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Angles and quasiconformal mappings on Riemannian manifolds

by MARIA WOJCIECHOWSKA (Łódź)

Abstract. In this paper the authoress, following F. W. Gehring and S. B. Agard, introduces a definition of the measure of a topological angle on a Riemannian manifold and gives a characterization of quasiconformal mappings in terms of angles for Riemannian manifolds.

Introduction. Quasiconformal mappings can be studied as, in some sense, angle-preserving. Such mappings are almost everywhere differentiable, but not necessarily everywhere. Therefore an exceptional point p can lie on a differentiable curve which, under a quasiconformal mapping, is mapped onto a curve having no tangent line at the image of p. Then the usual measure between two curves which are images of differentiable curves under quasiconformal mapping, is meaningless. The problem of defining the measure of an angle, possibly without tangent lines at the vertex, has been studied in the plane by F. W. Gehring, S. B. Agard and O. Taari. The definition of measure for a topological angle in the n-dimensional Euclidean space given by Agard, [1], and on a Riemannian manifold, introduced in the present paper, are formally adapted from the plane.

There exist a large number of equivalent definitions for quasiconformal mappings in the n-dimensional Euclidean space. S. B. Agard gave in [1] a characterization of these mappings in terms of angles. In this paper we give such a characterization for quasiconformal mappings on Riemannian manifolds.

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1. Preliminaries. Throughout the whole paper the manifolds are supposed to be C^{∞}-differentiable, paracompact and connected. The tangent bundle of a differentiable manifold M is denoted by TM and the tangent space at $p \in M$ by $T_p M$. The derivative of a differentiable mapping $f\colon M \to N$ is a fibre mapping $Df\colon TM \to TN$. The set of real numbers is denoted by R.

Let M be a C^{m}-manifold. An open set $U \subset M$ is called a *C^{m}-coordinate* *neighbourhood of* M, $m \leqslant n$, if there is a C^m-diffeomorphism μ of U onto an open set in R^p; μ is then called a *coordinate mapping.*

By a curve on a manifold M we mean a continuous mapping γ from a closed interval $[a, b]$, $a \leqslant b$, to M. A curve γ is called *differentiable* if γ is continuously differentiable. A Borel measure in a manifold M is any measure defined in the family of Borel sets of M. Such a measure will be denoted by τ_M.

By a Riemannian manifold we mean a manifold which has a fixed Riemannian metric. The distance between points p and q of a Riemannian manifold M is denoted by $d_M(p, q)$. The symbol $l(\gamma)$ denotes the length of a piecewise differentiable curve γ on a Riemannian manifold M.

Let γ be rectifiable and let $s(t)$ denote the length of the restriction of γ to $[a, t], a \leqslant t \leqslant b$. For each rectifiable curve γ there is a unique curve γ_1: $[0, l(\gamma)] \to M$ with the property $\gamma = \gamma_1 \circ s$. The curve γ_1 is called the *parametrization of* γ by means of its arc length.

Let γ be a rectifiable curve and ϱ a Borel function on a Riemannian manifold M. Let γ_1 be the parametrization of γ by means of its arc length. The integral of ϱ along γ is defined by

$$\int_\gamma \varrho\, ds = \int_0^{l(\gamma)} \varrho \circ \gamma_1\, ds,$$

provided the latter integral exists. Otherwise the integral of ϱ along γ is undefined.

Suppose now that Γ is a family of curves in a Riemannian manifold M. We denote by $\operatorname{adm} \Gamma$ the class of all non-negative Borel functions ϱ in M which satisfy

$$\int_\gamma \varrho\, ds \geqslant 1$$

for all rectifiable curves $\gamma \in \Gamma$.

For each positive real number p we define the p-module of Γ as follows:

$$\operatorname{mod}_p \Gamma = \inf \int_M \varrho^p\, d\tau_M,$$

where the infimum is taken over all $\varrho \in \operatorname{adm} \Gamma$. If Γ is empty, we put $\operatorname{mod}_p \Gamma = \infty$.

