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On eigenvalues and eigenfunctions of an operational equation

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Introduction. Let G be a bounded Jordan-measurable domain in the space E^m of m variables $X = (x_1, \ldots, x_m)$ which can be approximated by an increasing sequence of domain G_n with regular boundaries (the boundary ∂G_n of G_n is a surface of class C^1_σ; for the definition of a surface of class C^1_σ see [6], p. 132). We do not require any regularity properties of the boundary of G.

We shall consider an operational equation of the form

$$L(u)+\mu K(u) = 0, \tag{1}$$

where

$$L(u) = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(X)\frac{\partial u}{\partial x_j}\right] - q(X)u$$

is a selfadjoint differential operator and μ is a real parameter. We make the following assumptions: $a_{ij}(X) = a_{ji}(X)$ $(i, j = 1, \ldots, m)$ are of class C^1 in $\bar{G}$, $q(X) \geqslant 0$ is continuous in $\bar{G}$ and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(X)\xi_i\xi_j$ is positive definite in $\bar{G}$. Concerning the operator K we make the following assumptions:

1° K: $\mathscr{L}^2(G) \to \mathscr{L}^2(G)$, is a linear bounded operator,

2° the subspace $\mathscr{L}^2(G) \cap C(G)$ of continuous functions is invariant with respect to K,

3° K is symmetric, i.e.

$$\big(\varphi, K(\psi)\big) = \int_G \varphi(X)K(\psi)\,dX = \int_G \psi(X)K(\varphi)\,dX = \big(\psi, K(\varphi)\big)$$

for $\varphi, \psi \in \mathscr{L}^2(G)$,

4° K is positive, i.e. $\big(\varphi, K(\varphi)\big) > 0$ for $\varphi \neq 0$.

We shall also consider a generalized boundary condition (cf. [1] and [2]) which in the case where the boundary ∂G is regular may be written in the form

$$(2) \qquad \frac{du}{d\nu} - h(X)u = 0 \quad \text{on } \partial G - \Gamma, \qquad u = 0 \quad \text{on } \Gamma,$$

where Γ denotes an $(m-1)$-dimmensional part of ∂G (Γ being connected or not); in extreme cases Γ may be the whole boundary of G, or an empty set. Here $h(X)$ is a non-negative continuous function in $\bar{G}$ and $du/d\nu$ is the transversal derivative of u with respect the operator $L(u)$, i.e.,

$$(3) \qquad \frac{du}{d\nu} = \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial u}{\partial x_i}\cos(n, x_j),$$

n being the interior normal to ∂G.

We will consider the eigenvalues and eigenfunctions corresponding to equation (1) and condition (2) (we shall shortly say: eigenvalues and eigenfunctions of problem (1), (2)). The eigenvalues and eigenfunctions of problem (1), (2) will be defined variationally. For this purpose let us write

$$(4) \qquad D(\varphi, \psi) = \int_G \Big[\sum_{i,j=1}^{m} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + q\varphi\psi\Big]dX + \int_{\partial G - \Gamma} h\varphi\psi\, dS,$$

$$(5) \qquad H(\varphi, \psi) = \int_G \varphi K(\psi)\, dX = \big(\varphi, K(\psi)\big).$$

The bilinear forms (4) and (5) are defined in the space $\mathscr{D}$ (for the definition of the space $\mathscr{D}$ see [1]) and have all the fundamental properties mentioned in [1].

1. EIGENVALUES AND EIGENFUNCTIONS OF PROBLEM (1), (2).

1. The first eigenvalue λ_1 of problem (1), (2) is defined as (comp. [1] and [4])

$$(6) \qquad \lambda_1 = \min_{\varphi \in \mathring{\mathscr{D}}} \frac{D(\varphi)}{H(\varphi)},$$

where $\mathring{\mathscr{D}}$ is the subclass of $\mathscr{D}$ of functions φ such that $\varphi = 0$ on Γ (in the generalized sense), and the first eigenfunction u_1 is that φ at which the minimum (6) is attained.

