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AN EXAMPLE OF A NON-COMPACT LOCALLY COMPACT ARCWISE CONNECTED METRIC SPACE WITH THE FIXED POINT PROPERTY

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Although most results of the theory of fixed points concern the case of compact spaces, there is an interest in investigating the fixed point property for spaces which are non-compact. In some cases, however, this property seems to be incompatible with non-compactness. For instance, the following two theorems are well-known (see [2], p. 32 and 35):

THEOREM 1. *If X is a normal space which contains a homeomorphic image of the half-open interval as a closed subset, then there exists a null-homotopic mapping $f\colon X \to X$ such that f has no fixed point.*

THEOREM 2. *If X is a non-compact, locally compact, locally connected metric space, then X does not have the fixed point property.*

As a consequence of Theorem 1 we obtain the following corollary:

COROLLARY. *If X is a non-compact, locally compact, normal space with the fixed point property, then the one-point compactification of X is not arcwise connected.*

A well-known example (ibidem) of the cone over an infinite discrete space shows that local compactness is a necessary condition in Theorem 2. Observe that this example is arcwise connected. The question has been raised by William Bonnice (oral communication) as to whether local connectedness also is necessary in Theorem 2. We answer the question in the affirmative by constructing an example of a non-compact, locally compact, arcwise connected metric space X with the fixed point property(1). By the corollary, the one-point compactification $Y = X \cup \{p\}$ of X is not arcwise connected. Thus the space Y is an example of a continuum

(1) The author wishes to thank Dr. W. Bonnice for interesting discussions we had at the Middle East Technical University, Ankara, Turkey. The subject of this paper was also discussed with a group of topologists at the University of Stockholm, Stockholm, Sweden.

which is not arcwise connected, but becomes arcwise connected after removing the point p. The latter phenomenon is not possible when arcwise connectedness is replaced by local connectedness.

Example. Let P be the pseudo-arc and let $p', p'' \in P$ be points such that P is an irreducible continuum between p' and p''. We take a sequence $p_0, p_1, p_2, \ldots$ of different points on the plane such that $p_1, p_2, \ldots$ converge to p_0. Let us find a sequence $P_1, P_2, \ldots$ of topological copies of P such that $p_i, p_{i+1} \in P_i$, where p_i and p_{i+1} correspond to p' and p'', respectively, the sets $P_1, P_2, \ldots$ converge to $\{p_0\}$, but $p_0 \notin P_i$ and $P_i \cap P_{i+1} = \{p_{i+1}\}$, $P_i \cap P_j = \emptyset$ $(i, j = 1, 2, \ldots)$ and $|i-j| > 1$. Then $C = \{p_0\} \cup P_1 \cup P_2 \cup \ldots$ is a chainable (2) irreducible continuum between p_0 and p_1.

Let us denote $I = [0, 1]$, $D_0 = I \times I$ and $A_0 = I \times \{\frac{1}{2}\}$. It follows (see [1], p. 654) that there exists a sequence $D_1, D_2, \ldots$ of disks contained in D_0 such that $(0, \frac{1}{2})$, $(1, \frac{1}{2}) \in D_i$, $D_{i+1} \subset D_i$ for $i = 1, 2, \ldots$ and

$$\bigcap_{i=0}^{\infty} D_i \underset{\text{top}}{=} C,$$

where $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ correspond to p_0 and p_1, respectively. By a slight modification of D_i, we can also guarantee that some segments of the boundary of D_0 having the centers at $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ are contained in D_i for $i = 1, 2, \ldots$ Let $A_i \subset D_i$ be an arc joining $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ such that A_i has only its end points on the boundary of D_i. Observe that D_i is a (closed) neighbourhood of A_i in D_0 for $i = 0, 1, \ldots$ We denote

$$E_i = D_i \times [0, 2^{-i}], \qquad F_i = D_0 \times [2^{-i-i}, 2^{-i}] \quad \text{and} \quad G_i = E_i \cap F_i \quad \text{for } i = 0, 1, \ldots$$