Let M and N be n-dimensional Riemannian manifolds. A homeomorphism f: $M \to N$ is called a *quasiconformal mapping* if there is a constant Q such that

$$\text{(1)} \qquad Q^{-1} \operatorname{mod}_n \Gamma \leqslant \operatorname{mod}_n f[\Gamma] \leqslant Q \operatorname{mod}_n \Gamma$$

for each family Γ of curves in M and its image $f[\Gamma]$. If (1) is satisfied, f is said to be *Q-quasiconformal.*

By a simple curve on a manifold M we understand the homeomorphic image of an interval $[a, b)$, $[a, b]$ or $(a, b]$, where $a < b$. The jacobian of f will be denoted by J_f.

Let M be a manifold. A set $E \subset M$ is a *null set* if $\mu(E \cap U)$ has Lebesgue measure zero for each coordinate neighbourhood $U \subset E$ and each coordinate mapping μ: $U \to R^n$ (of class C^∞). We say that a condition holds almost everywhere in M, if it holds everywhere except for a null set.

2. Topological angles on Riemannian manifolds and their measure. Let M be a Riemannian manifold. By a topological angle at a point $p_0 \in M$ we mean a pair of simple curves (γ_1, γ_2), with a common initial point p_0, termed the vertex.

The measure A_M of a topological angle $\alpha = (\gamma_1, \gamma_2)$ on M at p_0 is defined by:

$$A_M(\alpha) = \liminf_{\substack{p_1 \in \gamma_1 \\ p_2 \in \gamma_2 \\ p_1, p_2 \to p_0}} 2 \arcsin \frac{d_M(p_1, p_2)}{d_M(p_1, p_0) + d_M(p_2, p_0)}. \tag{2}$$

Let μ: $U \to M$ be a coordinate mapping such that $p_0 \in U$. If simple curves $\mu(\gamma_1 \cap U)$ and $\mu(\gamma_2 \cap U)$ have unique tangent lines at $\mu(p_0)$, then the angle (γ_1, γ_2) is said to be *ordinary*.

Definition (2) coincides with the usual one in the case when (γ_1, γ_2) at p_0 is an ordinary angle (cf. Theorem 2.2 in [1]).

Let N be a Riemannian manifold and f: $M \to N$ be a continuous mapping. By $f(\alpha)$ we will denote the angle $(f(\gamma_1), f(\gamma_2))$ on N at $f(p_0)$. We say that α is *non-zero* if $A_M(\alpha) > 0$. If $M \subset R^n$, we write $A_M = A$.

LEMMA 1. *Suppose that:*

(i) *M and N are n-dimensional Riemannian manifolds,*

(ii) *f: $M \to N$ is a C^1-diffeomorphism,*

(iii) *p_0 is a point of M,*

(iv) *$(Df)(p_0)$: $T_{p_0}M \to T_{f(p_0)}N$ is an isometry.*

Then, for each $\varepsilon > 0$, there exists an open set V containing p_0, such that for any $p_1, p_2 \in V$

$$d_N(f(p_1), f(p_2)) \leqslant (1+\varepsilon) d_M(p_1, p_2). \tag{3}$$

Proof. If $(Df)(p_0)$ is an isometry, we have $\|Df\|(p_0) = 1$. Then for each $\varepsilon > 0$, there exists an open and convex set V containing p_0 such that for each $p \in V$

$$\|Df\|(p) \leqslant 1+\varepsilon.$$

Let p_1 and p_2 be any points of V and let γ be a geodesic joining points p_1 and p_2 such that

$$l(\gamma) = d_M(p_1, p_2).$$

Then (cf. Theorem 4.11 in [5])

$$d_N(f(p_1), f(p_2)) \leqslant l(f(\gamma)) \leqslant \int_0^{l(\gamma)} \|Df\| \, ds \leqslant (1+\varepsilon) d_M(p_1, p_2),$$

whence we obtain (3).

