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Covariant differentiation of geometric objects

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CONTENTS

I. INTRODUCTION

The beginning of the theory of covariant differentiation which was preceded by Riemann's and Christoffel's papers ((1868) [35] and (1869) [7]) may be taken back to the turn of the nineteenth century, when Ricci and Levi Cività began a systematical investigation of parallel displacement of vector fields (1901) [24]. Then formulae for covariant derivatives of covariant vectors and tensors of arbitrary valence appeared (J. A. Schouten (1922) [36]). These methods were of experimental nature. The idea of general definition of the covariant derivative of any geometric object appeared simultaneously with the notion of such an object in the monograph of Schouten and Struik (1935) [38]. The authors noticed that the transformation rules of Pfaff derivatives of the geometric objects in question were complicated in comparison with those rules for the original objects. But the transformation rules of the corresponding covariant derivatives were linear. The authors concluded that the operation of the covariant differentiation did not change the type of the object. This fact has its contemporary geometrical interpretation, when we postulate the invariancy of the rigging of a fibre. Nevertheless we now observe that the transformation rules of covariant derivatives of non-linear objects have a complicated form in general.

Schouten and Struik proposed to define the covariant derivative of a geometric object by six axioms. We repeat the first of them:

A. *The covariant derivative of a geometric object is a geometric object of the same type as the original object.*

However, a profound analysis contained in S. Gołąb [12]-[15] and in S. Gołąb and J. Aczél [1] showed that the axiom A together with the following:

B. *The covariant derivative of a geometric object is a concomitant of that object which depends on the components of the original object, on their first Pfaff derivatives and on any auxiliary objects,*

which yields a more accurate description of the admissible type, define the covariant derivatives in the cases investigated. But the above definition does not guarrantee uniqueness, because the concomitants of the covariant derivative also satisfy these axioms.

An analysis of the papers of E. Cartan and a rapid development of the theory of the fibre bundles [39] suggested a different treatment of these questions. There appeared a notion of connection in a principal fibre bundles (C. Ehresmann (1950) [10]), and in the associated bundles (K. Nomizu (1956) [34]). After the papers of Haantjes and Laman (1953) [16], the fields of geometric objects may be investigated as cross-sections in convenable fibre bundles. This point of view permits the use of an intrinsic method. Such a formulation of the notion of the covariant derivative was given by R. Crittenden (1962) [8]. A connection bound with an invariant rigging of surfaces was investigated by G. F. Laptěv (1959) [28].

The purely differential objects form a separate and perhaps the most important topic in the theory of geometric objects and in differential geometry. Monograph [1] is devoted principally to those objects. The structure of the corresponding groups, which are named differential groups, was investigated by V. V. Vagněr and Ě. B. Dynkin (cf. [45] [46] [9]). Ehresmann's theory of jets (cf. [11] [25]) appeared as a useful algorithm for investigating groups and pseudogroups which are bound with the transformations of local coordinates of the base. A. Nijenhuis, who had defined natural bundles, gave a possibility of intrinsic treatment of holonomic cases (1960) [33]. The investigation of connections in the bundles with higher order differential groups was initiated by V. Hlavatý (1949) [17]. The computation of the corresponding connection form in the case of an imbedded manifold is due to P. I. Shvěykin, (1955) [42], (1958) [43].

The Lie differentiation of geometric objects, which was initiated by W. Ślebodziński (1931) [40] (cf. also [49]), is strongly bound with the covariant differentiation (cf. [8]). B. L. Laptěv investigated the Lie derivatives in bundles with a field of a resistant element (1956) [27]. An interesting connection between the equations of invariancy of an object and its Lie derivative was found by L. Ě. Ěvtushik, (1960) [19].

This brief historical sketch has indicated those turning points in the development of the theory which are known to the author. The following part of the present paper is devoted to the covariant differentiation of the geometric objects based on the theory of connections in the fibre bundle. We start with an intrinsic expression, which consequently implies the expression by coordinates. In the third chapter we treat the Lie derivatives and their relations to the covariant ones. The fourth chapter is devoted to the connections in bundles with the differential structure group. We touch upon the prolongation theory there. In the fifth chapter we present a solution of a problem posed by Schouten and Gołąb, namely how to define the covariant derivative of a geometric object with the aid of a functional equation. We also add a brief sketch of Laptěv's theory of prolongations.

The author wishes to express his gratitude to Professor S. Gołąb and to Professor G. F. Laptěv for their valuable advice.

II. COVARIANT DIFFERENTIATION IN FIBRE BUNDLES

In the present chapter we assume that all manifolds, mappings, etc. are at least twice continuously differentiable. Thus if we write "differentiable", then we mean the differentiability of a definite class, no less than C^2.

We shall denote the n-dimensinal Euclidean space by R^n.

Let M be a manifold and p one of its points. Let $\mathscr{F}_p$ be the algebra of differentiable functions defined in an open neighbourhood of p. Let p_t ($t \geqslant 0$, $p_0 = p$) be a differentiable arc in M. The vector tangent to the arc p_t at p_0 is defined as a mapping $x\colon \mathscr{F}_p \to R^1$ as follows: if $\sigma \in \mathscr{F}_p$ then we put

$$\sigma_* x = \big(d\sigma(p_t)/dt\big)\big|_{t=0}. \tag{1}$$

In other words, $\sigma_* x$ is a derivative of σ in the direction of the arc p_t at p. Let $\xi^1, \ldots, \xi^n$ be the local coordinates in a neighbourhood $\mathcal{O}$ of p. Then n coordinate lines pass through p: $p_{\xi^1}, \ldots, p_{\xi^n}$. Each of them defines a vector which will be denoted by $(\partial/\partial\xi^\alpha)_p$ ($\alpha = 1, \ldots, n$). We shall show that the set of all vectors at p constitutes a vector space $T_p(M)$, the tuple $(\partial/\partial\xi^1)_p, \ldots, (\partial/\partial\xi^n)_p$ being its basis. Given any curve p_t ($p_0 = p$), we express it in terms of local coordinates, viz. $p_t = \{\xi^1(t), \ldots, \xi^n(t)\}$. Thus we have

$$\big(d\sigma(p_t)/dt\big)\big|_{t=0} = (\partial\sigma/\partial\xi^\alpha)_p\big(d\xi^\alpha(t)/dt\big)\big|_{t=0},$$

which proves that every vector at p is a linear combination of $(\partial/\partial\xi^1)_p, \ldots$ $\ldots, (\partial/\partial\xi^n)_p$. Conversely, if we are given a linear combination $v^\alpha(\partial/\partial\xi^\alpha)_p$, then we define a curve p_t by its coordinates

$$\xi^\alpha(t) = (\xi^\alpha)_p + v^\alpha t.$$

The vector tangent to this curve at $t = 0$ is equal to $v^\alpha(\partial/\partial\xi^\alpha)_p$. The independence of the basic vectors $\partial/\partial\xi^\alpha$ may be proved as follows: Suppose that $a^\alpha(\partial/\partial\xi^\alpha)_p = 0$. Then we have

$$0 = (\xi^\mu)_*(a^\alpha\partial/\partial\xi^\alpha) = a^\alpha\delta^\mu_\alpha = a^\mu$$

and consequently all a^α vanish.

The set of tangent vectors at p is called the *tangent space at p*, and we denote it by $T_p(M)$.

Let f be a mapping of the manifold M into a manifold N (which may be M itself). Then f induces the following tangential mapping of $T_p(M)$ into $T_{f(p)}(N)$. If an arc $p_t \subset M$ defines a vector $\boldsymbol{x}$, then we put $\boldsymbol{y} = f_* \boldsymbol{x}$, where $\boldsymbol{y}$ is a vector tangent to the arc $q_t = f(p_t)$. Thus $\boldsymbol{y}$ is the induced tangential map of $\boldsymbol{x}$ by f and f_* is called the *induced mapping*. We shall sometimes write simply

$$(2) \qquad \boldsymbol{x} = (dp_t/dt)|_{t=0} \quad \text{and} \quad \boldsymbol{y} = \big(df(p_t)/dt\big)|_{t=0}.$$

We see that this notation is consistent with (1).

We consider also the linear spaces of linear forms on $T_p(M)$ at each p. If we have such a form ϑ, then its value on the vector $\boldsymbol{v}$ will be denoted by

$$\vartheta \boldsymbol{v} \quad \text{or by} \quad \langle \vartheta, \boldsymbol{v} \rangle.$$

The mapping $f\colon M \to N$ considered above induces also a mapping of linear forms on $T_p(M)$ into those on $T_{f(p)}(N)$. This induced mapping of linear forms will be denoted by f^*. It is defined by the formula

$$\langle f^* \omega, \boldsymbol{x} \rangle = \langle \omega, f_* \boldsymbol{x} \rangle.$$

The differentials of the local coordinates $(d\xi^1)_p, \ldots, (d\xi^n)_p$ may be assumed as a base of the space of linear forms at p. Then the following equalities hold:

$$\left\langle d\xi^\alpha, \frac{\partial}{\partial \xi^\beta} \right\rangle = \delta^\alpha_\beta.$$

We take into considerations a principal fibre bundle $P = P(B, G, \pi)$, where the differentiable manifold B is its base, G is a structure group and π is the canonical projection of P onto B. We denote by ϱ the canonical right action of G on P. We denote by $\varrho_g p$ the result of this action by an element $g \in G$ on a point $p \in P$. ϱ induces a tangential mapping of $T_e(G)$ (e denotes the unity in G) into $T_p(P)$ at each $p \in P$. This tangential mapping is invariant under the commutator of the corresponding Lie algebra and it may be extended to the so-called *natural homomorphism* of the Lie algebra $\mathscr{G}$ of G into the Lie algebra based on $T_p(P)$ (cf. Pontriagin, *Topological Groups*, Chapter X). That homomorphism, when restricted to a fibre, is an isomorphism. Sometimes we shall write simply $p \cdot g$ instead of $\varrho_g p$.

A *connection in P* is a distribution of $T_p(P)$ at every p into a direct sum of subspaces $T^|_p(P)$ and $T^-_p(P)$ so that $T^|_p(P)$ is tangent to the fibre through p. This distribution is invariant under ϱ and it depends differentiably on the point p.

A given connection implies the existence of a field of linear forms $\omega_p\colon T_p(P) \to T_e(G)$ which maps $T^-_p(P)$ onto a zero vector in $T_e(G)$. Moreover, ω establishes the natural isomorphism between the corresponding

Lie algebras. The invariancy of the connection implies the following property of the form ω: if $\boldsymbol{x} \in T_p(P)$, $g \in G$ then we have

$$\langle \omega, \varrho_g \boldsymbol{x} \rangle = \operatorname{adj}(g^{-1}) \langle \omega, \boldsymbol{x} \rangle$$

where $\operatorname{adj}(g) = \operatorname{Adj}_*(g)$ and $\operatorname{Adj}(g)h = ghg^{-1}$ for any $g, h \in G$.

Let p_t and q_t ($t \in [a, b] \subset R^1$) be two differentiable curves such that $\pi(p_t) = \pi(q_t)$ and $t_1 \neq t_2$ implies $p_{t_1} \neq p_{t_2}$ and $q_{t_1} \neq q_{t_2}$. Thus there exists a curve $g_t \subset G$ such that we have $p_t = \varrho_{g_t} q_t$.

PROPOSITION 1. *We consider the vector fields tangent to the above defined curves, $\boldsymbol{x}_t = dp_t/dt$ and $\boldsymbol{y}_t = dq_t/dt$. Then the following equality holds*

$$\langle \omega, \boldsymbol{y}_t \rangle = \operatorname{adj}(g_t^{-1}) \langle \omega, \boldsymbol{x}_t \rangle + \langle g_t^{-1} dg_t, \boldsymbol{x}_t \rangle .$$

A curve in P is said to be *horizontal* if and only if every vector tangent to it is horizontal, i.e. if it belongs to the *horizontal subspace* $T_p^-(P)$.

PROPOSITION 2. *Every differentiable curve $p_t \subset P$ may be lifted into a horizontal one, i.e. there exists a horizontal curve of the form $\bar{p}_t = p_t g_t^{-1}$, the curve $g_t \subset G$ being determined by the equation*

$$g_t^{-1} dg_t = \omega_{p_t} .$$

For more adequate definitions and the proofs the reader may consult the book by K. Nomizu.