Thus $E_0, E_1, \ldots$ are 3-cells such that

$$E_{i+1} \subset E_i \quad \text{and} \quad K = \bigcap_{i=0}^{\infty} E_i = \bigcap_{i=0}^{\infty} D_i \times \{0\} \underset{\text{top}}{=} C \quad \text{for} \quad i = 0, 1, \ldots$$

There exists a homeomorphism g_i of D_i onto itself such that $g_i(A_i) = A_{i+1}$ and g_i is the identity on the boundary of D_i (see [3], p. 535). The set G_i is also a 3-cell, and we define a homeomorphism g_i' of the boundary of G_i onto itself by setting

$$g_i'(x, 2^{-i-1}) = \big(g_i(x), 2^{-i-1}\big) \quad \text{for } x \in D_i$$

and taking g_i' the identity on the set $\mathrm{bd}\, G_i \setminus (D_i \times \{2^{-i-1}\})$. Let h_i' be a homeomorphism of G_i onto itself such that h_i' is an extension of g_i'.

Then the mapping $h_i \colon E_i \to E_i$ defined by the formula

$$h_i(x, t) = \begin{cases} \big(g_i(x), t\big) & \text{for } 0 \leqslant t \leqslant 2^{-i-1}, \\ h_i'(x, t) & \text{for } 2^{-i-1} \leqslant t \leqslant 2^{-i} \end{cases}$$

(2) The class of chainable continua coincides with that of *arc-like* curves, i. e. those which can be mapped onto arcs by means of mappings with arbitrarily small diameters of point-inverses. Sometimes, chainable continua are also called "snake-like".

is a homeomorphism of E_i onto itself such that h_i is an extension of h'_i. Moreover, h_i is the identity on the set $H_i = \mathrm{bd}\, E_i \setminus (D_i \times \{0\})$ for $i = 0, 1, \ldots$ and we have

$$h_{i-1} \ldots h_0(A_0 \times [2^{-i-1}, 2^{-i}]) = A_i \times [2^{-i-1}, 2^{-i}] \quad \text{for } i = 1, 2, \ldots$$

Clearly, the frontier of G_i in F_i is contained in H_i and $h_i|G_i = h'_i|G_i$ is a homeomorphism. Since D_i is a neighbourhood of A_i in D_0, it follows that G_i is a neighbourhood of

$$h_i \ldots h_0(A_0 \times [2^{-i-1}, 2^{-i}]) = h_i(A_i \times [2^{-i-1}, 2^{-i}])$$

in F_i, for $i = 0, 1, \ldots$ But $A_0 \times [2^{-i-1}, 2^{-i}] = I \times \{\frac{1}{2}\} \times [2^{-i-1}, 2^{-i}]$, and thus there exist numbers $a_i, b_i \in I$ such that $a_i < \frac{1}{2} < b_i$ and

$$h_i \ldots h_0(I \times [a_i, b_i] \times [2^{-i-1}, 2^{-i}]) \subset G_i \quad \text{for } i = 0, 1, \ldots$$

The points $q_i = (\frac{1}{2}, 2^{-i})$ and $q_{i+1} = (\frac{1}{2}, 2^{-i-1})$ lie on the boundary of the rectangle $[a_i, b_i] \times [2^{-i-1}, 2^{-i}]$ in which we can find a topological copy Q_i of the pseudo-arc P such that $q_i, q_{i+1} \in Q_i$, where q_i and q_{i+1} correspond to p' and p'', respectively. We also require that the sets $Q_0, Q_1, \ldots$ converge to $\{(\frac{1}{2}, 0)\}$ and $Q_i \cap Q_{i+1} = \{q_{i+1}\}$ for $i = 0, 1, \ldots$ Then the union $\{(\frac{1}{2}, 0)\} \cup \cup Q_0 \cup Q_1 \cup \ldots$ is homeomorphic to the continuum C. We get

$$L = \{(0, \tfrac{1}{2}, 0)\} \cup \Big(\{0\} \times \bigcup_{i=0}^{\infty} Q_i\Big) \underset{\text{top}}{=} C$$

