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QUASI-ALGEBRAIC REPRESENTABILITY OF SETS IN R^n

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1. R-quasi-algebraic structures in R^n

Let A be any set of real functions defined on subsets of R^n. Denoting the union of all domains D_α of functions $\alpha \in A$ by Points$A$ and, for a given set S, the constant function on S with value c by c_S, we assume that $\alpha+\beta$, $\alpha\cdot\beta$, $c_{\mathrm{Points}\,A} \in A$ for $\alpha, \beta \in A$ and $c \in R$. In [5] similar sets A with values in a given field K were considered.

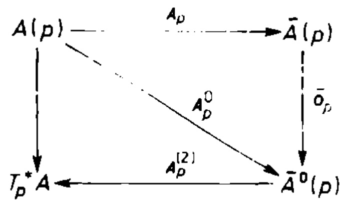
Such sets A are called *K-quasi-algebraic spaces* (K-q.a.s.'s). In the particular case where $K = R$ we consider R-q.a.s.'s. For any set A of functions the smallest topology on Points$A$ such that all D_α, $\alpha \in A$, are open will be denoted by top$A$. If A is an R-q.a.s. then top$A$ will be regarded as the topology of the space A. For every set A of functions with values in the field K we have the smallest of all K-q.a.s.'sA_0, containing A and such that PointsA_0 = Points$A$ (see [5]). The K-q.a.s. A_0 is called the K-q.a.s. *generated by A*.

We have got the concept of an *A-germ* defined in the usual way as a coset of the equivalence relation $\equiv$, where $(\alpha, p) \equiv (\beta, q)$ iff $\alpha, \beta \in A$, $p = q$ and there exists a $U \in \mathrm{top}A$ such that $p \in U \subset D_\alpha \cap D_\beta$ and $\alpha|U = \beta|U$. Each A-germ ξ has the source $a\xi$ and the target $b\xi$, $(\alpha, a\xi) \in \xi$, $\alpha(a\xi) = b\xi$, $\alpha \in A$. Denoting the set of all $\alpha \in A$ such that $p \in D_\alpha$ by $A(p)$, the set of all A-germs ξ with $a\xi = p$ by $\bar{A}(p)$, and assigning to each $\alpha \in A(p)$ the A-germ $A_p(\alpha)$ including (α, p), we have the mapping

$$A_p\colon A(p) \to \bar{A}(p).$$

$\bar{A}(p)$ is, in the natural way, an R-algebra, where $\xi+\eta = A_p(\alpha+\beta)$, $\xi\cdot\eta = A_p(\alpha\cdot\beta)$, $c\xi = A_p(c\alpha)$, $\xi = A_p(\alpha)$, $\eta = A_p(\beta)$, $\alpha, \beta \in A(p)$ and c in R.

We denote the set of all $\alpha \in A(p)$ such that $\alpha(p) = 0$ by $A^0(p)$ and, similarly, we denote the set of all germs ξ of $\bar{A}(p)$ such that $b\xi = 0$ by $\bar{A}^0(p)$. $\bar{A}^0(p)$ is the ideal of the R-algebra $\bar{A}(p)$. Taking the vector space $T_p^*A = \bar{A}^0(p)/(\bar{A}^0(p))^2$, where $(\bar{A}^0(p))^2$ is the square of the ideal $\bar{A}^0(p)$, and considering the canonical mapping $A_p^{(2)}\colon \bar{A}^0(p) \to T_p^*A$, we have the commutative diagram of epimorphisms



where $\bar{o}_p(\xi) = \xi - A_p(b\xi_{\mathrm{Points}\,A})$ for ξ in $A(p)$. The dual vector space to $T_p^* A$ is usually denoted by $T_p A$ and called the *Zariski tangent space to A at the point p*. We evidently have the natural isomorphism