Let F be a differentiable manifold. We assume that G acts on F associatively from the left. This means that there exists a mapping $\Lambda\colon G \times F \to F$ such that if $g, h \in G$ and $s \in F$, then we have

$$\Lambda_g(\Lambda_h s) = \Lambda_{gh} s .$$

Sometimes we shall write simply gs instead of $\Lambda_g s$.

We consider a fibre bundle $W = W(B, G, F, \Pi)$ with the standard fibre F and the canonical projection Π, B and G being as above. We say that W is *associated with P* if and only if there exists a mapping A of $P \times F$ onto W such that we have for every $g \in G$, $s \in F$

$$A(p \cdot g^{-1}, gs) = A(p, s) \tag{3}$$

and if $p \neq q$, then

$$A(p, s) \neq A(q, s) . \tag{4}$$

A pair (p, s) will be called a *representation of the point $A(p, s)$*.

EXAMPLES. If B is an n-dimensional manifold and L_n is a linear group of automorphisms of R^n, then we consider $P(B, L_n, \pi)$ as a *bundle of linear frames over B*. The points of P are the n-tuples of vectors $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_n$. Every vector $\boldsymbol{i}_a$ is a certain linear combination of the basic vectors $\partial/\partial\xi^1, \ldots, \partial/\partial\xi^n$. A field of frames $(\boldsymbol{i}_1)_b, \ldots, (\boldsymbol{i}_n)_b$ in a certain domain

$\mathcal{O} \subset B$ $(b \in \mathcal{O})$ is called *holonomic* if and only if there exists such a parametrization of $\mathcal{O}$ that the $(\boldsymbol{i}_a)_b$ are respectively tangent to the parametrical curves passing through the point b. In other cases we call that field of frames $(\boldsymbol{i}_a)_b$ a *non-holonomic* one. We construct an associated bundle if we assume R^n as its standard fibre. If a point $\in R^n$ has coordinates $v^1, \ldots, v^n$, then we put

$$A(\boldsymbol{i}_1, \ldots, \boldsymbol{i}_n;\ v^1, \ldots, v^n) = v^a \boldsymbol{i}_a.$$

In such a way A maps $P \times R$ onto a *vector bundle* over B.

Another example will be obtained if we replace the group L_n of the previous example by its subgroup $P_{n-1} = L_n/C_n$, C_n being the group of homoteties of R^n.

We now take the projective space $\mathscr{P}^{n-1}$ as the standard fibre of the associated bundle. The canonical mapping of L_n onto P_{n-1} induces a mapping of the bundle $P(B, L_n, \pi)$ onto its proper subbundle $P(B, P_{n-1}, \pi)$. We now introduce the following relation $\sim$ among the points of the Cartesian product $P(B, P_{n-1}, \pi) \times \mathscr{P}^{n-1}$:

$$(a, x) \sim (b, y) \Leftrightarrow \exists_{g \in G} (b = a \cdot g^{-1}, y = g \cdot x).$$

We define A as the canonical mapping of $P(B, P_{n-1}, \pi) \times \mathscr{P}^{n-1}$ onto $P(B, P_{n-1}, \pi) \times \mathscr{P}^{n-1}/\sim$. Thus the map by A is a *bundle of Penzov's objects* (cf. [1]).

A field of objects is a cross-section in a convenable bundle.

Let us turn to the general case. We shall consider a differentiable cross-section $X\colon B \to W(B, G, F, \Pi)$. It will not cause a misunderstanding if we use the same symbol for a mapping of B onto a cross-section and for the cross-section viewed as a set of points.

PROPOSITION 3. (Cf. [8].) *If X is a cross-section in an associated bundle $W(B, G, F, \Pi)$ then there exists a mapping $\mathscr{X}\colon P(B, G, \pi) \to F$ defined as follows: if $p \in P$ then we have*

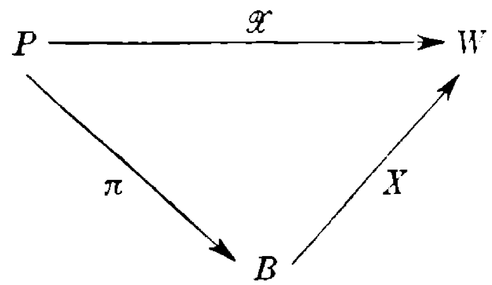
$$\mathscr{X}_p = s \Leftrightarrow A(p, s) = X(\pi p). \tag{5}$$

This mapping $\mathscr{X}$ satisfies the equation

$$\mathscr{X} \circ \varrho_{g^{-1}} = \Lambda_g \mathscr{X} \quad (g \in G,\ arbitrary). \tag{6}$$

Conversely, if we have a differentiable mapping $f\colon P \to F$ which satisfies $f \circ \varrho_{g^{-1}} = \Lambda_g f$, then the set of points $A\big(p, f(p)\big)$ is a differentiable cross-section in W.

Formula (5) asserts the commutativity of the following diagram:



Each mapping $\mathscr{X}$ corresponding to a section X induces a tangential mapping $(\mathscr{X}_*)_p$: $T_p(P) \to T_s(F)$. Implication (5) yields the following:

(7) $$(\mathscr{X}_*)_p \boldsymbol{x} = \boldsymbol{u} \Leftrightarrow A_*(\boldsymbol{x}, \boldsymbol{u}) \in T_{A(p,s)}(F)$$

where $\boldsymbol{x} \in T_p(P)$ and $\boldsymbol{u} \in T_s(F)$.

Let $\boldsymbol{y} = \boldsymbol{y}_b$ $(b \in B)$ be a differentiable vector field on B. We lift $\boldsymbol{y}$ onto a horizontal vector field on P. Thus we obtain a vector field $\bar{\boldsymbol{y}}_p \in T_p^-(P)$ $(p \in P)$ such that $\pi_* \bar{\boldsymbol{y}}_p = \boldsymbol{y}_{\pi_p}$. We see that this horizontal lifting is a linear operation which maps $\bigcup_b T_b(B) \times P$ onto $\bigcup_p T_p(P)$. It depends of course on the connection in P which we have chosen. We denote the *operation of horizontal lifting* by $\boldsymbol{H}$.

DEFINITION 1. The linear form

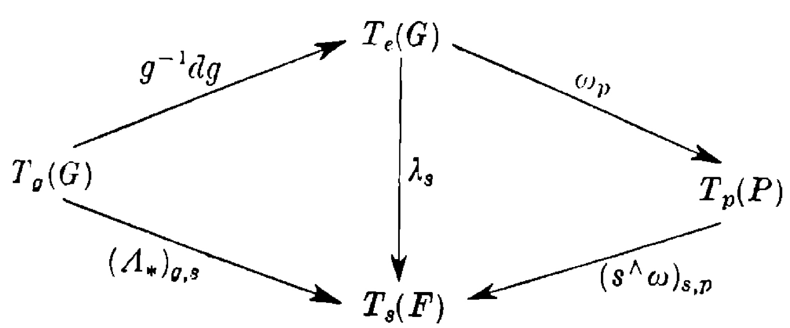
$$\nabla \mathscr{X} \stackrel{\text{def}}{=} \mathscr{X}_* \circ H$$

is called the *covariant differential of the cross-section X with respect to the given connection.* The value of this form on a vector $\boldsymbol{y}$, i.e. $\langle \nabla \mathscr{X}, \boldsymbol{y} \rangle = \langle \mathscr{X}_* \circ H, \boldsymbol{y} \rangle$, is called the *covariant derivative of X with respect to $\boldsymbol{y}$* and is denoted by $\nabla_{\boldsymbol{y}} \mathscr{X}$.

In order to compute an expression for $\nabla \mathscr{X}$ we have to investigate the tangential mapping $\mathscr{X}_*$.

The action A: $G \times F \to F$ induces a tangential mapping λ of $T_e(G)$ into $T_s(F)$ at every $s \in F$. (We recall that e is the unity of G.) Moreover, λ defines a natural homomorphism of the Lie algebra $\mathscr{G}$ into the Lie algebra based on $T(F)$. (Cf. Nomizu [34].) Let us look at the diagram.

(8)



The left-hand "triangle" is commutative. We define the mapping $s^\wedge\omega$ by requiring the commutativity of the right-hand "triangle". Thus we have

$$(s^\wedge\omega)_{p,s} = \lambda_s \circ \omega_p .$$

LEMMA 1. *If g varies in G and p varies in P, then the induced tangential mapping of $A_g\mathscr{X}_p$ is equal to*

(9) $$(A_g\mathscr{X}_p)_* = (\mathscr{X}_*)_p + (\lambda_{A_g\mathscr{X}_p}) \circ g^{-1}dg .$$

Proof. We recall Proposition 3 and Implication (7). The rules of differentiation of functions of many variables imply the equalities

$$(A_g\mathscr{X}_p)_* = (\mathscr{X}_*)_{pg^{-1}} + (A_g\mathscr{X}_p)_*|_{p-\mathrm{const}} = (\mathscr{X}_*)_{pg^{-1}} + (A_*)_{g,\mathscr{X}_p} .$$

In view of diagram (8) we have

$$(A_*)_{g,\mathscr{X}_p} = \lambda_{A_g\mathscr{X}_p} \circ g^{-1}dg$$

and consequently (9) holds.

THEOREM 1. *If X is a differentiable cross-section in $W(B, G, F, H)$ and $\mathscr{X}$ is the corresponding function of Proposition 3, then we have*

$$(\nabla\mathscr{X})_p = (\mathscr{X}_*)_p + (\mathscr{X}^\wedge\omega)_{p,\mathscr{X}_p} \circ L_p ,$$

where L_p is an arbitrary lifting of $T_{\pi_p}(B)$ into $T_p(P)$.

Proof. Let $\boldsymbol{y}$ be any vector tangent to B at p. Let L_p be any lifting which sends $\boldsymbol{y}$ to a vector $\boldsymbol{x} \in T_p(P)$. We have of course $\pi_*\boldsymbol{x} = \boldsymbol{y}$. Then we have to show that the following equality holds:

(10) $$(\mathscr{X}_* \circ H)_p\boldsymbol{x} = \langle(\mathscr{X}_*)_p + (\mathscr{X}^\wedge\omega)_{\mathscr{X}_p}, \boldsymbol{x}\rangle .$$

Let p_t ($t \geq 0$, $p_0 = p$) be an arc in P such that $\boldsymbol{x} = (dp_t/dt)$. Let $\bar{p}_t$ be a horizontal lift of p_t such that $\bar{p}_0 = p_0 = p$. Thus we have $H\boldsymbol{x} = (d\bar{p}_t/dt)|_{t=0}$, and in virtue of Proposition 2, $\bar{p}_t = p_t g_t^{-1}$ where

(11) $$g_t^{-1}dg_t = \omega_{p_t} .$$

Hence we obtain

$$(\mathscr{X}_* \circ H)_p\boldsymbol{x} = (d\mathscr{X}_{p_t g_t^{-1}}/dt)|_{t=0} = \big(d(A_{g_t}\mathscr{X}_{p_t})/dt\big)|_{t=0} .$$

We apply Lemma 1. This yields

(12) $$(\mathscr{X}_* \circ H)_p\boldsymbol{x} = (\mathscr{X}_*)_p\boldsymbol{x} + \left\langle \left(\lambda \circ g_t^{-1}\frac{dg_t}{dt}\right)\Big|_{t=0}, \boldsymbol{x}\right\rangle$$

where $\lambda = \lambda_{\mathscr{X}_p}$. In view of (8) and (11) we have

$$\left(\lambda \circ g_t^{-1}\frac{dg_t}{dt}\right)\Big|_{t=0} = \lambda \circ \omega_p = (\mathscr{X}^\wedge\omega)_{p,\mathscr{X}_p} .$$

Thus the right-hand member of (12) assumes the form (10), q.e.d.

THEOREM 2. *The covariant derivative $\nabla_{\boldsymbol{y}}\mathscr{X}$ satisfies the following equation:*

$$\nabla_{\boldsymbol{y}}\mathscr{X}_{pg^{-1}} = \big((\Lambda_*)_g \circ \nabla_{\boldsymbol{y}}\big)\mathscr{X}_p .$$

Proof. Let $\bar{\boldsymbol{x}}_p$ (and resp. $\bar{\boldsymbol{x}}_{pg^{-1}}$) be the horizontal lift of $\boldsymbol{y}$ to p (resp. to pg^{-1}). Thus we have $\bar{\boldsymbol{x}} = (\varrho_*)_{g^{-1}}\bar{\boldsymbol{x}}$. In view of Theorem 1 and Proposition 3 we have

$$\nabla_{\boldsymbol{y}}\mathscr{X} = \mathscr{X}_* \bar{\boldsymbol{x}}_{pg^{-1}} = \mathscr{X}_* \circ (\varrho_*)_{g^{-1}}\bar{\boldsymbol{x}}_p = \big((\Lambda_*)_g \circ \mathscr{X}_*\big)\bar{\boldsymbol{x}}_p ,$$

q.e.d.