Having defined the eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenfunctions $u_1, \ldots, u_n$, we put

$$(7) \qquad \lambda_{n+1} = \min_{\varphi \in \mathscr{K}_n} \frac{D(\varphi)}{H(\varphi)},$$

where $\mathscr{K}_n$ is the subclass of $\mathring{\mathscr{D}}$ consisting of the functions φ satisfying the orthogonality conditions

$$(8) \qquad H(\varphi, u_i) = 0 \quad \text{for } i = 1, \ldots, n,$$

and u_{n+1} is that $\varphi \in \mathscr{K}_n$ at which minimum (7) is attained.

We shall need the following assumption:

HYPOTHESIS Z. *Given* (1) *and* (2), *there exist a sequence of eigenvalues of* (1), (2)

$$(9) \qquad 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$$

and a corresponding sequence of eigenfunctions

$$(10) \qquad u_1(X), u_2(X), u_3(X), \ldots$$

which belong to $\mathscr{F}$ (¹).

The problem whether Hypothesis Z is satisfied under the assumptions which we have made concerning the coefficients of operator L of equation (1) will not be considered in this paper. Of course, this problem depends essentially on the form and the properties of the operator K of equation (1).

In the sequel we shall use the following formula:

$$(11) \qquad D(\varphi, \psi) + \int_G L(\varphi)\psi dX + \int_{\partial G - \Gamma} \psi\left(\frac{d\varphi}{d\nu} - h\varphi\right) dS = 0,$$

for every $\varphi \in \mathscr{F}$ and $\psi \in \mathring{\mathscr{D}}$. The proof of formula (11) is quite similar to the proof of an analogous formula in [1].

THEOREM 1. *If Hypothesis Z is satisfied, then each function u_n of sequence* (10) *satisfies equation* (1) *for* $\mu = \lambda_n$, $n = 1, 2, 3, \ldots$, *and* $u_n \in \mathscr{F}_{h,\Gamma}(G)$ (²).

The proof of this theorem is quite similar to that of an analogous theorem in [1], and is omitted.

2. The maximum-minimum property of eigenvalues of (1), (2). We shall now give another definition of eigenvalues of problem (1), (2). Let $\mathscr{V}_n$ denote a set of n functions $v_1(X), \ldots, v_n(X)$ belonging to $\mathscr{L}^2(G)$

(¹) By $\mathscr{F}$ we denote the subspace of $\mathscr{D}$ of all functions φ of class C^2 in G such that $L(\varphi) \in \mathscr{L}^2(G)$ (see [1]).

(²) By $\mathscr{F}_{h,\Gamma}(G)$ we denote the subspace of $\mathscr{F}$ of all functions φ satisfying condition (2) in the generalized sense (see [1]).

and let

$$d[\mathscr{V}_n] = \min_{u \in \overline{\mathscr{K}}_n} \frac{D(u)}{H(u)},$$

where $\overline{\mathscr{K}}_n$ is the subclass of $\mathring{\mathscr{D}}$ consisting of functions $u(X)$ satisfying the orthogonality conditions

$$H(u, v_i) = 0 \quad \text{for } i = 1, \ldots, n.$$

THEOREM 2. *If Hypothesis Z and the above assumptions are satisfied, then*

$$\lambda_{n+1} = \sup_{\mathscr{V}_n} d[\mathscr{V}_n], \tag{12}$$

where $\mathscr{V}_n$ is defined above.

The proof of this theorem is quite similar to the proof of an analogous theorem in the case of differential equations (cf. [3], p. 405 or [6], p. 289).

The proofs of the following theorems are also similar to the proofs of analogous theorems for differential equations and are omitted.

THEOREM 3. *If $\{\mu_n\}$, $\{\lambda_n\}$ and $\{\nu_n\}$ are the sequences of eigenvalues of equation* (1) *with boundary condition $u = 0$ on ∂G, with boundary condition* (2) *and with boundary condition $du/d\nu = 0$ on ∂G, respectively, then*

$$\nu_n \leqslant \lambda_n \leqslant \mu_n \quad (n = 1, 2, 3, \ldots).$$

THEOREM 4. *If $K_2 - K_1$ is a positive operator and if $\{\lambda_n^{(2)}\}$ and $\{\lambda_n^{(1)}\}$ are the sequences of eigenvalues of problem* (1), (2), *where $K = K_2$ and $K = K_1$, respectively, then*