LEMMA 2. *Let N and N' be Riemannian manifolds of dimension n. Suppose $v\colon N \to N'$ is a C^1-diffeomorphism such that $(Dv)(q_0)\colon T_{q_0}N \to T_{v(q_0)}N'$ is an isometry, where $q_0 \in N$. Then for any $\varepsilon > 0$ there exists an open set $V \subset N$ containing q_0 such that for any $q \in V$ and any ordinary angle β in q*

$$(1+\varepsilon)^{-2} A(v(\beta)) \leqslant A_N(\beta) \leqslant A(v(\beta))(1+\varepsilon)^2. \tag{4}$$

In particular, if $q = q_0$, we get

$$A(v(\beta)) = A_N(\beta). \tag{5}$$

Proof. From Lemma 1 it follows that, for any $\varepsilon > 0$, there exist open sets $V_1 \subset N$ and $V_2 \subset N'$ such that $q_0 \in V_1$, $v(q_0) \in V_2$, $v(V_1) = V_2$, and for any $p, p_1, p_2 \in V_2$ we have

$$\begin{aligned} &d_N(v^{-1}(p_1), v^{-1}(p_2)) \leqslant (1+\varepsilon) d_{N'}(p_1, p_2), \\ &d_N(v^{-1}(p_i), v^{-1}(p)) \leqslant (1+\varepsilon) d_{N'}(p_i, p), \quad i = 1, 2, \end{aligned} \tag{6}$$

while for every $q, q_1, q_2 \in V_1$

$$\begin{aligned} &d_{N'}(v(q_1), v(q_2)) \leqslant (1+\varepsilon) d_N(q_1, q_2), \\ &d_{N'}(v(q_i), v(q)) \leqslant (1+\varepsilon) d_N(q_i, q), \quad i = 1, 2. \end{aligned} \tag{7}$$

Inequalities (6) hold for every $p, p_1, p_2 \in V_2$, in particular for $p_1 = v(q_1)$, $p_2 = v(q_2)$ and $p = v(q)$. Then we get

$$\begin{aligned} &d_N(q_1, q_2) \leqslant (1+\varepsilon) d_{N'}(v(q_1), v(q_2)), \\ &d_N(q_i, q) \leqslant (1+\varepsilon) d_{N'}(v(q_i), v(q)), \quad i = 1, 2. \end{aligned} \tag{8}$$

Hence by (7) we have

$$\frac{d_N(q_1, q_2)}{d_N(q_1, q) + d_N(q_2, q)} \leqslant (1+\varepsilon)^2 \frac{d_{N'}(v(q_1), v(q_2))}{d_{N'}(v(q_1), v(q)) + d_{N'}(v(q_2), v(q))} \tag{9}$$

and

$$\frac{d_{N'}(v(q_1), v(q_2))}{d_{N'}(v(q_1), v(q)) + d_{N'}(v(q_2), v(q))} \leqslant (1+\varepsilon)^2 \frac{d_N(q_1, q_2)}{d_N(q_1, q) + d_N(q_2, q)}. \tag{10}$$

Inequalities (9) and (10) imply (4). Let now $q = q_0$; since ε can be chosen arbitrarily near zero, (5) follows.

THEOREM 1. *Definition* (2) *coincides with the usual one in the case when (γ_1, γ_2) is an ordinary angle at p_0.*

Proof. This follows from Lemma 2 and Theorem 2.2 in [1].

3. Quasiconformal mappings. Now we are going to give a characterization of Q-quasiconformal mapping in terms of angles.

THEOREM 2. *Let M and N be Riemannian manifolds of dimension n. A homeomorphism $f\colon M \to N$ is Q-quasiconformal if and only if*

(a) *for every vertex p_0 in M and every non-zero ordinary angle α at p_0*

$$A_N(f(\alpha)) > 0,$$

(b) *for almost every vertex p_0 in M and every ordinary angle α at p_0*

$$A_N(f(\alpha)) \geqslant Q^{-2/n} A_M(\alpha).$$

Proof. Suppose that M and N are diffeomorphic to open subsets M' and N' of R^n by means of C^1-diffeomorphisms $\mu\colon M \to M'$ and $\nu\colon N \to N'$. Let p_0 be a point of M and let $q_0 = f(p_0)$. We may choose μ and ν so that $(D\mu)(p_0)$ and $(D\nu)(q_0)$ are isometries. Suppose ε is any positive number and M and N are so small that μ and ν are $(1+\varepsilon)$-quasiconformal and the inequalities of Lemma 2 hold on M and N.