We now define a new bundle associated with P. Let $\bar{A}(p, \boldsymbol{w})$ be the tangential mapping induced by $A(p, s)$ if s varies in F and $p \in P$ is arbitrary but fixed during the induction $A(p, s) \to \bar{A}(p, \boldsymbol{w})$.

PROPOSITION 4. *The map of $P \times \bigcup_s T_s(F)$ by $\bar{A}$ is a fibre bundle $\overline{W} = \overline{W}\big(B, G, T(F), \overline{\Pi}\big)$ associated with P.*

Proof. The group action Λ on F induces Λ_* on $T(F)$. $\bar{A}$ has the properties of Proposition 3, namely we have

$$\bar{A}\big(\varrho_{g^{-1}}p, (\Lambda_*)_g\boldsymbol{w}\big) = \bar{A}(p, \boldsymbol{w})$$

and

$$\bar{A}(p, \boldsymbol{w}) \neq \bar{A}(q, \boldsymbol{w}) \quad \text{if} \quad p \neq q .$$

We define the canonical projection $\overline{\Pi}$: $\overline{W} \to B$ as follows:

$$\overline{\Pi}\bar{A}(p, \boldsymbol{w}) = \pi p .$$

We see that the map by $\bar{A}$ has all the properties of an associated fibre bundle, q.e.d.

Theorem 2 and Proposition 4 imply at once the following theorem (cf. [8].):

THEOREM 3. *If we are given a vector field $\boldsymbol{y}$ on B, then the mapping*

$$\overline{\mathscr{X}}_p \stackrel{\text{def}}{=} \bar{A}\big(p, (\nabla_{\boldsymbol{y}}\mathscr{X})_p\big)$$

has the property of Proposition 3 and consequently it defines a certain cross-section in $\overline{W}$.

We now look for relations between covariant differentiation and the connections in the associated bundles. We consider the vector space $T_w(W)$ at an arbitrary point $w \in W$. Let us put

$$T_w^{|}(W) = \Big\{x\colon\ x = A_*(0, u),\ u \in \bigcup_s T_s(F)\Big\},$$

$$T_w^{-}(W) = \Big\{x\colon\ x = A_*(v, 0),\ v \in \bigcup_p T_p(P)\Big\}.$$

PROPOSITION 5. *A connection in P implies the existence of the distribution*

$$T_w(W) = T_w^{|}(W) + T_w^{-}(W) \quad (\textit{direct sum}).$$

Proof. Let $\boldsymbol{x}$ be any vector $\in T_w(W)$ and let z_t $(t \geqslant 0)$ be an arc such that $\boldsymbol{x} = (dz_t/dt)|_{t=0}$. Let (p_t, s_t) be a representation of z_t. Thus z_t may be considered as a local cross-section of W over the arc z_t. Let $\varrho g_t^{-1} p_t$ be a horizontal lift of p_t (see Proposition 2). Thus $(\varrho g_t^{-1} p_t, \Lambda_{g_t} s_t)$ is an equivalent representation of z_t. If we put

$$\boldsymbol{v} = (d\varrho g_t^{-1} p_t/dt)|_{t=0}, \qquad \boldsymbol{u} = (d\Lambda_{g_t} s_t/dt)|_{t=0},$$

then we have $\boldsymbol{x} = A_*(\boldsymbol{v}, \boldsymbol{u})$ where $\boldsymbol{v} \in T_w(P)$. Then we compute with the aid of Lemma 1:

$$\boldsymbol{u} = (ds_t/dt)|_{t=0} + \langle s_0^{\wedge}\omega, \pi_* \boldsymbol{x}\rangle.$$

The right-hand term is the covariant derivative of $\boldsymbol{x}$ in the direction $\boldsymbol{x}$. By the linearity of A_* we have

$$\boldsymbol{x} = A_*(\boldsymbol{v}, \boldsymbol{u}) = A_*(\boldsymbol{v}, 0) + A_*(0, \boldsymbol{u}),$$

which is the distribution in question. It may easily be proved that the above distribution is unique, that it determines a splitting of $T_w(W)$ into the direct sum of $T_w^{|}(W)$ and $T_w^{-}(W)$ and that it depends differentiably on the point $w \in W$.

In order to show that this distribution is a connection in the associated bundle W it would be necessary to prove that for any curve b_t from b_0 to b_1 in B there is an integral curve w_t such that $(dw_t/dt) \in T_{w_t}^{-}(W)$ and that w_t starts at any given point of the fibre $\Pi^{-1} b_0$ and $\Pi w_t = b_t$. Moreover, w_t should define an isomorphism of the fibre $\Pi^{-1} b_0$ to the $\Pi^{-1} b_t$, in such a way as to show that this isomorphism depends piecewise differentiably on t. We shall not deal with this problem and we refer the reader to the book by Nomizu [34].

We now deal with computing the expression of the covariant differential in coordinates. We choose a domain $\mathcal{O} \subset B$ such that there exists a system of local coordinates (ξ^a): $\mathcal{O} \to R^n$ and that $P|\pi^{-1}\mathcal{O}$ is a homeomorphic map of the Cartesian product $\mathcal{O} \times G$. Consequently, the same holds for $W|\Pi^{-1}\mathcal{O}$. Thus the group parameters $v^1, \ldots, v^r$ may be viewed as the local coordinates in the restricted bundle $P|\pi^{-1}\mathcal{O}$. Let us fix a section $\mathcal{R}$ in P. Thus we have on $\mathcal{R}|\pi^{-1}\mathcal{O}$ the equalities $v^k = v^k(\xi^1, \ldots, \xi^n)$. We see that if a point $p \in \mathcal{R}|\pi^{-1}\mathcal{O}$ then it may be represented by the local coordinates ξ^a. Thus the mapping $\mathcal{X}|\mathcal{R}$ (see formula (5)) can be represented by N functions $\Omega^K(\xi^1, \ldots, \xi^n)$, where Ω^K are the local coordinates in a convenable domain in F. Thus we may represent $d\mathscr{X}|\mathscr{R}$ as a triple of differentials $\frac{\partial \Omega^K}{\partial \xi^a} d\xi^a$. A coordinate representation of the term $\mathscr{X}^\wedge \omega$, which appears in formula (10) is somewhat more complicated. The action Λ of G on F may be represented by the triple of N functions $\varphi^K(\Omega^1, \ldots, \Omega^N; v^1, \ldots, v^r)$. We assume that the coordinate domain in G under consideration contains the group unity e. Thus if a group element g has coordinates (group parameters) $v^1, \ldots, v^r$, and the point $s \in F$ has coordinates $\Omega^1, \ldots, \Omega^N$, then the N numbers $\varphi^K(\Omega^1, \ldots, \Omega^N, v^1, \ldots, v^r)$ are the coordinates of the point $\Lambda_g s$. We return to diagram (8). We see that λ_s is represented by N differential forms

(13) $$\hat{\Omega}^K_j \, dv^j$$

where

$$\hat{\Omega}^K_j = \left(\frac{\partial \varphi^K(\Omega^1, \ldots, \Omega^N; v^1, \ldots, v^r)}{\partial v^j} \right) \Bigg|_{g=e}.$$

Now we have to find a representation of the connection form ω. Let $e_1, \ldots, e_r$ be a holonomic base of vectors in $T_e(G)$, $e_i = (\partial/\partial v^i)|_e$. Thus we may put $\omega = \omega^i e_i$. Then we have for the chosen section $\mathscr{R}$

$$\omega|\mathscr{R} = (\omega^i|\mathscr{R}) e_i.$$

Moreover, after restriction to the section $\mathscr{R}$, $\omega|\mathscr{R}$ may be written as a linear combination of the differentials $d\xi^a$, namely

(14) $$\omega|\mathscr{R} = \Gamma^i_a d\xi^a e_i$$

where Γ^i_a $(= \Gamma^i_a(\mathscr{R}))$ are the components of the connection object (cf. [26] II, 31). If we apply formula (14) to (13), then we obtain the following coordinate representation of $\lambda_s^\wedge \omega$:

$$\hat{\Omega}^K_j \Gamma^j_a d\xi^a.$$

We establish the holonomical bases of vectors in the domain in F in question by putting $I_K = \partial/\partial \Omega^K$. Thus Theorem 1 and the above considerations imply directly the following proposition:

PROPOSITION 5. *The holonomic local coordinate expression of the covariant differential is the following*:

(15) $$\nabla \mathscr{X} = (\partial_a \Omega^K + \hat{\Omega}^K_j \Gamma^j_a) d\xi^a I_K.$$

III. CONNECTION WITH THE LIE DERIVATION

We cite the definition of the natural fibre bundle according to A. Nijenhuis. These bundles are the most important in geometrical investigations.

DEFINITION 2. A fibre bundle $\mathcal{N}(B, G, F, \pi)$ is named a *natural bundle* if:

1^N. *The bundle space $\mathcal{N}$ and the standard fibre F are manifolds and the canonical projection π is differentiable;*

2^N. *With every diffeomorphism $f\colon \mathcal{O} \to B$ of an open set $\mathcal{O} \subset B$ into B there is associated a differentiable mapping $f_{\mathcal{N}}\colon \pi^{-1}\mathcal{O} \to \mathcal{N}$ such that*

(a) *$f_{\mathcal{N}}$ sends fibres into fibres by admissible mappings (i.e. mappings which belong to the structural group G); if $f(x) = y$, then $f_{\mathcal{N}}(F_x) = F_y$ (F_x (resp. F_y) denotes the fibre over x (resp. over y)), or equivalently: $\pi \circ f_{\mathcal{N}} = f \circ \pi$ on $\pi^{-1}(\mathcal{O})$;*

(b) *if $\mathcal{U}$ is an open subset of $\mathcal{O}$, then we have $(f|\mathcal{U})_{\mathcal{N}} = f_{\mathcal{N}}|\pi^{-1}(\mathcal{U})$;*

(c) *if id_B denotes the identity map of B, then $(\mathrm{id}_B)_{\mathcal{N}}$ is equal to the identity map of $\mathcal{N}$;*

(d) *if f and g are diffeomorphisms of B and $f \circ g$ is meaningful, then $(f \circ g)_{\mathcal{N}} = f_{\mathcal{N}} \circ g_{\mathcal{N}}$;*

3^N. *Every admissible map of any fibre F_x into itself can be obtained as the restriction to F_x of a certain $f_{\mathcal{N}}$ where f is a certain diffeomorphism of B such that $f(x) = x$.*

A natural bundle is at most of order r if $f_{\mathcal{N}}|F_x$ is the identity map whenever f is such that every differentiable arc $c\colon R^1 \to B$ with $c(0) = x$ has contact of order r at x with its transform $f \circ c$.

It is easy to see that a bundle which is associated with a natural bundle is also a natural bundle.

We shall apply the above definition to the above example of a bundle of frames. Let $\mathcal{O} \subset B$ be a coordinate domain. Then there exists a holonomic field of frames associated with every coordinate system $(\xi^a)\colon \mathcal{O} \to R^n$, namely $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_n$ where $\boldsymbol{i}_a = \partial/\partial\xi^a$. Let f be a diffeomorphism which maps $\mathcal{O}$ into a domain $\mathcal{Q} \subset B$. We assign to every point $x \in \mathcal{Q}$ the coordinates which the point $f(x)$ has. Let (η^a) be any coordinates on $\mathcal{Q}$ and let $\boldsymbol{j}_a$ be the corresponding field of holonomic frames. Then we have the relations $\boldsymbol{j}_a = A_a^\mu \boldsymbol{i}_\mu$ where

$$A_a^\beta = \frac{\partial\eta^\beta}{\partial\xi^a}.$$

The partial derivatives A_a^β are the parameters of a linear group, which is the structural group in a bundle of frames. Since every frame over any point $x \in B$ may be extended to a holonomic field of frames, we see that every frame may be sent into another by the manner described above, which means that 3^N is satisfied. It would be too obvious to verify in detail that our example satisfies axiom 2^N of Definition 2.

