 5.** We have defined the operation of exponentiation $p^x$ for every $p > 0$ and every real $x$:

$$(29) \qquad p^x = \eta\big(\eta^{-1}(p)x\big)\,.$$

Now we shall prove some further properties of this operation.

First we note that

$$(30) \qquad p^x > q^x \quad \text{whenever} \quad 0 < q < p\,,\ x > 0\,,$$

$$(31) \qquad p^x < q^x \quad \text{whenever} \quad 0 < q < p\,,\ x < 0\,.$$

Relations (30) and (31) result immediately from (29) in view of the fact that the functions $\eta(x)$ and $\eta^{-1}(x)$ are both strictly increasing.

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[2] Of course, we ask for extensions that are exponential functions in the sense of § 2.

Next we show that for every positive $p, q$ and real $x$ we have

$$(pq)^x = p^x q^x . \tag{32}$$

Let us write

$$\eta^{-1}(p) = a , \quad \eta^{-1}(q) = b , \quad a+b = c .$$

Then, according to (10),

$$\eta(c) = \eta(a+b) = \eta(a)\eta(b) = pq ,$$

i.e.

$$c = \eta^{-1}(pq) .$$

Hence, in virtue of (29),

$$(pq)^x = \eta\big(\eta^{-1}(pq)x\big) = \eta(cx) = \eta(ax+bx) = \eta(ax)\eta(bx)$$
$$= \eta\big(\eta^{-1}(p)x\big)\eta\big(\eta^{-1}(q)x\big) = p^x q^x ,$$

i.e. (32).

Lastly we prove that for every positive $p$ and every real $x, y$ we have

$$(p^x)^y = p^{xy} . \tag{33}$$

Put $q = p^x$. Then

$$(p^x)^y = q^y = \eta\big(\eta^{-1}(q)y\big) .$$

But $q = \eta\big(\eta^{-1}(p)x\big)$. Hence $\eta^{-1}(q) = \eta^{-1}(p)x$ and

$$(p^x)^y = \eta\big(\eta^{-1}(p)xy\big) = p^{xy} ,$$

i.e. (33) holds.

**§ 6.** The idea of such a characterization of exponential functions is inherent in the following characterization of the number $e$, due to W. Sierpiński ([3], § 47). Sierpiński proves that *there exists exactly one real number* $a$ *such that*

$$a^x \geqslant 1+x$$

*holds for all real* $x$. This number is denoted by $e$ and is given by formula (19). This definition of $e$ has led me to the theorem of the present paper. Sierpiński's definition of $e$ is essentially equivalent to that given in § 2 of the present paper. But the definition of exponentiation in Sierpiński's book is different.

## References


[1] M. Kuczma, *A characterization of the exponential and logarithmic functions by functional equations*, Fund. Math. 52 (1963), pp. 283-288.

[2] — *On the functional characterization of the logarithm*, Funkc. Ekvacioj 10 (1967), pp. 67-73.

[3] W. Sierpiński, *Działania nieskończone*, Warszawa 1948.


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