(1.1) $$w \mapsto \bar{w}\colon\ T_p A \to \mathrm{Der}\,A(p),$$

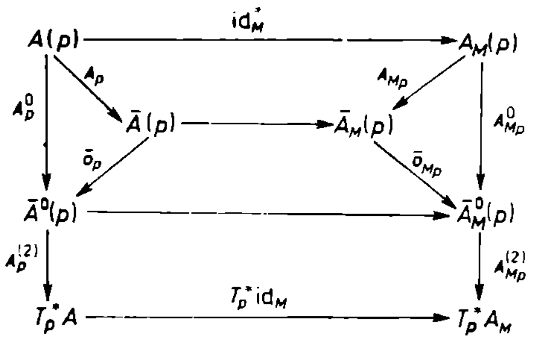
where $\mathrm{Der}\,A(p)$ denotes the vector space of all derivations of $A(p)$, and, for any w in $T_p A$, we have $\bar{w}$ in $\mathrm{Der}\,A(p)$ defined as follows:

(1.2) $$\bar{w}(\alpha) = w\big(A_p^{(2)}(A_p^0(\alpha))\big) \quad \text{for} \quad \alpha \in A(p).$$

Let us take the mapping $f\colon M \to N$ of sets. Setting $f^*(\beta) = \beta \circ f$ for any real function β defined on the subset D_β of N we obtain the mapping $f^*\colon \boldsymbol{R}(N) \to \boldsymbol{R}(M)$, where $\boldsymbol{R}(M)$ stands for the set of all real functions defined on subsets of M. We have the smooth mapping $f\colon A \to B$ of q.a.s. A into q.a.s. B iff f maps the set Points A into the set Points B and $\beta \circ f \in A$ for $\beta \in B$. In other words, $f^*[B] \subset A$.

For any $\boldsymbol{R}$-q.a.s. A and any subset M of Points A we have the $\boldsymbol{R}$-q.a.s. A_M defined as the set of all real functions β such that for any $p \in D_\beta$ there exist $U \in \mathrm{top}\,A$ and $\alpha \in A$ fulfilling the condition $p \in U \cap M$ and $\beta | U \cap M = \alpha | U \cap M$. It is easy to check that $\mathrm{top}\,A_M$ is the topology induced to the set M by $\mathrm{top}\,A$ and we have the commutative diagram (cf. [5]).

(1.3)



As a direct consequence of the definition of isomorphism (1.1) and the commutativity of diagram (1.3) we get

PROPOSITION 1.1. *If* $p \in M \subset \mathrm{Points}\,A$, *then we have the morphism*

$$v \mapsto v \circ \mathrm{id}_M^*\colon\ \mathrm{Der}\,A_M(p) \to \mathrm{Der}\,A(p)$$

and the commutative square

$$\begin{array}{ccc} T_p A_M & \xrightarrow{T_p \mathrm{id}_M} & T_p A \\ \downarrow & & \downarrow \\ \mathrm{Der}\, A_M(p) & \longrightarrow & \mathrm{Der}\, A(p) \end{array}$$

where the vertical arrows are natural isomorphisms.

Let A_1 and A_2 be any sets of functions with values in $\boldsymbol{R}$. Let us set

$$\pi_i\colon \mathrm{Points} A_1 \times \mathrm{Points} A_2 \to \mathrm{Points} A_i, \qquad \pi_i(q_1, q_2) = q_i$$

for $(q_1, q_2) \in \mathrm{Points} A_1 \times \mathrm{Points} A_2$, $i = 1, 2$. The smallest $\boldsymbol{R}$-q.a.s. $A_1 \times A_2$ containing the set $\pi_1^*[A_1] \cup \pi_2^*[A_2]$ such that $\mathrm{Points}(A_1 \times A_2) = \mathrm{Points} A_1 \times \mathrm{Points} A_2$ is called (cf. [5]) the *Cartesian product* of the $\boldsymbol{R}$-q.a.s.'s A_1 and A_2. It is easy to verify the following

PROPOSITION 1.2. *We have $A_1 \times A_2 = A_{10} \times A_{20}$, where A_{i0} is the $\boldsymbol{R}$-q.a.s. generated by A_i, $i = 1, 2$.*

The set A of real functions is said to be *locally bounded* iff for any $\alpha \in A$ and any $p \in D_\alpha$ there exist $U \in \mathrm{top} A$ and $c > 0$ such that $p \in U \subset D_\alpha$ and

$$|\alpha(q)| \leqslant c \quad \text{for} \quad q \in U. \tag{1.4}$$

PROPOSITION 1.3. *If A is locally bounded, then the $\boldsymbol{R}$-q.a.s. A_0 generated by A is locally bounded.*