Let A_a^β be the parameters of a certain transformation of a fibre F_x onto itself. Let (ξ^a) be coordinates in a neighbourhood of x. We extend A_a^β onto this neighbourhood in such a way that the equalities

$$\frac{\partial {}^*A_a^\beta}{\partial \xi^\gamma} = \frac{\partial {}^*A_\gamma^\beta}{\partial \xi^a}, \quad \text{where} \quad {}^*A_a^\mu A_\mu^\beta = \delta_a^\beta,$$

hold. Then we consider the system of differential equations $\partial\eta^a/\partial\xi^\beta = {}^*A_\beta^a$ with the initial condition $\eta^a|_x = \xi^a|_x$. This system is solvable with respect to (η^a). Thus we obtain a new coordinate system in a neighbourhood of x. We define a transformation f of this neighbourhood into B by mapping a point z into such a point v that we have $\eta^a|_y = \xi^a|_x$. Thus we have a transformation which is required by 3^N.

We return to the general case. Let $X\colon B \to \mathcal{N}$ be a differentiable cross-section. Let $\boldsymbol{v}$ be a vector field on B. Thus $\boldsymbol{v}$ generates a one-parameter group of motions $m_t\colon B \to B$, $t \in R^1$, such that we have

$$(\sigma_* \boldsymbol{v})_x = \lim_{t\to 0} t^{-1}\big(\sigma(m_t x) - \sigma(x)\big),$$

σ being any differentiable scalar in a neighbourhood of $x \in B$. That group is additive, i.e. we have $m_t \circ m_s = m_{t+s}$ if it makes sense. In virtue of 2^N there exists a one-parameter set of diffeomorphisms $(m_t)_{\mathcal{N}}\colon F_x \to F_{m_t x}$. We denote $(m_t)_{\mathcal{N}}$ by M_t. Thus $M_t|X$ maps X into a certain one-parameter set of cross-sections in $\mathcal{N}$.

DEFINITION 3. The mapping $M_t|X\colon X \to N$, induced by the vector field $\boldsymbol{v}$, is called a *dragging of X in the direction $\boldsymbol{v}$.*

If there exists a limit $\lim_{t\to 0} t^{-1}({}'u_t - u)$ where $u \in X$ and ${}'u_t = M_t u$, then we call it the *Lie derivative of X at u* and we denote it by $\mathfrak{L}_{\boldsymbol{v}} u$ or by $(\mathfrak{L}_{\boldsymbol{v}} \mathscr{X})_u$.

We apply this definition to a field of differential objects of the rth class (cf. [16], [1], [46]).

The corresponding fibre bundle is $\mathcal{N}(B, L_n^r, R^N, \pi)$ where L_n^r is the differential group of order r and dimension n $(= \dim B)$, R^N is the standard fibre. Let $A_\beta^a, \ldots, A_{\beta_1\ldots\beta_r}^a$ denote the parameters of the group L_n^r. We describe the action of L_n^r on R^N by the system on N differentiable functions $\varphi^K(\Omega^1, \ldots, \Omega^N; A_1^1, \ldots, A_{\beta_1\ldots\beta_k}^a, \ldots)$ or, more briefly, by $\varphi^K(\Omega; A)$. $\mathcal{N}$ is a natural bundle if we require that every change of local coordinates in B, say $(\xi^a) \to (\xi^{a'})$ implies a transformation of the fibre $\Omega^K \to \varphi^K(\Omega; A)$ where

$$A_\beta^{a'} = \frac{\partial \xi^{a'}}{\partial \xi^\beta}, \quad \ldots, \quad A_{\beta_1\ldots\beta_r}^{a'} = \frac{\partial^r \xi^{a'}}{\partial \xi^{\beta_1} \ldots \partial \xi^{\beta_r}}.$$

The unity of the group L_n^r has the coordinates $A_\beta^a = \delta_\beta^a$, $A_{\beta_1\beta_2\ldots}^a = 0$. We write

(16) $$\hat{\Omega}^K|_a^{\beta_1\ldots\beta_k} = \left(\frac{\partial\varphi^K(\Omega, A)}{\partial A_{\beta_1\ldots\beta_k}^a}\right)\Bigg|_{A_\beta^a=\delta_\beta^a,\ A_{\beta_1\beta_2\ldots}^a=0}.$$

If $\boldsymbol{v} = v^\mu \boldsymbol{i}_\mu$ is a vector field in $\mathcal{C}$, then the corresponding group of motions m_t may be expressed in coordinates as follows:

(17) $$\xi^a \to \xi^a + tv^a + o(t).$$

Thus the corresponding element of L_n^r has the parameters

(18) $$A_\beta^a = \delta_\beta^a + t(\partial_\beta v^a), \quad \ldots, \quad A_{\beta_1\ldots\beta_k}^a = t\partial_{\beta_1\ldots\beta_k} v^a$$

where we leave out the terms of order $o(t)$ and $\partial_{\beta_1\ldots\beta_k} = \partial/\partial\xi^{\beta_1}\ldots\partial\xi^{\beta_k}$. The reciprocal element to (18) has the parameters

$$^*A_\beta^a = \delta_\beta^a - t(\partial_\beta v^a), \quad \ldots, \quad ^*A_{\beta_1\ldots\beta_k}^a = -t\partial_{\beta_1\ldots\beta_k} v^a$$

(+ terms of order $o(t)$). Thus the Kth coordinate of the dragged object is equal to

$$'\Omega_t^K = \varphi^K(\Omega_t^1, \ldots, \Omega_t^N;\ \delta_1^1 - t\partial_1 v^1, \ldots, -\partial_{\beta_1\ldots\beta_K} v^a, \ldots),$$

where Ω_t^K is the Kth coordinate of the point ϵX over the point of the base with coordinates (17). Simple computation yields us the result

$$\mathfrak{L}_v\Omega^K = \lim_{t\to 0} t^{-1}('\Omega_t^K - \Omega_0^K) = (v^\mu\partial_\mu\Omega^K) - \sum_{k=1}^{r}\hat{\Omega}^K|_a^{\beta_1\ldots\beta_k}\partial_{\beta_1\ldots\beta_k}v^a.$$

We obtain in this way the well-known formula for the Lie derivative in the classical case (cf. [49], [32], [27]). Thus we have

PROPOSITION 6. *Definition 3 is consistent with the classical one in the case of differential objects.*

If the fibre bundle in question is not a natural one but there is defined an infinitesimal connection, then it is also possible to define a kind of dragging. Let X be a cross-section in a fibre bundle $W(B, G, F, \pi)$ with a connection which was considered in the previous chapter. Let $\boldsymbol{v}$ be a vector field in B. Thus each point $x\epsilon B$ is the origin of some arc $x_t = m_t x$ (see formula (17)). Let w_t be a horizontal lift of X_t in the principal bundle P. Then we assume the following dragging: $'\mathscr{X}_t = A_{g_t^{-1}}\mathscr{X}_t$, where g_t is from Proposition 2 and $\mathscr{X}_t \epsilon F_{x_t} \cap X$. If we compare this with Definition 1, then we obtain the following proposition [8]:

PROPOSITION 7. *The limit* $\lim_{t\to 0} t^{-1}('\mathscr{X}_t - \mathscr{X})$ *exists and is equal to the covariant derivative* $\nabla_{\boldsymbol{v}}\mathscr{X}$.

We observe that this kind of dragging, which is implied by connections, is of infinitesimal nature, while the usual one, implied by transformations, is local.

IV. CONNECTIONS IN THE BUNDLES OF DIFFERENTIAL OBJECTS

In this chapter we make use of the notion of jet (cf. [11], [25]). To begin with we consider a set of C^q-regular mappings of a domain $\mathscr{U} \subset R^n$ into R^m, such that a fixed point a is mapped into b. Let r be a natural number, $r \leqslant q$. We assign the two mappings f and g to one class if $f(a) = g(a) = b$ and all partial derivatives up to the order r of $f-g$ are equal to 0 at a. Such a class of mappings is named a *jet of order* r, a is its *source* and b its *target*. Such a jet will be denoted by $j^r_a(f)$, f being its representative.

This notion may immediately be extended onto the case of differentiable manifolds. Let M be a manifold of the regularity class C^q. If $x \in \mathcal{O} \subset M$, $\mathcal{O}$ is a coordinate domain, then there exists a reversible mapping h of $\mathcal{O}$ into R^n ($n = \dim M$). The components of h are local coordinates in $\mathcal{O}$. Let φ be any mapping of $\mathcal{O}$ into R^m. Thus φ may be expressed by local coordinates with the aid of h, namely there exists a mapping f of a convenable domain of R^n into R^m such that $\varphi = f \circ h$. If we change the coordinate system in a neighbourhood of x, i.e. if we take a mapping h' instead of h, then we write $\varphi = f' \circ h'$, where $f' = h \circ f \circ h'^{-1}$. One may verify that f' and f determine the same jet. Thus the jets on differentiable manifold may be defined independently of the coordinates.

We consider the jets of mappings of a neighbourhood of $0 \subset R^n$ into $\mathcal{O} \subset M$, 0 being mapped onto x. We observe that two such mappings determine the same holonomic frame at x if and only if they determine the same jet of order one. This leads inmediately to the following generalization of the definition of the frame: *A frame of order r and dimension n at $x \in M$ is a jet of order r with the source $0 \subset R^n$ and the target x.*

We take into considerations a principal fibre bundle $P^r = P(B, L^r_n)$ where L^r_n is a differential group of order r and dimension n, B is n-dimensional manifold of class C^q and the canonical projection (which is not explicitly written here) maps each frame into its origin. The points of P^r are frames of order r, which we identify with rth order jets of local mappings $R^n \to B$. The structure of the group L^r_n may be investigated by considering jets of inversible mappings $R^n \to R^n$, the source and target being at 0. If we have two such mappings f and f' and the corresponding jets $j^r_0(f)$ and $j^r_0(f')$, then the group product of these jets will be a jet $j^r_0(f \circ f')$. A unity in L^r_n is a $j^r_0(\text{id.})$. The parameters $A^{\alpha}_{\beta_1 \ldots}$ which were dealt with in the previous chapter, were the coordinates in the group L^r_n.

We denote the Lie algebra of L^r_n by $\mathscr{L}^r_n$. We assume the following vectors as constituting a base in $\mathscr{L}^r_n$

$$e^{\alpha_1 \ldots \alpha_k}_{\alpha} = \left(\frac{\partial}{\partial A^{\alpha}_{\alpha_1 \ldots \alpha_k}} \right) \Bigg|_e .$$

According to a result of V. V. Vagněr [45] the Poisson bracket in $\mathscr{L}_n^r$ is expressed by the formulae

$$(19)\qquad \begin{cases} [e_a^{a_1\ldots a_s}, e_\beta^{\beta_1\ldots\beta_t}] \\ = \dfrac{(s+t-1)!}{s!(t-1)!} e_\beta^{(a_1\ldots a_s)(\beta_1\ldots\beta_{t-1}}\delta_a^{\beta_t)} - \dfrac{(s+t-1)!}{(s-1)!t!} e_a^{(\beta_1\ldots\beta_t)(a_1\ldots a_{s-1}}\delta_\beta^{a_s)}, \\ e_a^{a_1,\ldots,a_{s+t+1}} = 0, \\ [e_a^{a_1}, e_\beta^{\beta_1}] = (e_\beta^{a_1}\delta_a^{\beta_1} - e_a^{\beta_1}\delta_\beta^{a_1}). \end{cases}$$

Let $\theta = \sum_{s=1}^{r} \theta_{a_1\ldots a_s}^a e_a^{a_1\ldots a_s}$ denote the fundamental left invariant form on L_n^r. The Maurer-Cartan equations may be computed from the formula

$$d\theta = -[\theta, \theta] = \sum_k \sum_h [e_a^{a_1\ldots a_k}, e_\beta^{\beta_1\ldots\beta_h}]\theta_{a_1\ldots a_k}^a \wedge \theta_{\beta_1\ldots\beta_h}^\beta$$

and from (19). After some simple computations we obtain

$$(20)\qquad d\theta_{a_1\ldots a_k}^a = \sum_{h=1}^{k} \frac{k!}{h!(k-h)!} \theta_{(a_1\ldots a_h}^\mu \wedge \theta_{a_{h+1}\ldots a_k)\mu}.$$

Let $\vartheta^1, \ldots, \vartheta^n$ constitute a system of basic forms in a certain domain $\mathcal{O} \subset B$. If E_a are basic vectors in R^n, then the form $\vartheta^{(n)} = \vartheta^a E_a$ maps $T(\mathcal{O})$ into R^n. We assume that the basic forms satisfy the *structure equations of E. Cartan*:

$$\begin{aligned} d\vartheta^a &= \vartheta^\nu \wedge \vartheta_\nu^a, \\ d\vartheta_\beta^a &= -\vartheta_\nu^a \wedge \vartheta_\beta^\nu + \vartheta^\nu \wedge \vartheta_{\beta\nu}^a, \\ \vartheta_{\beta\gamma}^a &= \vartheta_{\gamma\beta}^a. \end{aligned}$$

A prolongation of these equations by G. F. Laptěv's method(¹) yields a sequence of linear forms $\vartheta_{a_1a_2a_3}^a, \ldots, \vartheta_{a_1\ldots a_r}^a$ which we assume to be symmetric with respect to the lower indices. These forms are subject to the following structure equations:

$$d\vartheta_{a_1\ldots a_k}^a = \sum_{h=1}^{k} \frac{k!}{h!(k-h)!} \vartheta_{(a_1\ldots a_h}^\mu \wedge \vartheta_{a_{h+1}\ldots a_k)\mu} + \vartheta^\nu \wedge \vartheta_{a_1\ldots a_k\nu}^a$$

which may be written more briefly if we use L. Ě. Ěvtushik's symbols [18]:

$$d\vartheta_{a_1\ldots a_k}^a = \vartheta_{\{a_1\ldots a_h}^\mu \wedge \vartheta_{a_{h+1}\ldots a_k\}\mu}^a + \vartheta^\nu \wedge \vartheta_{a_1\ldots a_k\nu}^a$$

The signs $\{\cdot\}$ denote here a symmetrization with summing with respect to h from 1 to k.