and $h_i \ldots h_0$ is the identity on the set $\{0\} \times Q_i$. Moreover, we obtain the inclusions $h_i \ldots h_0(I \times Q_i) \subset G_i \subset E_i$, for $i = 0, 1, \ldots$, which guarantee that the set

$$Z = K \cup \bigcup_{i=0}^{\infty} h_i \ldots h_0(I \times Q_i)$$

is compact and $L \subset Z$. The set L has only the point $(0, \frac{1}{2}, 0)$ in common with the side $D_0 \times \{0\}$ on which the set K is located. Thus $K \cap L = \{(0, \frac{1}{2}, 0)\}$ and the sets K and L are both homeomorphic to C. Under these homeomorphisms, the points $(0, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, 0)$ and $(1, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, 1)$ correspond to the points p_0 and p_1, respectively. Consequently, there exists a homeomorphism h of K onto L such that $h\big((0, \frac{1}{2}, 0)\big) = (0, \frac{1}{2}, 0)$ and $h\big((1, \frac{1}{2}, 0)\big) = (0, \frac{1}{2}, 1)$.

Let us distinguish yet another copy of C. It is the set

$$M = \{(1, \tfrac{1}{2}, 0)\} \cup \Big(\{1\} \times \bigcup_{i=0}^{\infty} Q_i\Big)$$

and $h_i \ldots h_0$ is the identity on $\{1\} \times Q_i$, whence $M \subset Z$ and $K \cap M = \{(1, \frac{1}{2}, 0)\}$, $L \cap M = \emptyset$.

The segment $N = I \times \{\frac{1}{2}\} \times \{1\}$ is contained in H_0 and, therefore, h_0 is the identity on N. It follows that $N \subset Z$, $K \cap N = \emptyset$, $L \cap N = \{(0, \frac{1}{2}, 1)\}$ and $M \cap N = \{(1, \frac{1}{2}, 1)\}$.

Now, let R denote the equivalence relation in Z such that the equivalence classes under R are 1° the one-point sets $\{z\}$ for $z = (0, \frac{1}{2}, 0)$ or $z \notin K \cup L \cup M \cup N$, 2° the two-point sets $\{z, h(z)\}$ for $z \in K$ and $(0, \frac{1}{2}, 0) \neq z \neq (1, \frac{1}{2}, 0)$, and 3° the set $M \cup N$.

We define $Y = Z/R$ and let $f\colon Z \to Y$ be the quotient mapping generated by R.

Put $p = f\big((0, \frac{1}{2}, 0)\big)$, $q = f\big((1, \frac{1}{2}, 0)\big)$.

By setting $X = Y \setminus \{p\}$, we complete the construction of our example. It is rather apparent that the quotient space generated by R remains metrizable. Thus Y is a compact metric space, and X is a non-compact locally compact metric space which will be shown to be arcwise connected and to possess the fixed point property.

Indeed, let us first notice that the mapping f is one-to-one on K. Therefore, we can consider K as a subset of Y if we identify z with $f(z)$ for $z \in K$. Next, observe that $h(x)$ is a point of L different from $(0, \frac{1}{2}, 0)$ for $x \in K \setminus \{p, q\}$. Then we also have

$$h(x) = \big(0, u(x), v(x)\big) \neq (0, \tfrac{1}{2}, 1),$$

where $\big(u(x), v(x)\big) \in Q_{i(x)}$, and the set

$$J(x) = fh_{i(x)} \ldots h_0\big(I \times \{u(x)\} \times \{v(x)\}\big)$$

is an arc in X with end points x and q, for any point $x \in K \setminus \{p, q\}$. Since $f(M \cup N) = \{q\}$, the set X is the union of the arcs $J(x)$, where $x \in K$ and $p \neq x \neq q$. It follows that X is arcwise connected. Furthermore, since h is one-to-one, the common part of two different arcs $J(x_1)$ and $J(x_2)$ consists of the only point q. Then $J(x_1) \cup J(x_2)$ is an arc again. If we prove that subarcs of such arcs are the only possible arcs in X, this will imply that the union of any increasing sequence of arcs in X is contained in an arc. It is known that the latter property implies the fixed point property (see [5], p. 493). Thus what remains to be done is the following lemma:

