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ON NULL GEODESIC COLLINEATIONS IN SOME RIEMANNIAN SPACES

BY

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1. Introduction. A non-flat n-dimensional Riemannian space is said to be *of recurrent curvature* [5] (briefly, a *recurrent space*) if its curvature tensor satisfies the condition

$$R_{hijk,l} = c_l R_{hijk} \tag{1}$$

for some non-zero vector c_j, where the comma indicates covariant differentiation with respect to the metric of the space.

As an immediate consequence of (1) we get

$$R_{ij,l} = c_l R_{ij}. \tag{2}$$

Spaces whose Ricci tensor satisfy (2) for some non-zero vector c_j, where $n > 2$ and R_{ij} is non-zero, are called *Ricci-recurrent* [3]. Thus every recurrent space ($n > 2$) with non-vanishing Ricci tensor is Ricci-recurrent.

Differentiating (2) covariantly and contracting with g^{ij}, we obtain

$$R(c_{l,k} - c_{k,l}) = R_{,lk} - R_{,kl} = 0.$$

Hence, if the scalar curvature of a Ricci-recurrent space is non-zero, then the vector of recurrence c_j is a gradient. Consequently, as it has been proved ([4], Lemma 2), we have

$$R_{ri} R^r{}_j = \tfrac{1}{2} R R_{ij}. \tag{3}$$

An n-dimensional Riemannian space is called *Ricci-symmetric* if its Ricci tensor is non-zero and satisfies

$$R_{ij,k} = 0. \tag{4}$$

Spaces of this kind are, evidently, generalizations of so-called *symmetric* (in the sense of E. Cartan) spaces, i.e., non-flat Riemannian n-spaces characterized by the condition

$$R_{hijk,l} = 0. \tag{5}$$

It follows easily from (4) that the scalar curvature of a Ricci-symmetric space and, therefore, of a symmetric space, is constant.

According to Chaki and Gupta [1], an n-dimensional $(n > 3)$ Riemannian space is said to be *conformally symmetric* if its Weyl's conformal tensor

$$(6)\qquad C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

satisfies

$$(7)\qquad C^h{}_{ijk,l} = 0.$$

It can be easily verified that every conformally n-space flat $(n > 3)$ as well as every symmetric Riemannian n-space $(n > 3)$ is necessarily conformally symmetric, but the converse is, in general, not true.

Differentiating (6) covariantly, summing over h and l and taking into account (7) and the well-known relations

$$R^r{}_{ijk,r} = R_{ij,k} - R_{ik,j}, \qquad R^r{}_{j,r} = \tfrac{1}{2}R_{,j}$$

we obtain

$$(8)\qquad R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik}).$$

It follows from this equation that a conformally symmetric space is of constant scalar curvature if and only if

$$(9)\qquad R_{ij,k} = R_{ik,j}.$$

Spaces satisfying (9) will be called *almost Ricci-symmetric spaces* or, briefly, ARS_n-spaces.

Evidently, every Ricci-symmetric as well as every conformally symmetric space with constant scalar curvature is an ARS_n-space.

Katzin and Levine [2] have introduced the concept of a null geodesic collineation. It is defined as an infinitesimal point transformation $\bar{x}^i = x^i + v^i\,\delta t$ for which

$$(10)\qquad L\Gamma^h_{ij} = g^{hr}g_{ij}Q_{,r},$$

where Q is a certain function, and $L\Gamma^h_{ij}$ denotes the Lie derivative of Christoffel symbols with respect to v^i (for geometrical interpretation, see [2]).

If $Q = \mathrm{const}$, the null geodesic collineation is an affine one.

The purpose of this paper is to prove that null geodesic collineations in recurrent as well as in ARS_n-spaces with $R_{ij} \neq 0$ are necessarily affine.

Throughout this note we assume that metrics of considered spaces are indefinite.

2. Preliminary results. First, we shall obtain some results on null geodesic collineations in general Riemannian spaces.

LEMMA 1. *If a Riemannian space admits a null geodesic collineation, then*

$$LR_{ij} = A^r{}_{,r} g_{ij} - A_{i,j}, \tag{11}$$

$$a_{ri} R^r{}_{hjk} + a_{rj} R^r{}_{hki} + a_{rk} R^r{}_{hij} = 0, \tag{12}$$

$$2A_r R^r{}_{ijh} + A_i R_{hj} + A_h R_{ij} + a^r{}_i R^s{}_{jhr,s} + a^r{}_h R^s{}_{jir,s} = 2A_{j,hi} - g_{ij} A^r{}_{,hr} - g_{hj} A^r{}_{,ir}, \tag{13}$$

where $A^h = g^{hr} Q_{,r}$ *and* $a_{ij} = Lg_{ij}$.