(¹) The reader may consult [28] — [31], [18], [47] or the Appendix in this paper.

The fixing of any fibre in $\pi^{-1}(\mathcal{O})$ consists in choosing a certain solution of the system $\vartheta^1 = \ldots = \vartheta^n = 0$. Then if we fix a fibre, the structure equations of the forms $\vartheta^a_{a_1\ldots}$ become identical with the Maurer-Cartan equations (20) of the group L^r_n. We see that the forms $\vartheta^a_\beta, \ldots, \vartheta^a_{a_1\ldots a_r}$ may be assumed to be basic forms in the bundle $P^r|\pi^{-1}\mathcal{O}$, provided that they are not zeros and that they are linearly independent. A linear transformations of the forms ϑ^a implies passing from one local section to another. If the forms ϑ^a are direct differentials of the local coordinates, then the forms are their linear combinations [18]. In the following we assume that such a special case does not occur. We assume that the sequence of the forms $\{\vartheta^a, \vartheta^a_\beta, \ldots, \vartheta^a_{\beta_1\ldots\beta_r}\}$ constitutes a base of linear forms in P^r. Thus the connection form $\omega^{(r)}$ may be expressed by the formula

$$\omega^{(r)} = \vartheta^{(r)} + \chi^{(r)} \tag{21}$$

where

$$\vartheta^{(r)} = \sum_{k=1}^{r} \vartheta^a_{a_1\ldots a_k} e_a^{a_1\ldots a_k}, \tag{22}$$

$$\chi^{(r)} = \sum_{k=1}^{r} B^a_{a_1\ldots a_k \varkappa} \vartheta^\varkappa e_a^{a_1\ldots a_k}. \tag{23}$$

We shall investigate the object B. We shall derive its infinitesimal equations from the curvature equations

$$d\omega^a_{a_1\ldots a_k} + [\omega, \omega]^a_{a_1\ldots a_k} = \boldsymbol{R}^a_{a_1\ldots a_k \mu\nu} \vartheta^\mu \wedge \vartheta^\nu \tag{24}$$

where $\boldsymbol{R}$ is a curvature tensor of the connection $\omega^{(r)}$. The transformation rule of $\boldsymbol{R}$ is of the type "adj" [26]. The substitution of (21), (22) and (23) into (24) yields the following equations for B:

$$dB^a_{a_1\ldots a_k\varkappa} + B^\mu_{\{a_1\ldots a_l|\varkappa|} \vartheta^a_{a_{l+1}\ldots a_k\}\mu} - B^a_{\{a_1\ldots a_l|\mu\varkappa|} \vartheta^\mu_{a_{l+1}\ldots a_k\}} - B^a_{a_1\ldots a_k\mu} \vartheta^\mu_\varkappa - \vartheta^a_{a_1\ldots a_k\varkappa} = h^a_{a_1\ldots a_k\varkappa\mu} \vartheta^\mu. \tag{25}$$

We denote the left-hand member of (25) by $\Delta B^a_{a_1\ldots a_k\varkappa}$. Then we state that the system of equations

$$\vartheta^1 = \cdots = \vartheta^n = 0,$$

$$\Delta B^a_{a_1\ldots a_k} = 0 \quad (a, a_1, \ldots, a_k = 1, \ldots, n;\ k \leqslant r)$$

is completely integrable. This follows from the fact that $d\Delta B = 0$ if $\vartheta^a = 0$. Thus system (25) defines a field of geometric objects. We get the following theorem:

THEOREM 4. *Equations* (22), (23) *and* (25) *define a family of connections over a domain* $\mathcal{O}$.

Now we deduce the formulae for the covariant derivative of a connection object Γ in P^r. We make use of the following formal relations between the infinitesimal equations of a geometric object and the covariant and Lie derivatives. Let $\boldsymbol{v}$ be a vector field in the domain $\mathcal{O}$. Let v^a denote the coordinates of $\boldsymbol{v}$ with respect to the frames which are dual to ϑ^a. If a field of objects Ω is given by its infinitesimal equations

$$\Delta\Omega^K = d\Omega^K + \sum_{k=1}^{r} \hat{\Omega}^K|_a^{a_1\ldots a_k} \vartheta^a_{a_1\ldots a_k} = \Omega^K_\nu \vartheta^\nu,$$

then the covariant derivative with respect to $\boldsymbol{v}$ is equal to

$$\Delta_{\boldsymbol{v}} \Omega^K = v^a \Big(\Omega^K_a + \sum_{k=1}^{r} \hat{\Omega}^K|_a^{a_1\ldots a_k} \Gamma^a_{a_1\ldots a_k\nu}\Big)$$

and the Lie derivative is

$$\mathfrak{L}_{\boldsymbol{v}} \Omega^K = v^a \Omega^K_a - \sum_{k=1}^{r} \hat{\Omega}^K|_a^{a_1\ldots a_k} v^a_{a_1\ldots a_k}$$

where $v^a_{a_1\ldots a_k}$ denote anholonomic derivatives of the field $\boldsymbol{v}$, which are obtained by the prolongation method (cf. [19]). Thus the covariant derivative of B may be derived from the infinitesimal equations (25). We have

$$\nabla_{\boldsymbol{v}} B^a_{a_1\ldots a_k\varkappa} = v^\nu h^a_{a_1\ldots a_k\nu} + B^\mu_{(a_1\ldots a_l|\varkappa|} \Gamma^a_{a_{l+1}\ldots a_k)\mu\nu} v^\nu - B^a_{(a_1\ldots a_l|\mu\varkappa|} \Gamma^\mu_{a_{l+1}\ldots a_k)\nu} v^\nu -$$
$$- B^a_{a_1\ldots a_k\mu} \Gamma^\mu_{\varkappa\nu} v^\nu - \Gamma^a_{a_1\ldots a_k\nu} v^\nu.$$

The formula just obtained is valid in an arbitrary frame. We pass continuously to holonomic coframes on the base $B|\mathcal{O}$, namely $\vartheta^a \to d\xi^a$. Then the forms $\vartheta^a_{\beta_1\ldots}$ pass into linear combinations of the differentials $d\xi^a$. We can choose them all to be zeros. In such a case formula (21) takes the form

$$\omega^{(r)} = \sum_{k=1}^{r} \Gamma^a_{a_1\ldots a_k\mu} d\xi^\mu e_a^{a_1\ldots a_k},$$

where Γ are the coordinates of the connection object in a holonomic frame. We see that Γ may be treated as a boundary case of the object B. The following theorem results from the infinitesimal equations of B and their formal relations to the covariant and Lie derivatives.

PROPOSITION 8. *The covariant and Lie derivartives of the connection object $\Gamma^a_{a_1\ldots a_k}$ of the order r are expressed by the formulae*

$$\nabla_{\boldsymbol{v}} \Gamma^a_{a_1\ldots a_k\mu} = v^\nu \partial_\nu \Gamma^a_{a_1\ldots a_k\mu} + \Gamma^\varkappa_{(a_1\ldots a_l|\mu|} \Gamma^a_{a_{l+1}\ldots a_k)\varkappa\nu} v^\nu -$$
$$- \Gamma^a_{(a_1\ldots a_k|\mu\varkappa|} \Gamma^\varkappa_{a_{l+1}\ldots a_k)\nu} v^\nu - \Gamma^a_{a_1\ldots a_k\varkappa} \Gamma^\varkappa_{\mu\nu} v^\nu - \Gamma^a_{a_1\ldots a_{k+1}\mu\nu} v^\nu,$$

$$\mathfrak{L}_{\boldsymbol{v}} \Gamma^a_{a_1\ldots a_k\mu} = v^\nu \partial_\nu \Gamma^a_{a_1\ldots a_k\mu} - \Gamma^\mu_{(a_1\ldots a_l|\varkappa|} \partial_{a_{l+1}\ldots a_k)\mu} v^\varkappa +$$
$$+ \Gamma^a_{(a_1\ldots a_l|\mu\varkappa|} \partial_{a_{l+1}\ldots a_k)} v^k + \Gamma^a_{a_1\ldots a_k\varkappa} \partial_\mu v^\varkappa + \partial_{a_1\ldots a_k\mu} v^a.$$

V. DEFINITION OF THE COVARIANT DERIVATIVE IN TERMS OF FUNCTIONAL EQUATIONS

We now restrict ourselves to considering natural fibre bundles (cf. Chapter III). In particular, the subsequent results are valid for differential objects, and they do not depend on the results of Chapters III and IV.

We have remarked above that a mapping $\mathscr{X}$ of a principal bundle P into a standard fibre F of an associated bundle determines a field of geometric objects if and only if the following functional equation is satisfied:

$$\mathscr{X}_{pg^{-1}} = A_{g^{-1}}\mathscr{X}_p \quad \text{where} \quad g \in G. \tag{26}$$

The point p is thus a frame and the point $\Omega = \mathscr{X}_p$ is a value of an object. In order to investigate the object by functional equations we have to expand (26) in (local) coordinates. But we do not need any representation of the frames, because, in agreement with our assumption, a frame is relatively determined by local coordinates of the base. Fixing a representation of a field of geometric objects consists in a choice of a field of frames $\mathscr{S} \subset P$ and in finding $\mathscr{X}_p$, where $p \in \mathscr{S}$.

The associativity of the action $A\colon G \times F \to F$ implies the identity $A_{g^{-1}} \circ A_{h^{-1}} = A_{(hg)^{-1}}$. An expansion in local coordinates in G yields the system of functional equations

$$\varphi^K\big(\varphi(\Omega, u), w\big) = \varphi^K\big(\Omega, \theta(u, w)\big), \tag{27}$$

where the system of functions $\varphi = \{\varphi^1, \ldots, \varphi^N\}$ expresses the action A and $\Omega^1, \ldots, \Omega^N$ denote the local coordinates in a certain domain $\subset F$, u^k and w^k $(k = 1, \ldots, r, r = \dim G)$ denote respectively local coordinates of g and h respectively in a certain neighbourhood of the unity $e \in G$, and $\theta(u, w)$ express the multiplication of the group elements. If we have to do with classical cases, G being a differential group and F being a Euclidean space R^N, then all the coordinates in question are defined globally.

Let us notice that system (27) is defined on $G \times F$. Its solutions are called *general geometric objects*, and if they are considered from an algebraical point of view, we call them *representations of the group G*.

Let (ξ^a) and $(\xi^{\bar{a}})$, $a = 1, \ldots, n$, be two different maps of a certain domain $\mathcal{O} \subset B$ into R^N. Simultaneously the functions

$$A^{\bar{a}}_{\beta} = \frac{\partial \xi^{\bar{a}}}{\partial \xi^{\beta}}$$

are defined, because we take into consideration differentiable maps only. Also our field of objects always have ordinary Pfaff derivatives with respect to the local coordinates.

