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OPERATORS ON VN(G) *COMMUTING WITH* $A(G)$

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1. Introduction. Let G be a locally compact group; we denote by VN(G) the von Neumann algebra generated by the operators of left translations by elements of G, and by $A(G)$ its predual, as in [2]. For every v in $A(G)$ we consider the linear operator $T \to vT$ on VN(G) defined by requiring the equation

$$(vT, u) = (T, uv)$$

to hold for every u in $A(G)$. We call $\mathfrak{A}$ the *set of bounded linear operators* Φ on VN(G) commuting with $A(G)$, i.e. such that

$$v\Phi(T) = \Phi(vT)$$

for each v in $A(G)$ and T in VN(G). It is immediate to check that $\mathfrak{A}$ is an algebra.

In Section 2 we identify the space $\mathscr{R}$ of those elements in $\mathfrak{A}$ which are weak-* continuous on VN(G) with the algebra $B_2(G)$ of multipliers of $A(G)$, and we show that $\mathscr{R}$ is a subset of the center of $\mathfrak{A}$. In Section 3 an equivalence chain is proved, connecting the commutativity of $\mathfrak{A}$, the discreteness of G, and the fact that $\mathfrak{A}$ coincides with $\mathscr{R}$. Section 4 is devoted to the identification of $\mathfrak{A}$ with the dual of the subspace of $\mathfrak{A}$ consisting of uniformly continuous functionals (UBC) on $A(G)$, defined as in [3] and [4], in the particular case in which G is an amenable group.

The problems discussed in Section 2 are in a sense the "dual" problems of the ones treated in [6] and [7]. Some parts of this paper*, in particular Section 4, have been proved independently by Lau [4].

2. In this section we prove two theorems relating the algebra $B_2(G)$ of multipliers of $A(G)$ with $\mathfrak{A}$.

THEOREM 1. *There is a natural isomorphism λ between $B_2(G)$ and the subalgebra $\mathscr{R}$ of the weak-* continuous elements in $\mathfrak{A}$, where $\lambda(F) = \Phi$, Φ being*

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a weak- continuous element of* $\mathfrak{A}$, *and* F *a multiplier of* $A(G)$ *defined by the equation*

$$(1)\qquad \big(F(u), T\big) = \big(u, \Phi(T)\big)$$

for every u *in* $A(G)$ *and* T *in* $\mathrm{VN}(G)$.

Moreover, for any F *in* $B_2(G)$ *we have*

$$[\lambda(F)](L_x) = F(x)L_x,$$

where L_x *is the left translation by* x *on* $L_2(G)$.

Proof. Relation (1) establishes an isomorphism between the Banach algebra of continuous linear operators on $A(G)$ and the Banach algebra of weak-* continuous linear operators on $\mathrm{VN}(G)$. We must show that if F is a multiplier of $A(G)$, then $\lambda(F)$ is in $\mathfrak{A}$ and, conversely, that if Φ is in $\mathfrak{A}$, then $\lambda^{-1}(\Phi)$ is in $B_2(G)$.

The first assertion follows from the fact that for all u, v in $A(G)$ and T in $\mathrm{VN}(G)$ we have

$$\begin{aligned}\big(v[\lambda(F)](T), u\big) &= \big([\lambda(F)](T), uv\big) = \big(T, F(uv)\big) = \big(T, vF(u)\big)\\ &= \big(vT, F(u)\big) = \big([\lambda(F)](vT), u\big)\end{aligned}$$

and, therefore,

$$v[\lambda(F)](T) = [\lambda(F)](vt).$$

The second assertion follows from the fact that for all u, v in $A(G)$ and x in G we have

$$\begin{aligned}\big[[\lambda^{-1}(\Phi)](uv)\big](x) &= \big(L_x, [\lambda^{-1}(\Phi)](uv)\big)\\ &= \big(\Phi(L_x), uv\big) = \big(v\Phi(L_x), u\big) = \big(\Phi(vL_x), u\big)\\ &= \big(vL_x, [\lambda^{-1}(\Phi)](u)\big) = \big(L_x, v[\lambda^{-1}(\Phi)](u)\big)\\ &= \big[v[\lambda^{-1}(\Phi)](u)\big](x).\end{aligned}$$

The last statement in the theorem is true, since

$$\begin{aligned}\big((\lambda(F))(L_x), u\big) &= \big(L_x, F(u)\big) = \big(F(u)\big)(x) = F(x)u(x)\\ &= \big(F(x)L_x, u\big) \quad \text{for any } u \text{ in } A(G).\end{aligned}$$

COROLLARY 1. $\mathscr{R}$ *is abelian.*

Proof. $B_2(G)$ is obviously abelian, and so is $\mathscr{R}$ by Theorem 1.

THEOREM 2. $\mathscr{R} \subseteq \mathscr{Z}(\mathfrak{A})$ *if* $\mathscr{Z}(\mathfrak{A})$ *is the center of* $\mathfrak{A}$.

We need first the following lemmas:

LEMMA 1. *If* F *is in* $B_2(G)$, *then* $F(v)T = [\lambda(F)](vT)$ *for all* v *in* $A(G)$ *and* T *in* $\mathrm{VN}(G)$.