$$\lambda_n^{(2)} \leqslant \lambda_n^{(1)} \quad (n = 1, 2, 3, \ldots).$$

2. COMPLETENESS OF EIGENVALUES OF PROBLEM (1), (2)

Let us denote by $\{\varkappa_n\}$ the increasing sequence of eigenvalues for the differential equation

$$L(w) + \mu M w = 0 \tag{13}$$

with boundary condition (2), where $L(w)$ is the differential operator from equation (1) and

$$M = \|K\| = \sup\Big\{\int_G \varphi(X) K(\varphi) dX:\ \int_G \varphi^2(X) dX = 1\Big\}.$$

We assume the following

HYPOTHESIS Z_M. *For problem* (13), (2) *there exist a sequence of eigenvalues*

(14) $$0 \leqslant \varkappa_1 \leqslant \varkappa_2 \leqslant \varkappa_3 \leqslant \ldots$$

and a corresponding sequence of eigenfunctions

(15) $$w_1(X), w_2(X), w_3(X), \ldots$$

which belong to $\mathscr{F}$.

We shall prove the following

THEOREM 5. *If hypotheses* Z *and* Z_M *are satisfied and if the range* $R[K]$ *of operator* K *contains the space* $\mathscr{F}_{h,\Gamma}(G)$, *then* $\lim_{n\to\infty} \lambda_n = +\infty$.

Proof. To begin with, observe that for every function $\varphi(X) \in \mathscr{L}^2(G)$, we have the inequality

(16) $$H(\varphi) = \int_G \varphi(X) K(\varphi) dX \leqslant M \int_G \varphi^2(X) dX.$$

From (16) it follows that for every function $\varphi(X)$ belonging to $\mathring{\mathscr{D}}$ we have

$$\frac{D(\varphi)}{H(\varphi)} \geqslant \frac{D(\varphi)}{J(\varphi)},$$

where $J(\varphi) = M \int_G \varphi^2(X) dX$. Let $\bar{v}_i(X) = w_i(X)$ $(i = 1, \ldots, n)$, where $w_i(X)$ $(i = 1, \ldots, n)$ are the functions of the sequence (15), then

$$d[\bar{\mathscr{V}}_n] = \min_{u \in \bar{\mathscr{X}}_n} \frac{D(u)}{J(u)} = \varkappa_n,$$

where $\bar{\mathscr{V}}_n = \{\bar{v}_1(X), \ldots, \bar{v}_n(X)\}$. Let us denote by $\tilde{\mathscr{V}}_n = \{\tilde{v}_1(X), \ldots, \tilde{v}_n(X)\}$ a sequence of functions $\tilde{v}_i(X)$ $(i = 1, \ldots, n)$, where

(17) $$M\bar{v}_i(X) = K(\tilde{v}_i) \quad \text{for } i = 1, \ldots, n.$$

Since $\bar{v}_i(X)$ $(i = 1, \ldots, n)$ belong to $R[K]$, therefore there exist solutions of equations (17).

From (17) it follows that

(18) $$J(u, \bar{v}_i) = H(u, \tilde{v}_i) \quad \text{for } i = 1, \ldots, n,$$

where u is an arbitrary function of $\mathring{\mathscr{D}}$. From (18) it follows that the class $\tilde{\mathscr{X}}_n$ consisting of functions $\varphi(X)$ satisfying the conditions $H(\varphi, \tilde{v}_i) = 0$ $(i = 1, \ldots, n)$ coincides with the class of functions $\varphi(X)$ satisfying the conditions $J(\varphi, \bar{v}_i) = 0$ $(i = 1, \ldots, n)$. Therefore we have

$$d[\tilde{\mathscr{V}}_n] = \min_{u \in \tilde{\mathscr{X}}_n} \frac{D(u)}{H(u)} \geqslant \min_{u \in \tilde{\mathscr{X}}_n} \frac{D(u)}{J(u)} = \min_{u \in \bar{\mathscr{X}}_n} \frac{D(u)}{J(u)} = \varkappa_n,$$

whence

$$\lambda_n = \sup_{\mathscr{V}_n} d[\mathscr{V}_n] \geqslant d[\tilde{\mathscr{V}}_n] \geqslant \varkappa_n.$$

It is known ([4], p. 424) that the sequence of eigenvalues of problem (13), (2) tends to $+\infty$ for $n \to \infty$; therefore by the last inequality sequence (9) also tends to infinity.