Proof. Setting $A_1 = A \cup \{c_{\mathrm{Points} A};\ c \in \boldsymbol{R}\}$ and $A_{k+1} = A_k + A_k \cdot A_k$, $k = 1, 2, \ldots$, we have (see [5]) $A_0 = \bigcup_k A_k$ and $\mathrm{top} A_0 = \mathrm{top} A$. For any sets A and B of real functions we have denoted here the sets $\{\alpha+\beta;\ \alpha \in A \text{ and } \beta \in B\}$ and $\{\alpha \cdot \beta;\ \alpha \in A \text{ and } \beta \in B\}$ by $A+B$ and $A \cdot B$, respectively. Let us take $\alpha \in A_1$ and $p \in D_\alpha$. If $\alpha \in A$, then there exist $U \in \mathrm{top} A$ and $c > 0$ such that (1.4) is satisfied. If $\alpha = a_{\mathrm{Points} A}$, where $a \in \boldsymbol{R}$, then, setting $U = \mathrm{Points} A$ and $c = |a|$, we obtain (1.4) again. Now, for any $\alpha \in A_k$ and $p \in D_\alpha$, let there exist $U \in \mathrm{top} A$ and $c > 0$ such that (1.4) holds. Take $\alpha, \beta, \gamma \in A_k$ and $p \in D_\alpha \cap D_\beta \cap D_\gamma$. Then we have $U \in \mathrm{top} A$ such that the inequalities $|\alpha(q)| \leqslant c/2$, $|\beta(q)| \leqslant \sqrt{c/2}$ and $|\gamma(q)| \leqslant \sqrt{c/2}$ for $q \in U$ hold. Then $|(\alpha + \beta \cdot \gamma)(q)| \leqslant c$ for $q \in U$. Hence, by the definition of A_{k+1}, for any $\alpha \in A_{k+1}$ and $p \in D_\alpha$ there exist $U \in \mathrm{top} A$ and $c > 0$ such that (1.4) holds. This ends the proof.

PROPOSITION 1.4. *The Cartesian product of two locally bounded $\boldsymbol{R}$-q.a.s.'s is locally bounded.*

Proof. Let A_1 and A_2 be locally bounded $\boldsymbol{R}$-q.a.s.'s. We set $\pi_i(p_1, p_2) = p_i$ for $(p_1, p_2) \in \mathrm{Points} A_1 \times \mathrm{Points} A_2$. Then

$$\pi_i\colon \mathrm{Points} A_1 \times \mathrm{Points} A_2 \to \mathrm{Points} A_i, \qquad i = 1, 2,$$

and (cf. [5]) $A_1 \times A_2$ is the smallest of all the $\boldsymbol{R}$-q.a.s.'s containing $\pi_1^*[A_1] \cup \pi_2^*[A_2]$ such that the set of all points is equal to $\mathrm{Points} A_1 \times \mathrm{Points} A_2$. We have $\mathrm{top}(A_1 \times$

$\times A_2) = \operatorname{top} A_1 \times \operatorname{top} A_2$. According to Proposition 1.3 it suffices to prove that for any $\alpha \in \pi_1^*[A_1] \cup \pi_2^*[A_2]$ and $p \in D_\alpha$ there exist $U \in \operatorname{top} A_1 \times \operatorname{top} A_2$ and $c > 0$ such that (1.4) is satisfied. So, let us take any $\alpha \in \pi_1^*[A_1]$ and $p \in D_\alpha$. We have $\alpha = \alpha_1 \circ \pi_1$, where $\alpha_1 \in A_1$. Hence, $\pi_1(p) \in D_{\alpha_1}$. Thus there exist $U_1 \in \operatorname{top} A_1$ and $c > 0$ such that $|\alpha_1(q_1)| \leqslant c$ for $q_1 \in U_1$. Setting $U = U_1 \times \operatorname{Points} A_2$, we obtain $U \in \operatorname{top} A_1 \times \times \operatorname{top} A_2$ and $|\alpha(q_1, q_2)| = |\alpha_1(q_1)| \leqslant c$ for $(q_1, q_2) \in U$. This ends the proof.