Proof. Given v in $A(G)$ and T in $\mathrm{VN}(G)$, for all u in $A(G)$ we have

$$\big(F(v)T, u\big) = \big(T, F(v)u\big) = \big(T, F(vu)\big) = \big(T, vF(u)\big) = \big(vT, F(u)\big)$$
$$= \big([\lambda(F)](vT), u\big).$$

LEMMA 2. *If $vT = 0$ for all T in $A(G)$, then $T = 0$.*

For the proof see [2], Proposition (4.6).

Proof of Theorem 2. Let $\Phi_1 \in \mathfrak{R}$ and $\Phi_2 \in \mathfrak{A}$. By Theorem 1 there is an element F of $B_2(G)$ such that $\lambda(F) = \Phi_1$. Then, by Lemma 1, for any v in $A(G)$ and T in $\mathrm{VN}(G)$ we have

$$\begin{aligned} v\Big(\Phi_1\big(\Phi_2(T)\big)\Big) &= v\Big([\lambda(F)]\big(\Phi_2(T)\big)\Big) \\ &= [\lambda(F)]\big(v\Phi_2(T)\big) = F(v)\big(\Phi_2(T)\big) \\ &= \Phi_2\big(F(v)T\big) = \Phi_2\big([\lambda(F)](vT)\big) = \Phi_2\big(\Phi_1(vT)\big) \\ &= v\Big(\Phi_2\big(\Phi_1(T)\big)\Big) \end{aligned}$$

and therefore, by Lemma 1, $\Phi_1\Phi_2 = \Phi_2\Phi_1$.

COROLLARY 2. *The algebra of linear bounded operators on $\mathrm{VN}(G)$ commuting with $A(G)$ is the same as the algebra of bounded linear operators commuting with $B_2(G)$ (in the sense of the correspondence established in Theorem 1).*

3. The aim of this section is to study the commutativity of $\mathfrak{A}$.

LEMMA 3. *L_x is the unique norm-one element in $\mathrm{VN}(G)$ such that $vT = v(x)T$ for all v in $A(G)$.*

Proof. It follows from Theorem 1 that L_x has this property. Let T be any other element in $\mathrm{VN}(G)$ with norm one. There must be at least one point $x_0 \neq x$ in the support of T. Let us take a v in $A(G)$ such that $v(x) \neq 0$ and $v(x_0) = 0$. Then $x_0 \notin \operatorname{supp} vT$, and so T and vT have different supports, whence the property in the lemma cannot hold for T.

PROPOSITION 1. *The set of common eigenvectors for all elements in $\mathfrak{A}$ is the set of all elements of $\mathrm{VN}(G)$ of the form aL_x for x in G.*

Proof. Let $\Phi \in \mathfrak{A}$. Then

$$v\Phi(L_x) = \Phi(vL_x) = \Phi\big(v(x)L_x\big) = v(x)\Phi(L_x)$$

for all v in $A(G)$, and so, by Lemma 3,

$$\Phi(L_x) = \varphi(\Phi, x)L_x,$$

where φ is a complex number depending on Φ and x. If T in $\mathrm{VN}(G)$ is not of the form aL_x for some x in G, then its support contains at least two different points x_1 and x_2. Then, as in Lemma 3, T cannot be an eigenvector for any v in $A(G)$ such that $v(x_1) = 0$ and $v(x_2) \neq 0$.

THEOREM 3. *Thfollowing statements are equivalent:*

(I) *G is discrete,*

(II) $\mathfrak{A} = \mathscr{R}$,

(III) *$\mathfrak{A}$ is abelian.*

Proof. (I) $\Rightarrow$ (II). If G is discrete, then $VN(G) \subset L_2(G)$, for — denoting by δ_x the function whose value is 1 on x and 0 elsewhere, and by e the identity of G — we have

$$Tf = TL_{\delta_e}f = T(\delta_e * f) = (T\delta_e) * f,$$

$T\delta_e$ being an L_2-function. If we put $(T\delta_e)(x) = \hat{T}(x)$, we have $Tf = \hat{T}(x)f$ for any T in $VN(G)$. Since δ_x is in $A(G)$, we also have $T(x) = (T, \delta_x)$. Then it follows from Proposition 1 and the equality $(L_x, \delta_x) = 1$ for each Φ in $\mathfrak{A}$ and for each T in $VN(G)$ that

$$\begin{aligned}(\Phi T)^{\wedge}(x) &= \big(\Phi(T), \delta_x\big) = \big(\Phi(T), \delta_x \delta_x\big) \\ &= \big(\delta_x \Phi(T), \delta_x\big) = \big(\Phi\big(\delta_x(T)\big), \delta_x\big) = \big(\Phi\big(\hat{T}(x) L_{\delta_x}\big), \delta_x\big) \\ &= \big(\Phi\big(\hat{T}(x) L_x\big), \delta_x\big) = \hat{T}(x)\big(\varphi(\Phi, x) L_x, \delta_x\big) \\ &= \hat{T}(x)\varphi(\Phi, x).\end{aligned}$$