COROLLARY 1. *Every eigenvalue of* (1), (2) *has finite multiplicity.*

THEOREM 6. *Under the assumptions of Theorem* 5 *the set of eigenfunctions of problem* (1), (2) *is a complete system in the class* $\mathscr{L}^2(G)$ *with respect to the scalar product* $H(u, v)$.

Proof. Let $\{u_n(X)\}$ be the sequence of eigenfunctions of problem (1), (2) normalized so that $H(u_n) = 1$ $(n = 1, 2, 3, \ldots)$, and let $f(X)$ be any function in $\mathscr{L}^2(G)$. Let $c_n = H(f, u_n)$ $(n = 1, 2, 3, \ldots)$ and let $S_n(X) = \sum_{k=1}^{n} c_k u_k(X)$. In virtue of Theorem 5, by a reasoning similar to the proof of an analogous theorem in [7], p. 303, we have

$$\lim_{n\to\infty} H(f-S_n) = 0. \tag{19}$$

From (19) it follows that the sequence $\{u_n(X)\}$ is a complete system in $\mathscr{L}^2(G)$, with respect to the scalar product $H(u, v)$.

COROLLARY 2. *The sequence* (10) *of eigenvalues of problem* (1), (2) *contains all the eigenvalues of this problem.*

3. SOME PROPERTIES OF THE FIRST EIGENVALUE AND FIRST EIGENFUNCTION OF PROBLEM (1), (2)

In the sequel we shall need the following assumption:

HYPOTHESIS Z_1. 1° *The operator K, besides the properties formulated in the introduction, satisfies the following condition: if $\varphi(X) \geqslant 0$ in G and $\varphi \in \mathscr{L}^2(G)$, then $K(\varphi) \geqslant 0$ in G.* 2° *No eigenfunction of* (1), (2) *can vanish identically in any subdomain of domain G.*

It follows from the definition of $u_n(X)$, $n = 1, 2, 3, \ldots$ that

$$H(u_i, u_j) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases} \quad (i, j = 1, 2, 3, \ldots). \tag{20}$$

This implies, by hypothesis Z_1, that all functions of sequence (10) except at most one, change their sign in G. And, since the functions $u_n(X)$ are continuous, there exist zero sets in G for these functions.

LEMMA 1. *Under assumptions* Z *and* Z_1 *each function* $u \in \mathscr{F}_{h,\Gamma}(G)$ *satisfying equation* (1) *with* $\mu = \lambda_1$ *preserves its sign in* G.

Proof. Let $u(X)$ be a function satisfying the assumptions of Lemma 1, and let $\lambda_1 = \lambda_2 = \ldots = \lambda_s < \lambda_{s+1}$ (i.e. λ_1 is an s-fold eigenvalue of (1), (2)). Suppose that $u(X) \geqslant 0$ in $\bar{G}_1 \subset G$. Let us write $G_2 = G - \bar{G}_1$. We have $u(X) < 0$ in G_2. If G_2 is non-empty, then it is open. Put

$$U = \begin{cases} u(X) & \text{for } X \in \bar{G}_1, \\ 0 & \text{for } X \in G_2. \end{cases}$$

By (11) for $\varphi = u(X)$ and $\psi = U(X)$ we get

$$D(u, U) = \lambda_1 H(u, U). \tag{21}$$

We see that $D(u, U) = D(U)$ and $H(u, U) = H(U) + H(u-U, U) \leqslant H(U)$, since $H(u-U, U) \leqslant 0$. From this, by (21), we get

$$D(U) \leqslant \lambda_1 H(U). \tag{22}$$

From the definition of the function U, by assumption Z_1 (2°), it follows that the functions $U, u_1, \ldots, u_s$ are linearly independent in G. Put