PROPOSITION 1.5. *If A_1 and A_2 are sets of real functions defined on subsets of $\boldsymbol{R}^n$ and continuous in the usual sense, then $A_1 \times A_2$ is a set of continuous functions.*

Proof. Let us remark that, if every function belonging to A is continuous, then every function belonging to the $\boldsymbol{R}$-q.a.s. A_0 generated by A is also continuous. Taking any function of the form $\alpha \circ \pi_i$, where $\alpha \in A_i$, $i = 1, 2$, we state that it is continuous. Therefore every function of the set $\pi_1^*[A_1] \cup \pi_2^*[A_2]$ is continuous in its domain. Thus all functions belonging to $A_1 \times A_2$ are continuous. This ends the proof.

2. Cauchy P-decomposition property

Let A be a set of real functions and let P be a finite subset of A. We say that a function $\alpha \in A$ has the *Cauchy P-decomposition property in A* iff for every $p \in D_\alpha$ there exist functions $\alpha_1, \ldots, \alpha_n \in A \times A$ and a neighbourhood U of p open in $\operatorname{top} A$ such that $U \times U \subset D_{\alpha_1} \cap \ldots \cap D_{\alpha_n}$ and

$$\alpha(r) - \alpha(q) = \sum_{i=1}^{n} \alpha_i(r, q)\big(\pi_i(r) - \pi_i(q)\big) \quad \text{for} \quad r, q \in U, \tag{2.1}$$

where $P = \{\pi_1, \ldots, \pi_n\}$. According to Proposition 1.2 any function $\alpha \in A$ has the Cauchy P-decomposition property in A iff α has the Cauchy P-decomposition property in A_0. We will say that A has the *Cauchy P-decomposition property* (P-d.p.) iff every $\alpha \in A$ has the Cauchy P-decomposition property in A.

PROPOSITION 2.1. *If a set A of real functions has the P-*d.p., *then the $\boldsymbol{R}$-*q.a.s. *A_0 generated by A has the P-*d.p.

Proof. Setting $A_1 = A \cup \{a_{\operatorname{Points} A};\ a \in \boldsymbol{R}\}$ and $A_{k+1} = A_k + A_k \cdot A_k$, $k = 1, 2, \ldots$, we have $A_0 = \bigcup_k A_k$. It is evident that every $\alpha \in A_1$ has the Cauchy P-decomposition property in A_0. Assume that the condition

(k) every function of A_k has the Cauchy P-decomposition property in A_0,

holds. Let α and β have the P-d.p. in A_0. We may assume (2.1) and

$$\beta(r) - \beta(q) = \sum_{i=1}^{n} \beta_i(r, q)\big(\pi_i(r) - \pi_i(q)\big) \quad \text{for} \quad r, q \in U,$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in A \times A$, $p \in U \in \operatorname{top} A$ and $U \times U \subset D_{\alpha_1} \cap \ldots \cap D_{\alpha_n} \cap \cap D_{\beta_1} \cap \ldots \cap D_{\beta_n}$. Hence we get

$$\alpha(r)\beta(r)-\alpha(q)\beta(q) = (\alpha(r)-\alpha(q))\beta(r)+\alpha(q)(\beta(r)-\beta(q))$$
$$= \sum_{i=1}^{n} \gamma_i(r, q)\,(\pi_i(r)-\pi_i(q)) \quad \text{for} \quad r, q \in U,$$

where $\gamma_i(r, q) = \alpha_i(r, q)\beta(r)+\alpha(q)\beta_i(r, q)$ for $(r, q) \in \bigcap_i (D_{\alpha_i} \cap D_{\beta_i})$. It is evident that $\gamma_i \in A \times A$. Similarly, we check that $\alpha+\beta$ has the same property. Taking any $\gamma \in A_{k+1}$ we have $\gamma = \alpha+\beta \cdot \beta_1$, where $\alpha, \beta, \beta_1 \in A_k$. Thus, $\beta \cdot \beta_1$ has the P-d.p. in A_0 and γ has it, too. Therefore the condition $(k+1)$ holds. This ends the proof.