Proof. As a consequence of (10), we obtain

$$L\Gamma^h_{ij} = A^h g_{ij}. \tag{14}$$

Substituting (14) into the well-known formula

$$LR^h{}_{ijk} = (L\Gamma^h_{ij})_{,k} - (L\Gamma^h_{ik})_{,j},$$

we have

$$LR^h{}_{ijk} = A^h{}_{,k} g_{ij} - A^h{}_{,j} g_{ik}. \tag{15}$$

Relation (11) follows immediately from (15).

Taking now into account (14) and the formula (see [6])

$$L\Gamma^h_{ij} = \tfrac{1}{2} g^{hr} [(Lg_{rj})_{,i} + (Lg_{ri})_{,j} - (Lg_{ij})_{,r}],$$

we find

$$A_h g_{ij} = \tfrac{1}{2}(a_{hj,i} + a_{hi,j} - a_{ij,h}), \tag{16}$$

whence

$$A_i g_{hj} = \tfrac{1}{2}(a_{ij,h} + a_{ih,j} - a_{hj,i}).$$

The last equation together with (16) yield

$$a_{hi,j} = A_h g_{ij} + A_i g_{hj} \tag{17}$$

which, by covariant differentiation, implies

$$a_{hi,jk} - a_{hi,kj} = A_{h,k} g_{ij} + A_{i,k} g_{hj} - A_{h,j} g_{ik} - A_{i,j} g_{hk}.$$

Applying now the Ricci identity, we obtain

$$a_{ri} R^r{}_{hjk} + a_{hr} R^r{}_{ijk} = A_{h,j} g_{ik} + A_{i,j} g_{hk} - A_{h,k} g_{ij} - A_{i,k} g_{hj} \tag{18}$$

or, by a cyclic permutation of indices i, j and k,

$$a_{rj} R^r{}_{hki} + a_{hr} R^r{}_{jki} = A_{h,k} g_{ji} + A_{j,k} g_{hi} - A_{h,i} g_{jk} - A_{j,i} g_{hk},$$

$$a_{rk} R^r{}_{hij} + a_{hr} R^r{}_{kij} = A_{h,i} g_{kj} + A_{k,i} g_{hj} - A_{h,j} g_{ki} - A_{k,j} g_{hi}.$$

But (18), together with the last two equations, gives

$$a_{ri}R^r{}_{hjk}+a_{hr}R^r{}_{ijk}+a_{rj}R_{hki}+a_{hr}R^r{}_{jki}+a_{rk}R^r{}_{hij}+a_{hr}R^r{}_{kij} = 0$$

which, in view of

$$a_{hr}R^r{}_{ijk}+a_{hr}R^r{}_{jki}+a_{hr}R^r{}_{kij} = 0,$$

leads immediately to (12). Differentiating now (18) covariantly and making use of (17), we get

$$\begin{aligned}(19)\qquad g_{il}A_rR^r{}_{hjk}+g_{hl}A_rR^r{}_{ijk}+A_iR_{lhjk}+A_hR_{lijk}+a_{ri}R^r{}_{hjk,l}+a_{hr}R^r{}_{ijk,l}&\\ = A_{h,jl}g_{ik}+A_{i,jl}g_{hk}-A_{h,kl}g_{ij}-A_{i,kl}g_{hj}&\end{aligned}$$

or, by contraction with g^{kl},

$$\begin{aligned}A_rR^r{}_{hji}+A_rR^r{}_{ijh}+A_iR_{hj}+A_hR_{ij}+a^r{}_iR^s{}_{jhr,s}+a^r{}_hR^s{}_{jir,s}&\\ = A_{h,ji}+A_{i,jh}-g_{ij}A^r{}_{,hr}-g_{hj}A^r{}_{,ir}.&\end{aligned}$$