Assume now that a homeomorphism $f\colon M \to N$ is a Q-quasiconformal mapping and α is a non-zero ordinary angle at p_0. Since f is Q-quasiconformal, then $\nu \circ f \circ \mu^{-1}$ is $(1+\varepsilon)\,Q$-quasiconformal. Hence, by Theorem 5.2a in [1], we get

$$A_N(f(\alpha)) > 0.$$

Thus we have obtained inequality (a).

In order to prove (b), we assume that f is differentiable at p_0 with a non-zero jacobian. Consequently, $\mu(p_0)$ is a point of differentiability of $\nu \circ f \circ \mu^{-1}$ with a non-zero jacobian and, by Theorem 3.3 in [1],

$$A(\mu(\alpha))\,Q^{-1/n}(1+\varepsilon)^{-4/n} \leqslant A((\nu \circ f)(\alpha)).$$

Applying inequality (5), we get

$$A_M(\alpha)\,Q^{-1/n}(1+\varepsilon)^{-4/n} \leqslant A_N(f(\alpha)).$$

Since ε can be chosen arbitrarily near zero, (b) follows.

Suppose now that a homeomorphism $f\colon M \to N$ satisfies conditions (a) and (b) of Theorem 2. Let p' be a point of M', and α' any non-zero ordinary angle on M' at p'. Because μ is a diffeomorphism, the angle $\alpha = \mu^{-1}(\alpha')$ is a non-zero ordinary angle on M at p, where $p = \mu^{-1}(p')$. Then (a) implies

$$A_N(f(\alpha)) > 0.$$

Since ν is a diffeomorphism, we obtain

$$A((\nu \circ f \circ \mu^{-1})(\alpha')) > 0.$$

We shall say that $p \in T$ if inequality (b) holds at p. Let $p' \in \mu(T)$ and let α' be an ordinary angle at p'. Lemma 2 and (b) yield

$$\begin{aligned} A((\nu \circ f \circ \mu^{-1})(\alpha')) &\geqslant (1+\varepsilon)^{-2} A_N((f \circ \mu^{-1})(\alpha')) \\ &\geqslant Q^{-2/n}(1+\varepsilon)^{-2} A_M(\mu^{-1}(\alpha')) \geqslant Q^{-2/n}(1+\varepsilon)^{-4} A(\alpha'). \end{aligned}$$

Hence, owing to Theorem 5.2. in [1], $\nu \circ f \circ \mu^{-1}$ is $Q(1+\varepsilon)^{2n}$-quasiconformal. Since μ and ν are $(1+\varepsilon)$-quasiconformal, the homeomorphism f is $Q(1+\varepsilon)^{2n+2}$-quasiconformal. In order to prove our theorem in the general case, we may assume that M and N are covered by a countable family of coordinate neighbourhoods such that on each of them f is $Q(1+\varepsilon)^{2n+2}$-quasiconformal.

Let now Γ be a family of curves on M such that $\operatorname{mod}_n \Gamma \neq 0$ and let $\varrho \in \operatorname{adm} f[\Gamma]$. Then, by inequality 7.2 in [5], we have

$$\int_N \varrho^n d\tau_N \geqslant Q^{-1}(1+\varepsilon)^{-2n-2} \int_M (\varrho \circ f)^n \|Df\|^n d\tau_M .$$

As the function $(\varrho \circ f)\|Df\|$ is an element of $\operatorname{adm} \Gamma$ (cf. Lemma 7 in [7]), we get

$$\operatorname{mod}_n f[\Gamma] \geqslant Q(1+\varepsilon)^{2n+2} \operatorname{mod}_n \Gamma . \tag{11}$$

Since this remains valid for $\operatorname{mod}_n \Gamma = 0$, (11) holds for every family of curves in M. In view of the fact that f^{-1} is also $Q(1+\varepsilon)^{2n+2}$-locally-quasiconformal, we obtain

$$\operatorname{mod}_n f[\Gamma] \geqslant Q(1+\varepsilon)^{2n+2} \operatorname{mod}_n \Gamma . \tag{12}$$

Letting in (11) and (12) $\varepsilon \to 0$, we conclude that f is Q-quasiconformal.

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UNIVERSITY OF ŁÓDŹ, DEPARTMENT OF GEOMETRY

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