We shall look for a definition of the covariant derivative of the field of objects. It will be given in terms of functional equations. We start with defining a connection object by giving its transformation rule. If a connection object is defined on every local section $\mathscr{S}: B \to P$, then simultaneously the corresponding connection form is defined (see p. 15). The relating formula is $\omega|\mathscr{S} = \Gamma^k_a d\xi^a e_k$.

We denote by $j^1, \ldots, j^r$ the coordinates of the unity in G and by $^*w^1, \ldots, {}^*w^r$ the coordinates of the group element which is reciprocal to $w^1, \ldots, w^r$. We introduce the following symbols:

$$\theta^s_{|k}(u, w) = \frac{\partial \theta^s(u, w)}{\partial w^k}, \qquad \theta^s_{k|}(u, w) = \frac{\partial \theta^s(u, w)}{\partial u^k}.$$

The sign $\wedge$ will denote that after performing the operation $\partial/\partial t^k$ we substitute in the differentiated function $t^1 = j^1, \ldots, t^r = j^r$. We put

$$\vartheta^s_{|k}(u) = \theta^{\wedge s}_{|k}(u, t), \qquad \vartheta^s_{k|}(u) = \theta^{\wedge s}_{k|}(t, u).$$

Then we map G onto a matrix group by putting

$$\mathscr{A}^s_p(u) = \left(\frac{\partial}{\partial t^p}\theta^s\big(\overset{*}{u}, \theta(t, u)\big)\right)^{\wedge}.$$

at is easy to prove that $\mathscr{A}$ is a coordinatic representation of the operator adj g^{-1}. The following identities may be deduced directly from the above definition of $\mathscr{A}$:

$$\mathscr{A}^s_p(u) = \theta^s_{|m}(\overset{*}{u}, u)\vartheta^m_{p|}(u) = \theta^s_{m|}(\overset{*}{u}, u)\vartheta^m_{|p}(u), \tag{28}$$

$$\mathscr{A}^s_p(u)\mathscr{A}^p_q(w) = \mathscr{A}^s_q\big(\theta(u, w)\big), \tag{29}$$

$$\mathscr{A}^s_q(u)\mathscr{A}^q_p(\overset{*}{u}) = \delta^s_p. \tag{30}$$

The parameters

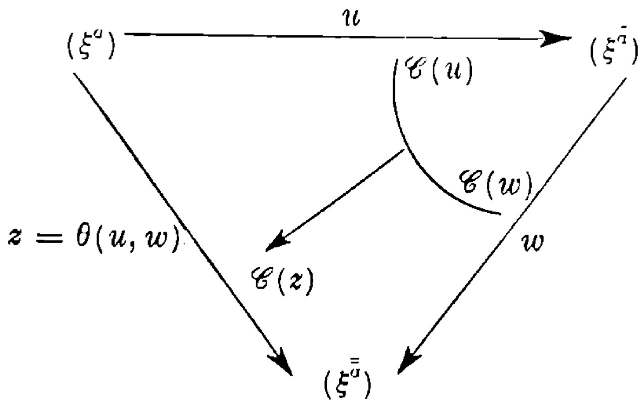
$$\mathscr{C}^k_\sigma(u) = \theta^k_{h|}(u, u^*)\partial_\sigma u^k$$

will be associated with every transformation

$$(\xi^a) \to (\xi^{\bar{a}}),$$

$$\Omega \to \bar{\Omega}, \qquad \bar{\Omega} = \{\Omega^{\bar{K}}\} = \{\varphi^K(\Omega, u)\}.$$

Suppose that an open neighbourhood $\mathscr{O} \subset B$ is covered by three local maps $(\xi^a), (\xi^{\bar{a}}), (\xi^{\bar{\bar{a}}})$ and that every transformation $(\xi^a) \to (\xi^{\bar{a}}) \to (\xi^{\bar{\bar{a}}}), (\xi^a) \to (\xi^{\bar{\bar{a}}})$ is associated with a certain group element which depends on a point of $\mathcal{O}$. These relations may be presented on the following diagram



Then $\mathscr{C}(z)$ may be computed by means of the following formula

(31) $$\mathscr{C}^k_\sigma(z) = \mathscr{C}^k_\sigma(w) + A^{\bar\varrho}_\sigma \mathscr{A}^k_l(w^*)\mathscr{C}^l_{\bar\varrho}(u),$$

where $A^{\bar\varrho}_\sigma = \partial\xi^{\bar\varrho}/\partial\xi^\sigma$.

We shall prove it. We have

(32) $$\begin{aligned}\mathscr{C}^k_\sigma(z) &= \mathscr{C}^k_\sigma(\theta(u, w)) \\ &= \theta^k_{j|}(\theta(u, w), \theta(w^*, u^*))\,\theta^j_{h|}(u, w)\,\partial_\sigma u^h + \\ &\qquad + \theta^k_{j|}(\theta(u, w), \theta(w^*, u^*))\,\theta^j_{|h}(u, w)\,\partial_\sigma v^h.\end{aligned}$$

We consider the first ingredient of the right-hand member of (32). In virtue of the associativity rule for the functions $\theta(u, w)$ we have

(33) $$\begin{aligned}&\theta^k_{j|}(\theta(u, w), \theta(u^*, w^*))\,\theta^j_{h|}(u, w) \\ &\qquad = \left(\frac{\partial}{\partial u^h}\theta^k(\theta(u, w), \theta(w^*, t))\right)\bigg|_{t=u^*} = \theta^k_{h|}(u, u^*).\end{aligned}$$

The last member is treated similarly, namely we have

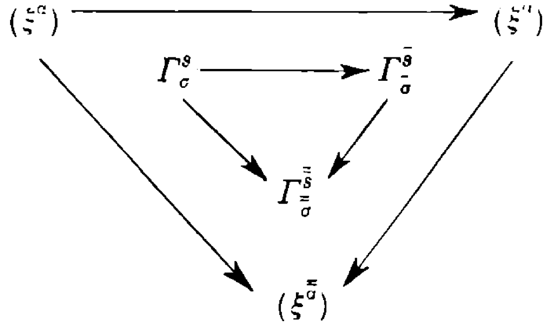
(34) $$\begin{aligned}\theta^k_{j|}(\theta(u, w), \theta(w^*, u^*))\,\theta^j_{|h}(u, w) &= \left(\frac{\partial}{\partial w^h}\theta^k(\theta(u, w), \theta(t, u^*))\right)\bigg|_{t=w^*} \\ &= \left(\frac{\partial}{\partial w^h}\theta^k\big(u, \theta(\theta(w, t), u^*)\big)\right)\bigg|_{t=w^*} \\ &= \mathscr{A}^k_l(u^*)\,\theta^l_{h|}(w, w^*).\end{aligned}$$

The substitution of the results of formulae (33) and (34) into (32) yields (31), q.e.d.

Now we define the connection object Γ by providing it with the following transformation rule:

$$\Gamma^{\bar{s}}_{\bar{\sigma}} = A^{\tau}_{\bar{\sigma}} \mathscr{A}^{s}_{p}(w)\left(\Gamma^{p}_{\tau} - \mathscr{C}^{p}_{\tau}(w)\right).$$

We have to prove that this transformation rule has a group property, in other words that the following diagram is commutative:



In view of formulae (29), (30) and (31) we have

$$\begin{aligned}
\Gamma^{\bar{\bar{s}}}_{\bar{\bar{\sigma}}} &= A^{\bar{\tau}}_{\bar{\bar{\sigma}}} \mathscr{A}^{\bar{\bar{s}}}_{\bar{p}}(u)\left(\Gamma^{\bar{p}}_{\bar{\tau}} - \mathscr{C}^{\bar{p}}_{\bar{\tau}}(u)\right) \\
&= A^{\bar{\tau}}_{\bar{\bar{\sigma}}} \mathscr{A}^{\bar{\bar{s}}}_{\bar{p}}(u)\left[A^{\pi}_{\bar{\tau}} \mathscr{A}^{\bar{p}}_{q}(w)\left(\Gamma^{q}_{\pi} - \mathscr{C}^{q}_{\pi}(w)\right) - \mathscr{C}^{\bar{p}}_{\bar{\tau}}(u)\right] \\
&= A^{\bar{\tau}}_{\bar{\bar{\sigma}}} A^{\pi}_{\bar{\tau}} \mathscr{A}^{\bar{\bar{s}}}_{\bar{p}}(u) \mathscr{A}^{\bar{p}}_{q}(w)\left[\Gamma^{q}_{\pi} - \left(\mathscr{C}^{q}_{\pi}(w)\right) + A^{\varrho}_{\pi} \mathscr{A}^{q}_{m}(w^{*}) \mathscr{C}^{m}_{\varrho}(u)\right] \\
&= A^{\pi}_{\bar{\bar{\sigma}}} \mathscr{A}^{\bar{\bar{s}}}_{q}(z)\left(\Gamma^{q}_{\pi} - \mathscr{C}^{q}_{\pi}(z)\right), \qquad \text{q.e.d.}
\end{aligned}$$

Of course the components of the object Γ which appear in formula (11) have the same transformation rule.

Now we turn to the axiomatic definition of the covariant derivative of the field of geometric objects. We assume that a connection object is defined in the space under consideration.

Definition 4. Let Ω and Y be two geometric objects which are subject to the same group G. A sequence of functions $\Psi^{M}(\Omega, Y)$ where $M = 1, \ldots, N_1$ will be called a *paracomitant of the object* Ω which depends on Y if and only if the sequence $\{\Psi^1, \ldots, \Psi^{N_1}, \Omega^1, \ldots, \Omega^N\}$ constitutes a sequence of coordinates of some geometric object.

For example, a paracomitant of the vector $\boldsymbol{v}$ in a space of affine connection is a (non-geometric) object $\Gamma^{\alpha}_{\beta\varkappa} v^{\varkappa}$.

Definition 5. Suppose that we have a *paracomitant* $\{\mathscr{K}^{I}_{\sigma}(\Omega, \Omega', \Gamma)\}$ of the field of the geometric object Ω and of its first Pfaff derivatives $\Omega' = \{\partial_\sigma \Omega^L\}$ which depends on the connection object Γ and satisfies the following axioms:

1^{∇}. $\mathscr{K}$ *is subject to the transformation rule*

$$\mathscr{K}^{\bar{I}}_{\bar{\sigma}}(\Omega, \Omega', \Gamma) = \mathscr{K}^{I}_{\sigma}(\bar{\Omega}, \bar{\Omega}', \bar{\Gamma}) = A^{\tau}_{\bar{\sigma}} \Omega^{\bar{I}}_{H} \mathscr{K}^{H}_{\tau}(\Omega, \Omega', \Gamma),$$

where

$$\Omega^I_{II} = \frac{\partial \Omega^I}{\partial \Omega^{II}} = \frac{\partial \varphi^I(\Omega, w)}{\partial \Omega^{II}}.$$

2^∇. *If the connection object* Γ *vanishes in a certain subset, then we have in that subset* $\mathscr{K}^I_\sigma(\Omega, \Omega', 0) = \partial_\sigma \Omega^I$.

Then such a paracomitant will be called the *covariant derivative of the field of object* Ω.

THEOREM 5. *Definition 5 determines uniquely a sequence of coefficients of the covariant differential of* Ω, *according to Definition* 1.

Before proving the theorem we have to prove some lemmas.