But the last equation, in virtue of

$$A_rR^r{}_{hji} = A_rR^r{}_{jhi}+A_rR^r{}_{ijh}$$

and

$$(20)\qquad A_{j,ih} = A_{j,hi}+A_rR^r{}_{jhi} = A_{i,jh},$$

can be written as

$$\begin{aligned}A_rR^r{}_{jhi}+A_rR^r{}_{ijh}+A_rR^r{}_{ijh}+A_iR_{hj}+A_hR_{ij}+a^r{}_iR^s{}_{jhr,s}+a^r{}_hR^s{}_{jir,s}&\\ = A_rR^r{}_{jhi}+A_{h,ji}+A_{j,hi}-g_{ij}A^r{}_{,hr}-g_{hj}A^r{}_{,ir}&\end{aligned}$$

which, because of $A_{h,ji} = A_{j,hi}$, proves the last part of our lemma.

Now we prove some results on null geodesic collineations in ARS_n-spaces.

LEMMA 2. *If an ARS_n-space admits a null geodesic collineation, then*

$$\begin{aligned}(21)\qquad 2A_rR^r{}_{ijh}+g_{ij}A_rR^r{}_h+g_{hj}A_rR^r{}_i+A_iR_{hj}+&\\ +A_hR_{ij}+g_{ij}B_{,h}+g_{hj}B_{,i} = 2A_{j,hi},&\end{aligned}$$

$$(22)\qquad A_rR^r{}_{ijk}+g_{ij}A_rR^r{}_k-g_{ik}A_rR^r{}_j+B_{,k}g_{ij}-B_{,j}g_{ik} = 0,$$

where $B = A^r{}_{,r}$.

Proof. It is easily seen that (9) gives

$$R^s{}_{ijk,s} = R_{ij,k}-R_{ik,j} = 0.$$

Moreover, as an immediate consequence of (20), we have

$$A^r{}_{,ir} = B_{,i} + A_r R^r{}_i. \tag{23}$$

Substituting now the last relations into (13), we obtain (21), which completes the proof of the first part of our lemma.

It follows easily from (9) that $L(R_{ij,k}) = L(R_{ik,j})$. Using now the well-known formula

$$L(T_{ij,k}) = (LT_{ij})_{,k} - T_{rj} L\Gamma^r_{ik} - T_{ir} L\Gamma^r_{jk}$$

and taking into consideration (11) and (14), we obtain

$$A_{i,kj} - A_{i,jk} + g_{ij} A_r R^r{}_k - g_{ik} A_r R^r{}_j + B_{,k} g_{ij} - B_{,j} g_{ik} = 0$$

which, in view of the Ricci identity, is equivalent to (22).

LEMMA 3. *The vector A_j of an ARS_n-space admitting a null geodesic collineation satisfies the conditions*

$$A_r R^r{}_k = -\tfrac{1}{2} R A_k, \tag{24}$$

$$B_{,k} = \frac{n}{2(n-1)} R A_k, \tag{25}$$

$$A_r R^r{}_{ijk} = \frac{1}{2(n-1)} R (A_j g_{ik} - A_k g_{ij}), \tag{26}$$

$$w R_{ij} = \frac{3}{2(n-1)} R w g_{ij} - \frac{n+2}{2(n-1)} R A_i A_j, \tag{27}$$

where $w = A^r A_r$.

Proof. The contraction of (21) with g^{ij} gives

$$(n+4) A_r R^r{}_h + R A_h + (n+1) B_{,h} = 2 A^r{}_{,hr}$$

or, in view of (23),

$$(n+2) A_r R^r{}_k + R A_k + (n-1) B_{,k} = 0. \tag{28}$$

On the other hand, contracting (22) with g^{ij}, we get

$$n A_r R^r{}_k + (n-1) B_{,k} = 0. \tag{29}$$

Comparing now (28) with (29), we obtain easily (24) which, together with (28), leads immediately to (25).

Relation (26) follows easily from (24), (25) and (22).

Substituting now (24), (25) and (26) into (21), we find

$$A_i R_{hj} + A_h R_{ij} + \frac{1}{2(n-1)} R A_i g_{hj} + \\ + \frac{2}{2(n-1)} R A_j g_{ih} - \frac{1}{2(n-1)} R A_h g_{ij} = 2 A_{j,hi}. \tag{30}$$

This equation, in view of $A_{j,hi} = A_{h,ji}$, yields

$$(31) \qquad A_h R_{ij} - A_j R_{ih} + \frac{3}{2(n-1)} R A_j g_{ih} - \frac{3}{2(n-1)} R A_h g_{ij} = 0,$$

which, by transvection with A^h and making use of (24), leads to (27). This completes the proof.