$$\Phi(X) = U + c_1 u_1 + \ldots + c_s u_s.$$

Then $\Phi \not\equiv 0$ in G. Let $c_i = -H(U, u_i)/H(u_i)$ $(i = 1, \ldots, s)$. Then Φ is orthogonal to $u_1, \ldots, u_s$ and therefore $\Phi \in \mathscr{K}_s$. Hence

$$D(\Phi) \geqslant \lambda_{s+1} H(\Phi). \tag{23}$$

On the other hand, $\varphi = c_1 u_1 + \ldots + c_s u_s$ belongs to $\mathscr{F}_{h,\Gamma}(G)$ and satisfies equation (1) with $\mu = \lambda_1$. Therefore, by (11) we have

$$D(\varphi, \psi) = \lambda_1 H(\varphi, \psi)$$

for every $\psi \in \mathring{\mathscr{D}}$. As a special case from the last equality we get

$$D(\varphi, U) = \lambda_1 H(\varphi, U), \tag{24}$$

$$D(\varphi) = \lambda_1 H(\varphi). \tag{25}$$

Because of the equality $\Phi = U + \varphi$, (22), (24) and (25) imply

$$D(\Phi) \leqslant \lambda_1 H(\Phi),$$

whence, by (23), $\lambda_1 \geqslant \lambda_{s+1}$, which is a contradiction.

LEMMA 2. *Under assumptions* Z *and* Z_1, *if* μ_1 *is a real number such that there exists a function* $u(X) \in \mathscr{F}_{h,\Gamma}(G)$ *which does not change its sign in* G *and which satisfies equation* (1) *with* $\mu = \mu_1$ *and* $u(X) \not\equiv 0$ *in* G, *then* μ_1 *is the first eigenvalue of problem* (1), (2).

Proof. We shall use formula (11) first for the pair u, u_1 and then for the pair u_1, u. Since both u and u_1 belong to $\mathscr{F}_{h,\Gamma}(G)$ and satisfy equation (1) with $\mu = \lambda_1$, $\mu = \mu_1$, respectively, we get

$$D(u, u_1) = \lambda_1 H(u, u_1) \quad \text{and} \quad D(u_1, u) = \mu_1 H(u_1, u),$$

whence by the symmetry of D and H

$$(\mu_1 - \lambda_1) H(u, u_1) = 0.$$

Since u and u_1 do not change their sign in G and u_1 does not vanish in any subdomain, we have $H(u, u_1) \neq 0$, and thus $\mu_1 = \lambda_1$.

THEOREM 7. *Under assumptions* Z *and* Z_1 *the first eigenfunction* $u_1(X)$ *of problem* (1), (2) *does not vanish at any point of* G.

Proof. From Lemma 1 it follows that the function $u_1(X)$ does not change its sign in G. Suppose that $u_1(X) \geqslant 0$ in G. From equation (1) it follows that the function $u_1(X)$ satisfies the equation

$$L(u_1) = -\lambda_1 K(u_1). \tag{26}$$

By assumption Z_1 (1°) and $\lambda_1 \geqslant 0$, we have $-\lambda_1 K(u_1) \leqslant 0$ in G. According to E. Hopf's theorem (see [5]), the value 0 cannot be attained by $u_1(X)$ in G if $u_1(X)$ does not vanish identically in G. Thus $u_1(X) > 0$ in G. In the case $u_1(X) \leqslant 0$ in G, the proof is analogous.

THEOREM 8. *Under assumptions* Z *and* Z_1 *each function* $\varphi(X) \in \mathscr{F}_{h,\Gamma}(G)$, *not vanishing identically in* G *and satisfying equation* (1) *with* $\mu = \lambda_1$ *is equal to the first eigenfunction of* (1), (2) *multiplied by a constant* $c \neq 0$.

The proof of this theorem is quite similar to an analogous theorem in [1], and is omitted.

COROLLARY 3. *The first eigenvalue of* (1), (2) *is a single eigenvalue, i.e.* $\lambda_1 < \lambda_2$.

Remark 1. The particular case of problem (1), (2), where $K(\varphi) = \varrho(X)\varphi$, $\varrho(X) > 0$ is a continuous function in $\bar{G}$, was considered in [1]. Other cases of problem (1), (2) and their applications will be published in the next paper.

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