LEMMA 2. *The derivatives of the functions* φ *which express the transformation rule of* Ω *satisfy the following identities:*

$$\varphi^{\bar K}_p(\Omega, w) = \varphi^{\bar{\bar K}}_L(\Omega, w)\hat{\varphi}^L_q(\Omega)\theta^q_{|p}(w^*, w), \tag{35}$$

$$\hat{\varphi}^{\bar{\bar K}}_p(\bar{\bar\Omega}) = \varphi^{\bar K}_L(\Omega, w)\hat{\varphi}^L_q(\Omega)\mathscr{A}^q_p(w^*), \tag{36}$$

$$\varphi^{\bar{\bar K}}_{II}\big(\Omega, \theta(t, w)\big) = \varphi^{\bar{\bar K}}_{\bar L}(\bar{\Omega}, t)\varphi^{\bar L}_{II}(\Omega, w), \tag{37}$$

where

$$\varphi^{\bar{\bar K}}_q = \partial\varphi^K/\partial w^q, \qquad \varphi^{\bar{\bar K}}_H = \Omega^{\bar{\bar K}}_{II} = \partial\varphi^K/\partial\Omega^{II}, \qquad \hat{\varphi}^K_q = \hat{\Omega}^K_q = (\partial\varphi^{\bar K}/\partial w^q)|_{w=j}.$$

Proof. We start from the functional identities

$$\varphi^{\bar{\bar K}}\big(\bar\varphi(\Omega, t), w\big) = \varphi^{\bar{\bar K}}\big(\Omega, \theta(w, t)\big) \tag{27'}$$

(see formula (27)). We differentiate them term by term with respect to t and afterwards we substitute $t^1 = j^1, \ldots, t^r = j^r$. We get

$$\varphi^{\bar{\bar K}}_L(\Omega, w)\hat{\varphi}^L_q(\Omega) = \varphi^{\bar K}_p(\Omega, w)\,\vartheta^p_{|q}(w). \tag{38}$$

In order to compute the matrix reciprocal to $\vartheta^p_{|q}$ we differentiate the evident equality

$$\theta^s\big(w^*, \theta(w, t)\big) = t^s$$

with respect to t^r and we substitute $t = j$. We have

$$\theta^s_{|p}(w^*, w)\,\vartheta^p_{|r}(w) = \delta^s_r.$$

If we apply this result to (38), then we obtain (35).

Similarly we obtain the identity

$$\hat{\varphi}^{\bar K}_h\big(\varphi(\Omega, w)\big) = \varphi^{\bar K}_s(\Omega, w)\,\vartheta^s_{h|}(w)$$

by differentiating the following

$$\varphi^{\bar{\bar K}}\big(\varphi(\Omega, w), t\big) = \varphi^{\bar K}\big(\Omega, \theta(t, w)\big)$$

with respect to t^h and then substituting $t = j$. The substitution of the right-hand member of (35) for $\varphi_s^{\overline{K}}$ yields

$$\hat{\varphi}_h^{\overline{K}}(\Omega) = \varphi_L^{\overline{K}}(\Omega, w)\hat{\varphi}_q^L(\Omega)\theta^q_{|s}(w^*, w)\vartheta^s_{r|}(w).$$

The last formula together with the identities

$$\theta^q_{|s}(w^*, w)\vartheta^s_{p|}(w) = \left[\frac{\partial}{\partial t^p}\theta^q(w^*, \theta(t, w^*))\right]^\wedge = \mathscr{A}^q_p(w)$$

implies (36).

Finally (37) will be obtained by differentiating (27') with respect to Ω^H.

LEMMA 3. *The object $\Omega' = \{\partial_\sigma \Omega^K\}$ is subject to the following transformation rule:*

$$\partial_{\bar\sigma}\Omega^{\overline{K}} = A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L[\partial_\tau\Omega^L + \hat{\varphi}^L_s(\Omega)\mathscr{C}^s_\tau(w)].$$

Proof. In virtue of the identities of Lemma 1 we have

$$\begin{aligned}
\partial_{\bar\sigma}\Omega^{\overline{K}} &= A^\tau_{\bar\sigma}\partial_\tau\varphi^{\overline{K}}(\Omega, w)\\
&= A^\tau_{\bar\sigma}\big(\Omega^{\overline{K}}_L\partial_\tau\Omega^L + \varphi^{\overline{K}}_p(\Omega, w)\partial_\tau w^p\big)\\
&= A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L[\partial_\tau\Omega^L + \hat{\varphi}^L_s(\Omega)\theta^s_{|p}(w^*, w)\partial_\tau w^p]\\
&= A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L[\partial_\tau\Omega^L + \hat{\varphi}^L_s(\Omega)\mathscr{C}^s_\tau(w)],
\end{aligned}$$

q.e.d.

LEMMA 4. *The object*

$$\mathscr{D}^K_\sigma = \partial_\sigma\Omega^K + \hat{\varphi}^K_s(\Omega)\Gamma^s_\sigma$$

is subject to the following transformation rule:

$$\mathscr{D}^{\overline{K}}_{\bar\sigma} = A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L\mathscr{D}^L_\tau.$$

(This lemma may easily be deduced from Theorem 2, but we want to obtain it independently by means of functional equations.)

Proof. In view of formulae (30), (35) and (36) we compute

$$\begin{aligned}
\partial_{\bar\sigma}\Omega^{\overline{K}} + \varphi^{\overline{K}}_{\bar s}(\overline{\Omega})\Gamma^{\bar s}_{\bar\sigma} &= A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L[\partial_\tau\Omega^L + \hat{\varphi}^L_s(\Omega)\mathscr{C}^s_\tau(w)] +\\
&\quad + \Omega^{\overline{K}}_L\hat{\varphi}^L_s(\Omega)\mathscr{A}^s_q(w^*)\mathscr{A}^q_p(w)A^\tau_{\bar\sigma}[\Gamma^p_\tau - \mathscr{C}^p_\tau(w)]\\
&= A^\tau_{\bar\sigma}\Omega^{\overline{K}}_L[\partial_\tau\Omega^L + \hat{\varphi}^L_s(\Omega)\Gamma^s_\tau],
\end{aligned}$$

q.e.d.

Proof of Theorem 5. We have to show that the unique system of functions which satisfies Axioms 1^∇ and 2^∇ is the following:

$$\mathscr{K}^K_\sigma(\Omega, \Omega', \Gamma) = \partial_\sigma\Omega^K + \hat{\varphi}^K_s(\Omega)\Gamma^s_\sigma.$$

Axiom 1^∇ implies the following system of functional equations:

$$\mathscr{K}^K_\sigma(\bar{\Omega}, \bar{\Omega}', \bar{\Gamma}) = A^\tau_{\bar{\sigma}} \Omega^{\bar{K}}_H \mathscr{K}^H_\tau(\Omega, \Omega', \Gamma). \tag{39}$$

We treat the variables $\partial_a w^k$ as independent. We observe that the system of variables $\{w^1, \ldots, w^r, \partial_1 w^1, \ldots, \partial_a w^s, \ldots\}$ is equivalent to the system $\{w^1, \ldots, w^r, \mathscr{C}^1_1, \ldots, \mathscr{C}^s_a, \ldots\}$. We differentiate system (39) with respect to any $\mathscr{C}^a_a$ and then we substitute $v^1 = j^1, \ldots, v^r = j^r$, $C^1_1 = \ldots = C^a_a \ldots = \mathscr{C}^r_n = 0$ and $A^a_{\bar{\beta}} = \delta^a_\beta$. We obtain the following system of equations:

$$\frac{\partial \mathscr{K}^K_\sigma(\Omega, \Omega', \Gamma)}{\partial(\partial_a \Omega^I)} \hat{\varphi}^I_a(\Omega) = \frac{\partial \mathscr{K}^K_\sigma(\Omega, \Omega', \Gamma)}{\partial \Gamma^a_a}$$

which implies that $\mathscr{K}$ is of the form

$$\mathscr{K}^K_\sigma(\Omega, \Omega', \Gamma) = \mathscr{H}^K_\sigma(\Omega, \mathscr{D})$$

where $\mathscr{D}^K_\sigma = \partial_\sigma \Omega^K + \hat{\varphi}^K_s(\Omega) \Gamma^s_\sigma$.

We substitute $\Gamma^1_1 = \ldots = \Gamma^s_\sigma = \ldots = 0$. Thus Axiom 2^∇ implies the equality

$$\mathscr{H}^K_\sigma(\Omega, \Omega') = \partial_\sigma \Omega^K.$$

It follows that $\mathscr{H}$ is of the form

$$\mathscr{H}^K_\sigma(\Omega, \mathscr{D}) = \mathscr{D}^K_\sigma + h^K_\sigma(\Omega, \mathscr{D}),$$

where h has the same transformation rule as $\mathscr{H}$ and, moreover, h vanishes if the connection object does. Then h is always zero and we have the unique solution

$$\mathscr{K}^K_\sigma(\Omega, \Omega', \Gamma) = \partial_\sigma \Omega^K + \hat{\varphi}^K_s(\Omega) \Gamma^s_\sigma,$$

which is consistent with Proposition 5.

APPENDIX

OUTLINE OF THE THEORY OF PROLONGATIONS

We give a short outline of the theory which was initiated by E. Cartan and further developed in Moscow. This theory has various applications to the investigation of imbedded manifolds, connections, etc. On the other hand, it is related to the theory of prolongations expounded in [11]. For the sake of brevity we omit here some details, proofs and extensions, which are to be found in the cited papers of G. F. Laptěv and his pupils.

1. Invariance equations of geometric object. Let F be an N-dimensional manifold. Let G be a Lie group which acts on F on the right. We denote by $w^1, \ldots, w^r$ the coordinates of a group element in a certain domain $\subset G$ which contains the unity e. The vectors $\boldsymbol{e}_i = (\partial/\partial w^i)|_e$ form a base of a Lie algebra $\mathscr{G}$ of G. The fundamental left invariant form $\boldsymbol{\Phi} = g^{-1}dg$ may be represented as $\boldsymbol{\Phi} = \Phi^i \boldsymbol{e}_i$ and depend on w and dw.

We take into considerations a point $\Omega \in F$. An element $g \in G$ transforms Ω into a point $\bar{\Omega}$, i.e. $\bar{\Omega} = A_g \Omega$. In virtue of the associativity of this action we may write simply $\bar{\Omega} = g \cdot \Omega$. If we differentiate formally this equation, then we obtain

$$dg \cdot \Omega + g \cdot d\Omega = 0$$

and hence

(40) $$d\Omega + (g^{-1}dg) \cdot \Omega = 0.$$

It is an *intrinsic form of the invariance equations*. In order to obtain an explicit form in coordinates we have to introduce the transformation functions $\{q^K\}$, $\bar{\Omega}^K = q^K(\Omega, w)$, which express A in terms of local coordinates. Then equation (40) may be written as a system of equations

(40′) $$d\Omega^K + \hat{\Omega}_s^K \Phi^s(w, dw) = 0$$

where the functions $\hat{\Omega}$ may be defined by the formula

$$\hat{\Omega}_a^K = \left(\frac{\partial q^K(\Omega, w)}{\partial w^a}\right)\Big|_{g=e}.$$

Conversely, if we have a system of forms which constitute the left-hand member of (40′), then we may ask under what conditions it is completely integrable. We compute the external differential of the left-hand member of (40′). We have

(41) $$\frac{\partial \hat{\Omega}_a^K}{\partial \Omega^H} d\Omega^H \wedge \Phi^a + \hat{\Omega}_a^K d\Phi^a = 0.$$

If we make use of equations (41) and of the Maurer-Cartan equations

$$d\Phi^a = \tfrac{1}{2} C_{bd}^a \Phi^b \wedge \Phi^d,$$

where C_{bd}^a are structure constants of the group G, we put (41) in the form

$$\frac{1}{2}\left(\frac{\partial \hat{\Omega}_a^K}{\partial \Omega^L} \hat{\Omega}_b^L - \frac{\partial \hat{\Omega}_b^K}{\partial \Omega^L} \hat{\Omega}_a^L\right) \Phi^a \wedge \Phi^b - \frac{1}{2} \hat{\Omega}_h^K C_{ab}^h \Phi^a \wedge \Phi^b = 0.$$

We observe that the expression obtained does not contain any differential $d\Omega^K$. The condition of Frobenius gives us a necessary and a sufficient condition for the complete integrability of system (40), namely

$$(42)\qquad \frac{\partial \hat{\Omega}^K_a}{\partial \Omega^L}\hat{\Omega}^L_b - \frac{\partial \hat{\Omega}^K_b}{\partial \Omega^L}\hat{\Omega}^L_a = C^h_{ab}\hat{\Omega}^K_h .$$

Conversely, if we are given a system of equations

$$(43)\qquad d\Omega^K + \xi^K_a(\Omega)\Phi^a(w, dw) = 0$$

on $G\times F$, then it is completely integrable if and only if equalities (42) hold after the substitution of ξ^K_a for $\hat{\Omega}^K_a$. In such a case it can be proved that the prime integrals of (43) satisfy the identity (associativity law)

$$\Psi^K(\Psi(\Omega, w), w') = \Psi^K(\Omega, \theta(w, w'));$$

θ expresses here a multiplication of group elements in a chosen coordinate domain in G. We conclude that Ψ expresses a certain representation of the group G on the manifold F.