3. Main results. Now we can proceed to main results of this paper.

THEOREM 1. *If an ARS_n-space with $R_{ij} \neq 0$ admits a null geodesic collineation, then this collineation is necessarily an affine one.*

Proof. Since the scalar curvature of an ARS_n-space is constant, equation (27) gives

$$w_{,k} R_{ij} + w R_{ij,k} = \frac{3}{2(n-1)} R w_{,k} g_{ij} - \frac{n+2}{2(n-1)} R A_{i,k} A_j - \frac{n+2}{2(n-1)} R A_i A_{j,k},$$

whence, by (9),

$$w_{,k} R_{ij} - w_{,j} R_{ik} = \frac{3}{2(n-1)} R w_{,k} g_{ij} + \frac{n+2}{2(n-1)} R A_{i,j} A_k - \frac{n+2}{2(n-1)} R A_{i,k} A_j - \frac{3}{2(n-1)} R w_{,j} g_{ik}.$$

This equation, by (27), can be written in the form

$$(32) \qquad R w_{,k} A_i A_j - R w_{,j} A_i A_k = R w A_{i,k} A_j - R w A_{i,j} A_k.$$

Transvecting now (32) with A^k and taking into account the equality

$$(33) \qquad A^r A_{r,i} = \tfrac{1}{2} w_{,i} = A^r A_{i,r},$$

we find

$$(34) \qquad R w^2 A_{i,j} = \tfrac{1}{2} R w w_{,i} A_j + R w w_{,j} A_i - R w_{,r} A^r A_i A_j.$$

This, by transvection with A^i and making use of (33), yields

$$R w^2 w_{,j} = R w A^r w_{,r} A_j.$$

Substituting this relation into (34), we get

$$(35) \qquad R w^3 A_{i,j} = \tfrac{1}{2} R w^2 w_{,i} A_j$$

or, by contraction with g^{ij},

$$(36) \qquad \tfrac{1}{2} R w^2 A^r w_{,r} = R B w^3.$$

On the other hand, since $A_{i,j} = A_{j,i}$, (34) gives

$$\tfrac{1}{2} R w^2 w_{,i} A_j = \tfrac{1}{2} R w^2 w_{,j} A_i,$$

whence, in view of (36), we have

$$(37) \qquad \tfrac{1}{2} R w^3 w_{,i} = R B w^3 A_i.$$

This together with (35) implies

$$Rw^4 A_{i,j} = RBw^3 A_i A_j. \tag{38}$$

Differentiating now (38) covariantly and using (37), (38) and (25), we obtain

$$Rw^5 A_{i,jk} = \frac{n}{2(n-1)} R^2 w^4 A_i A_j A_k.$$

Hence, $Rw^5(A_{i,jk} - A_{i,kj}) = 0$. But this equation implies $Rw^5 A_r R^r{}_{ijk} = 0$ which, by (24), gives finally

$$R^2 w^5 A_k = 0. \tag{39}$$

It follows easily from (24)-(26) and (21) that the assumption $R = 0$ yields

$$A_r R^r{}_{ijk} = 0 \tag{40}$$

and

$$A_i R_{hj} + A_h R_{ij} = 2A_{j,hi}. \tag{41}$$

On the other hand, from (27) we have $w = 0$, and, as a consequence of (33),

$$A^r A_{r,j} = A^r A_{j,r} = 0. \tag{42}$$

Therefore, transvecting (18) with A^k and using (40) and (42), we obtain

$$A_i A_{h,j} + A_h A_{i,j} = 0, \tag{43}$$

whence we have ([4], Lemma 1) $A_{i,j} = 0$. The last relation reduces (41) to the form $A_i R_{hj} + A_h R_{ij} = 0$. But this, since $R_{ij} \neq 0$, gives ([4], Lemma 1) finally $A_j = 0$. If, accordingly to (39), $w = 0$, then, in view of (27), we have $RA_j = 0$. Hence, in all cases, $A_j = 0$, which completes our proof.