Lemma. *Each arc contained in X is contained in the union of two arcs from the collection $\{J(x)\colon x \in K \setminus \{p, q\}\}$.*

Proof. Suppose on the contrary that there exists a bad arc $B \subset X$ which is not contained in the union of any two arcs from this collection. Then there exist three different points $x_0, x_1, x_2 \in K \setminus \{p, q\}$ such that $B \cap \big(J(x_j) \setminus \{q\}\big) \neq \emptyset$ for $j = 0, 1, 2$. Let $b_j \in B$ be points such that $b_j \in J(x_j)$ and $b_j \neq q$ $(j = 0, 1, 2)$. The point q may cut the arc B between only two, if any, pairs of the points b_0, b_1 and b_2. Without loss of generality, we can assume that q does not cut B between b_0 and b_1. Consequently, there exists an arc $A \subset B \setminus \{q\} \subset Y \setminus \{q\}$ such that $b_0 \in A$ and $b_1 \in A$. Since $x_0 \neq x_1$, we have $h(x_0) \neq h(x_1)$. Moreover, $h(x_0)$ is a point of L such that $(0, \frac{1}{2}, 0) \neq h(x_0) \neq (0, \frac{1}{2}, 1)$ and, by the definition of L, the set $\{0\} \times (Q_k \cup Q_{k+1})$ is a (closed) neighbourhood of $h(x_0)$ in L provided k is a suitable integer being either $i(x_0)$ or $i(x_0)-1$. Let us find a closed subset $W \subset Q_k \cup Q_{k+1}$ such that $\{0\} \times W$ is a neighbourhood of $h(x_0)$ in L which contains neither $h(x_1)$ nor $(0, \frac{1}{2}, 1)$. Then the set

$$F = fh_k \ldots h_0\big(I \times (Q_k \cap W)\big) \cup fh_{k+1} \ldots h_0\big(I \times (Q_{k+1} \cap W)\big)$$

contains the point b_0, and F does not contain the point b_1. It follows that $A \cap F$ is a proper closed subset of the arc A, and the component T of $A \cap F$ which contains b_0 must intersect the closure of $A \setminus F$ (see [3], p. 172). Let y_0 be a point of T such that y_0 belongs to the closure of $A \setminus F$; thus $y_0 \neq q$ and y_0 belongs to the frontier of F in Y.

Since h_i are homeomorphisms, we can treat the set F as being the image of $I \times W$ under f. Moreover, the action of f on $I \times W$ is restricted only to the set $\{1\} \times W$ which is transformed into the point q. In other words, F is homeomorphic to the cone over W and q is the vertex of the cone. Let π denote the projection of the set $F \setminus \{q\}$ from the vertex q onto the base $\{0\} \times W$ of the cone. We have

$$T \subset A \cap F \subset F \setminus \{q\}$$

and the set $\pi(T)$ contains the points $\pi(b_0)$ and $\pi(y_0)$. Finally, since $\{0\} \times W$ is a neighbourhood of $h(x_0)$ in L, the cone F is a neighbourhood of y in Y for any point $y \in J(x_0) \setminus \{q\}$. We conclude that y_0 does not belong to $J(x_0)$. On the other hand, b_0 belongs to $J(x_0) \setminus \{q\}$ which means that

$$\pi^{-1}\pi(b_0) = J(x_0) \setminus \{q\},$$

whence $\pi(b_0) \neq \pi(y_0)$, and $\pi(T)$ is non-degenerate. Being the continuous image of the arc T, the set $\pi(T) \subset \{0\} \times W$ must therefore contain an arc. As a result, we get an arc in $W \subset Q_k \cup Q_{k+1}$, where Q_k and Q_{k+1} are topological copies of the pseudo-arc which contains no arc. This is a contradiction and the lemma is proved.

Remark. The example as constructed above is 2-dimensional. However, utilizing the method which has been developed in [4], we could alter the construction so that the new modified continuum Y be 1-dimensional, and therefore embedable in the Euclidean 3-space. One can guess that no such continuum exists on the plane (**P 765**).

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