If we have an automorphism a of F and $\varphi = \{\varphi^K\}$ defines a representation of G on F, then a sequence of mappings $\{\tilde{\varphi}^K\} = \{a^{-1}\varphi^K(a\Omega, w)\}$ defines an equivalent representation. We have the following system of linear relations among the representations φ and $\tilde{\varphi}$: $\hat{\tilde{\varphi}}^K_a = M^K_H\hat{\varphi}^K_a$ where M is a non-singular matrix. The corresponding systems of infinitesimal equations are also linearly equivalent. We see that a system of equations of the form (43) determines a class of equivalent geometric objects, provided that it is completely integrable.

2. Bases of linear forms and structure equations of fibre bundles. Let $P(B, G, \pi)$ be a differentiable principal fibre bundle over a base B. B may be covered with a system of coordinate domains $\mathcal{O}_\varkappa$. On every $\mathcal{O}_\varkappa$ there exist n ($= \dim B$) fields of linear forms $\vartheta^1_\varkappa, \ldots, \vartheta^n_\varkappa$ which are linearly independent. Moreover, $\mathcal{O}_\varkappa$ may be chosen in such a way that every subbundle is locally trivial, i.e. $P|\pi^{-1}\mathcal{O}_\varkappa$ is a diffeomorphic map of $G\times\mathcal{O}_\varkappa$. Thus we can choose a basic system of linear forms $\vartheta^1_\varkappa, \ldots, \vartheta^n_\varkappa, \eta^1_\varkappa, \ldots, \eta^r_\varkappa$ ($r = \dim G$) in every bundle $P|\pi^{-1}\mathcal{O}_\varkappa$. Since subsequently we shall restrict ourselves to considerations of one such subbundle, we shall usually leave out the indices $\varkappa$. We choose our basic forms in such a way that they satisfy the following equations:

$$(44)\qquad d\vartheta^a = \vartheta^a_\mu \wedge \vartheta^\mu,$$

$$(45)\qquad d\eta^k = \tfrac{1}{2}C^k_{bd}\eta^b \wedge \eta^d + \eta^k_\nu\vartheta^\nu.$$

The coefficients C^a_{bd} are the structure constants of G. The existence of the forms ϑ^a_μ follows from a theorem of Frobenious, because the forms ϑ^a

are expressible by local coordinates and their differentials. The existence of the forms η^k and η^k_ν which satisfy (45) follows from the existence of trivial (flat) connections in $P|\pi^{-1}\mathcal{O}_\varkappa$. It may be verified that the form of equations (45) is invariant under the following transformations:

$$\eta \to \eta' = (\operatorname{adj} g^{-1})\eta + g^{-1}dg,$$

where $\eta = \eta^j e_j$ and g is an arbitrary element of G, or under a transformation

$$\eta \to \eta'' = \eta + \varrho,$$

where $\varrho = \gamma^a_\nu \vartheta^\nu e_a$ and γ^a_ν satisfies a system of equations of the form

$$d\gamma^a_\nu + \gamma^a_\mu \vartheta^\mu_\nu + C^a_{bd}\gamma^b_\nu \eta^d = \gamma^a_{\nu\lambda}\vartheta^\lambda,$$

$\gamma^a_{\nu\lambda}$ being a certain new object.

Conversely, suppose that we have a manifold M with a projection $\pi\colon M \to B$. Suppose that the manifold B is coverd by the system of domains $\mathcal{O}_\varkappa$ such that in every $\mathcal{O}_\varkappa$ there are defined fields of basic forms ϑ^ν and η^k, such that equations (44), (45) are satisfied, C^a_{bd} being structural constants of a certain group G. Then there exists a system of diffeomorphisms $h_\varkappa\colon G \times \mathcal{O}_\varkappa \to \pi^{-1}\mathcal{O}_\varkappa$ such that we have $\eta^a_\varkappa = h^*_\varkappa \Phi^a|_{\vartheta^\nu=0}$ where the form $\Phi = \Phi^a e_a$ is the left-invariant form on G and $h^*_\varkappa$ denotes a mapping induced by $h_\varkappa$. The invariancy of equations (45) under transformation (46) implies that the diffeomorphisms $h_\varkappa$ have the following property: If $h^{-1}_\varkappa \circ h_\lambda$ is defined at a certain point over $\mathcal{O}_\varkappa \cap \mathcal{O}_\lambda$ then $h^{-1}_\varkappa \circ h_\lambda \in G$. This property allows us to provide M with a structure of a fibre bundle, provided that the projection π is not trivial. Therefore equations (44), (45) are called the *structure equations of a principal fibre bundle.*

We turn to an associated fibre bundle $W(P, B, G, F, \Pi)$. Each of its points has a representation in $P \times F$, namely a pair (p, Ω) (i.e. a frame, value of the object). All other representations of the same point are of the form $(p\cdot g^{-1}, \Lambda_g \Omega)$, where g varies over G. We may speak also of a representations of a subbundle $W|\pi^{-1}\mathcal{O}$. We fix a section $\mathcal{S} \subset P|\pi^{-1}\mathcal{O}$ and we consider the points of $\mathcal{S} \times F$. The set of these points is a representation of $\Pi^{-1}\mathcal{O}$. The representation being fixed, we choose a base of linear forms $\vartheta^1, \ldots, \vartheta^n, \zeta^1, \ldots, \zeta^N$, where

$$\zeta^K = d\Omega^K + \hat{\Omega}^K_a \eta^a$$

and $\hat{\Omega}$ were defined above. We observe that if we fix a fibre by putting $\vartheta^1 = \ldots = \vartheta^n = 0$, then the forms h^* are equal to the components of the map by h^* of the left-hand invariant form on G. Thus $\zeta^1, \ldots, \zeta^N$ are maps of the left-hand member of the invariance equations.

In the geometrical applications we always use definite fields of geometric objects. Thus it is essential to define the sections in bundles of frames and in their associate bundles. A local section in P is an integral of the system

$$\eta^a = \Lambda^a_\nu \vartheta^\nu \tag{46}$$

and a local section in W, if a representation is fixed, is an integral of the system

$$d\Omega^K + \hat{\Omega}^K_a \eta^a = \Omega^K_\nu \vartheta^\nu; \tag{47}$$

Λ^a_ν and $\hat{\Omega}^K_a$ are some new objects.

3. Prolongations. The following lemma plays an essential role in all the following constructions.

LEMMA OF CARTAN-LAPTĚV. *Suppose that we have a system of external p-forms Ψ_ν, $\nu = 1, \ldots, n$, and a system of n linear forms σ^ν. If the identity $\Psi_\nu \sigma^\nu = 0$ holds, then there exists a system of forms $\Xi_{\mu\nu}$ such that we have*

$$\Psi_\nu = \Xi_{\mu\nu} \wedge \sigma^\mu,$$

$\Xi_{\mu\nu}$ being of degree $p-1$ and being symmetric.

The proof of this lemma does not essentially differ from the proof of Cartan's classical lemma for linear forms.

Now we shall prolong equations (44), (45). We differentiate externally (45) and, introducing simple reduction and using (44), we obtain the following equations:

$$(d\eta^a_\nu - \eta^a_\mu \wedge \vartheta^\mu_\nu - C^a_{bd}\eta^b_\nu \wedge \eta^d) \wedge \vartheta^\nu = 0.$$

In view of the above lemma we see that there exists a new system of forms $\eta^a_{\nu\mu}$ such that

$$d\eta^a_\nu = \eta^a_\mu \wedge \vartheta^\mu_\nu + C^a_{bd}\eta^b_\nu \wedge \eta^d + \eta^a_{\nu\lambda} \wedge \vartheta^\lambda. \tag{48}$$

If we fix a fibre by putting $\vartheta^1 = \ldots = \vartheta^\nu = 0$, then equations (48) become the Maurer-Cartan equations of a certain Lie group $\mathfrak{G}$, which contains G as a subgroup. Then the forms ϑ^ν, η^a, η^a_μ altogether constitute the fundamental system of forms of a certain principal fibre bundle $\bar{P}(B, \bar{G}, \bar{\pi})$, which includes the original bundle P. We have to note that if $\bar{P}(B, \bar{G}, \bar{\pi})$ and $\bar{P}'(B, \bar{G}', \bar{\pi})$ are two bundles obtained by the described prolongation of $P(B, G, \pi)$, then $\bar{P}$ and $\bar{P}'$ are isomorphic. Although the Cartan-Laptěv Lemma does not ensure the uniqueness of the forms η^a_μ. Two such systems of forms can differ only in a linear combination of the basic forms ϑ^ν. We have seen that two such systems define isomorphic groups.

An important special case is a linear one. The corresponding group is a multiplicative group of $n \times n$ matrices. We denote it by L^1_n. The structure equations in this case have the form

$$(49)\qquad \begin{aligned} d\vartheta^\nu &= \vartheta^\nu_\mu \wedge \vartheta^\mu, \\ d\vartheta^\mu_\lambda &= -\vartheta^\mu_\varrho \wedge \vartheta^\varrho_\lambda + \vartheta^\mu_{\lambda\nu} \wedge \vartheta^\nu. \end{aligned}$$

The first prolongation gives us the forms $\vartheta^\mu_{\lambda\nu\varrho}$, which appear in the equations

$$d\vartheta^\mu_{\lambda\nu} = -\vartheta^\mu_\varrho \wedge \vartheta^\varrho_{\lambda\nu} + \vartheta^\mu_{\lambda\varrho} \wedge \vartheta^\varrho_\nu - \vartheta^\mu_{\nu\varrho} \wedge \vartheta^\varrho_\lambda + \vartheta^\mu_{\lambda\nu\varrho} \wedge \vartheta^\varrho.$$

Together with (49) they define a group L^2_n and a corresponding bundle $P^2(B, L^2_n, \bar{\pi})$ (see p. 19).

We can repeat this process step by step. The result of this prolongation is a sequence of forms which determine the structure of the prolonged bundles $P^r = P(B, L^r_n)$. (Here we do not write the canonical projections $P \to B$.) The prolonged groups are isomorphic with differential groups of order r and dimension n.

We shall now explain the prolongation of associated structures. We turn to equations (47) of a field of geometric objects. We differentiate both members and we make use of formulae (42), (44), (45). We obtain

$$\hat{\Omega}^K_a \eta^a_\nu \wedge \vartheta^\nu = d\Omega^K_\nu \wedge \vartheta^\nu + \Omega^K_\mu \vartheta^\mu_\nu \wedge \vartheta^\nu.$$

By using the Cartan-Laptěv Lemma we obtain the system

$$(50)\qquad d\Omega^K_\nu + \Omega^K_\mu \vartheta^\mu_\nu - \Omega^K_a \eta^a_\nu = \Omega^K_{\nu\lambda} \vartheta^\lambda$$

where $\Omega^K_{\nu\lambda}$ are some new functions. This system is completely integrable and it defines a field of a certain prolonged object $\{\Omega^K, \Omega^K_a\}$. Repeating this process we obtain a sequence of prolonged objects $\{\Omega^K, \Omega^K_a, \ldots, \Omega^K_{a_1\ldots a_m}\}$. Such a process of prolongation may yield zeros at a certain step or may be continued to infinity. The prolonged components $\Omega^K_{a_1\ldots a_s}$ always constitute a paracomitant of the exit Ω^K (in the sense of Definition 4). It may be shown by examining the corresponding invariance equations that the objects $\{\Omega^K, \Omega^K_a\}$ and $\{\Omega^K, \partial_a \Omega^K\}$, which contain the first Pfaff derivatives of Ω, are equivalent. The prolonged equations (50) together with the exit equation (47) define a section in a prolonged bundle, which is associated with those obtained by prolongations of the exit principal bundle. It should be noted that the Cartan-Laptěv Lemma does not ensure the uniqueness of the prolongation of the structure equations. If η^a_ν and $\bar{\eta}^a_\nu$ are two systems of linear forms which are obtained by a prolongation method and both satisfy (48), then there holds a relation $\eta^a_\nu - \bar{\eta}^a_\nu = p^a_{\nu\mu}\vartheta^\mu$ where the coefficients $p^a_{\mu\nu}$ are symmetric. But if we once fix a certain prolongation of the principal bundle, then the prolongation of a field of objects in an associated bundle is uniquely determined.

Remark. In the above outline we have not dealt with the subgroups of L^r_n, or with the so-called *resistant elements*. This notion is an extension of the line elements of Cartan. The theory of the related connections has been treated by N. Takizawa [44].

For the general theory we refer the reader to the cited papers of G. F. Laptěv, P. I. Shvěykin, A. M. Vasilyev, V. Bliznikas and L. Ě. Ěvtushik.

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