As a consequence of Theorem 1, we have

COROLLARY 1. *If a Ricci-symmetric space admits a null geodesic collineation, then this collineation is an affine one.*

THEOREM 2. *If a symmetric space admits a null geodesic collineation, then this collineation is necessarily an affine one.*

Proof. If $R_{ij} \neq 0$, then our theorem follows immediately from Corollary 1. Suppose, therefore, that $R_{ij} = 0$. Then, in view of (26), we have $A_r R^r{}_{ijk} = 0$. Moreover, by (30), $A_{j,hi} = 0$. Hence, in virtue of (19), we obtain

$$A_i R_{lhjk} + A_h R_{lijk} = 0$$

which, evidently ([4], Lemma 1), completes the proof of the theorem.

It is easy to see that, for a conformally symmetric space with constant scalar curvature and non-vanishing Ricci tensor, Theorem 1 holds. If $R_{ij} = 0$, a non-flat conformally symmetric space is, evidently, symmetric. Thus, in view of Theorems 1 and 2, we get

COROLLARY 2. *If the scalar curvature of a non-flat conformally symmetric space admitting a null geodesic collineation is constant, then this collineation is an affine one.*

THEOREM 3. *If the scalar curvature of a Ricci-recurrent space admitting a null geodesic collineation is non-zero, then this collineation reduces to an affine one.*

Proof. The contraction of (12) with g^{hj} gives

$$a_{ri} R^r{}_k = a_{rk} R^r{}_i. \tag{44}$$

Differentiating now covariantly and using (17), we find

$$g_{ip} A_r R^r{}_k - g_{kp} A_r R^r{}_i + A_i R_{pk} - A_k R_{pi} = a_{rk} R^r{}_{i,p} - a_{ri} R^r{}_{k,p}$$

which, by (2) and (44), yields

$$g_{ip} A_r R^r{}_k - g_{kp} A_r R^r{}_i + A_i R_{pk} - A_k R_{pi} = 0. \tag{45}$$

From the last equation, by contraction with g^{ip}, we get

$$n A_r R^r{}_k = R A_k \tag{46}$$

or, by transvection with $R^k{}_h$,

$$n A_r R^r{}_s R^s{}_h = R A_r R^r{}_h. \tag{47}$$

But (47), in view of (3), can be written as $(n-2) R A_r R^r{}_h = 0$. Hence, $A_r R^r{}_h = 0$ which, together with (46), implies $A_k = 0$. Thus the proof of the theorem is complete.

THEOREM 4. *If a recurrent space admits a null geodesic collineation, then this collineation is necessarily an affine one.*

Proof. Differentiating (12) covariantly and taking into account (17) and (1), we find

$$\begin{aligned} g_{ip} A_r R^r{}_{hjk} + g_{jp} A_r R^r{}_{hki} + g_{kp} A_r R^r{}_{hij} + \\ + A_i R_{phjk} + A_j R_{phki} + A_k R_{phij} = 0 \end{aligned} \tag{48}$$

which, by contraction with g^{ip}, yields

$$(n-1) A_r R^r{}_{hjk} + A_j R_{hk} - A_k R_{hj} = 0. \tag{49}$$

If R is non-zero, then $R_{ij} \neq 0$. Theorem 4 is, therefore, an immediate consequence of Theorem 3. Suppose now that $R = 0$. Then, it follows from (46) that $A_r R^r{}_k = 0$. This reduces (45) to the form $A_i R_{pk} - A_k R_{pi} = 0$.

Substituting this equation into (49), we obtain easily (40) which, by equation (48), gives

$$A_i R_{phjk} + A_j R_{phki} + A_k R_{phij} = 0.$$

But the last relation implies

$$A^i A_i R_{phjk} = 0. \tag{50}$$

Since the space is not flat by the assumption, (50) yields $w = 0$ and, consequently, $A^r A_{j,r} = 0$. Transvecting now (18) with A^k and using (40) and the last result, we find easily (43). Hence, $A_{i,j} = 0$. Therefore, (18) can be written as

$$a_{ri} R^r{}_{hjk} + a_{rh} R^r{}_{ijk} = 0.$$

This, together with (1), (40), (19) and $A_{i,jk} = 0$, gives finally

$$A_i R_{lhjk} + A_h R_{lijk} = 0,$$

which completes the proof ([4], Lemma 1).

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