Therefore, to each Φ in $\mathfrak{A}$ there corresponds a multiplication by the function $\varphi(\Phi, x)$ on the space of the functions $\hat{T}(x)$. Now let T_n be a sequence in $VN(G)$ weakly converging to zero. For any Φ in $\mathfrak{A}$ we have

$$\lim_{n\to\infty} \big(\Phi(T_n), \delta_x\big) = \lim_{n\to\infty} \varphi(\Phi, x)\big(\hat{T}_n(x)\big) = 0,$$

and, simply by taking finite sums, we get

$$\lim_{n\to\infty} \big(\Phi(T_n), v\big) = 0$$

for any v with compact support in $A(G)$, since G is discrete. But the elements of $A(G)$ with compact support are norm dense in $A(G)$, whence the implication follows.

For (II) $\Rightarrow$ (III) see Corollary 1.

(III) $\Rightarrow$ (I). If G is not discrete, then there are two different means μ_1 and μ_2 on G by [5]. Let us write

$$\Phi_i(T) = \mu_i(T) I \quad (i = 1, 2).$$

The Φ_i's are different norm continuous linear operators on $VN(G)$ commuting with $A(G)$, since

$$\Phi_i(vT) = \Phi_i(vT) I = \Phi_i(T) v(e) I = \Phi_i(T) v I = v \Phi_i(T).$$

But Φ_1 and Φ_2 do not commute, since

$$\Phi_i\big(\Phi_j(T)\big) = \Phi_i\big(\mu_j(T) I\big) = \mu_j(T) I = \Phi_j(T).$$

4. In this section we deal with a particular case of an amenable group and show the relation between $\mathfrak{A}$ and the dual of a certain subspace of VN(G).

LEMMA 4. *Let G be an amenable group; then $\{vT\colon v\in A(G),\ T\in \mathrm{VN}(G)\}$ is a closed subspace of* VN(G). *It coincides with the set of elements in* VN(G) *for which*

$$\lim v_a T = T,$$

where v_a is an approximate identity in $A(G)$.

Proof. We apply Corollary 2.4 of [1] with $\mathfrak{A} = A(G)$, $X = \mathrm{VN}(G)$ and $\sigma(v)T = vT$ for v in $A(G)$ and T in VN(G). The mapping σ is faithful, since $vT = 0$ for each T implies $vL_x = v(x)L_x = 0$ for each x in G, and then $v\equiv 0$.

Definition 1. The subset of VN(G), described in Lemma 4, will be called UBC$(\hat{G})$, as in [3] and [4].

Remark. *If Ψ is in $\mathfrak{A}$ and $\Psi|_{\mathrm{UBC}(G)} = 0$, then Ψ is the zero operator.*

Proof. We have $v\Psi(T) = \Psi(vT) = 0$ for all v in $A(G)$ and T in VN(G). Therefore, by Lemma 2, $\Psi(T) = 0$ for all T and $\Psi\equiv 0$.

Definition 2. Let Φ be an element of UBC* (dual of UBC); then we can define the operator $\sigma(\Phi)$ on VN(G) by requiring that the equation

$$\big(\sigma(\Phi)(T), v\big) = (\Phi, vT)$$

hold for all T in VN(G) and v in $A(G)$.

THEOREM 4. *For any amenable group G the mapping σ defined above is an isometric bijection from* UBC$(\hat{G})^*$ *onto* $\mathfrak{A}$.

Proof. The operator $\sigma(\Phi)$ is in $\mathfrak{A}$. Its linearity is obvious; for all u in $A(G)$ and T in VN(G) we have

$$\begin{aligned}\big(v[(\sigma(\Phi))(T)], u\big) &= \big([(\sigma(\Phi))(T)], uv\big)\\ &= \big(\Phi, (uv)T\big) = \big(\Phi, u(vT)\big) = \big((\sigma(\Phi))(vT), u\big)\end{aligned}$$

and, therefore, $\sigma(\Phi)$ commutes with the action of v on VN(G). Finally,

$$\begin{aligned}\|\sigma(\Phi)\| &= \sup_{T\in[\mathrm{VN}(G)]_1}\|[\sigma(\Phi)](T)\| = \sup_{\substack{T\in[\mathrm{VN}(G)]_1\\ v\in[A(G)]_1}}\big([\sigma(\Phi)(T)], v\big)\\ &= \sup_{\substack{T\in[\mathrm{VN}(G)]_1\\ v\in[A(G)]_1}}(\Phi, vT) = \sup_{R\in[\mathrm{UBC}(\hat{G})]_1}(\Phi, R) = \|\Phi\|.\end{aligned}$$

The mapping σ is one-to-one, since if $\sigma(\Phi) = 0$, then for all T in VN(G) and v in $A(G)$ we have

$$0 = \big([\sigma(\Phi)(T)], v\big) = (\Phi, vT) \quad\text{and}\quad \Phi\equiv 0.$$

The mapping σ is onto, since the equation $\big(X(T), v\big) = (\Phi, vT)$ has clearly a solution in X for any Φ.

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