An element l of R^n is said to be a *direction of the subset M of* R^n *at the point* $p \in M$ iff there exist two sequences $p_1, p_2, \ldots$ and $p_1', p_2', \ldots$ of points of M tending to p such that $p_k \neq p_k'$, $k = 1, 2, \ldots$ and $(p_k-p_k')/|p_k-p_k'| \to l$, as $k \to \infty$ (see [7]). The vector subspace of R^n spanned by the set of all directions of M at p will be denoted by $\operatorname{Dir}_p M$.

We say that a locally bounded set A of real functions with $\operatorname{Points} A = R^n$, having the P-d.p., where $P = \{\pi_1, \ldots, \pi_n\}$, $\pi_i\colon R^n \to R$ being the standard projections, $i = 1, 2, \ldots, n$, and such that $\dim \operatorname{Dir}_p U = n$, when $p \in U \in \operatorname{top} A$, *establishes a quasi-algebraic structure in* R^n. Propositions 1.3, 1.4 and 1.5 yield

PROPOSITION 2.2. *If A establishes a quasi-algebraic structure in R^n, then the R-q.a.s. A_0 generated by A is a quasi-algebraic structure in R^n and every function belonging to A_0 is locally Lipschitzian, i.e., for every point p of the domain of $\alpha \in A_0$ there exist $U \in \operatorname{top} A$, $p \in U$ and $L > 0$ such that $|\alpha(q)-\alpha(q_1)| \leqslant L|q-q_1|$ for $q, q_1 \in U$. Thus all functions belonging to A_0, and all functions belonging to $A \times A$ as well, are continuous in the usual sense.*

Now we shall give some examples of quasi-algebraic structures in R^n.

EXAMPLE 1. Let A_1 be the set of all polynomial functions of n variables and all functions derived from them by restriction to open subsets of R^n as well.

EXAMPLE 2. Let A_2 be the set of all real functions α with domains which are open subsets in R^n such that for every point $p \in D_\alpha$ there exists a non-zero polynomial Q of $n+1$ variables such that $Q(x_1, \ldots, x_n, \alpha(x_1, \ldots, x_n)) = 0$ for $(x_1, \ldots, x_n)$ in some neighbourhood of p. These functions are called *Nash's functions*. For some interesting remarks concerning these functions see [1].

EXAMPLE 3. Let A_3 be the set of all real analytic functions of n variables with domains which are open subsets of R^n.

EXAMPLE 4. Let A_4 be the set of all C^∞-functions defined in open subsets of R^n.

We have the locally bounded R-q.a.s. A_h, $\operatorname{Points} A_h = R^n$, $\operatorname{top} A_h$ is the usual topology of R^n and A_h has the P-d.p., $h = 1, 2, 3, 4$. It seems interesting to find other examples of quasi-algebraic structures $A, A_1 \subset A \subset A_4$, different from A_1, A_2, A_3, A_4.

PROPOSITION 2.3. *If an* $\boldsymbol{R}$-q.a.s. A *is a quasi-algebraic structure in* $\boldsymbol{R}^n$, *then we have the isomorphism*

$$v \mapsto (v(\pi_1), \ldots, v(\pi_n)): \operatorname{Der}A(p) \to \boldsymbol{R}^n. \tag{2.2}$$

Hence we have the natural isomorphism

$$w \mapsto (\bar{w}(\pi_1), \ldots, \bar{w}(\pi_n)): T_pA \to \boldsymbol{R}^n, \tag{2.3}$$

where for any w *in* T_pA *the vector* $\bar{w}$ *of* $\operatorname{Der}A(p)$ *is defined by formula* (1.2).

Proof. Let $\alpha \in A(p)$, $p \in U \in \operatorname{top}A$ and let functions $\alpha_1, \ldots, \alpha_n$ belonging to $A \times A$ satisfy the equality

$$\sum_{i=1}^{n} \alpha_i(r, q)\,(\pi_i(r) - \pi_i(q)) = 0 \quad \text{for} \quad r, q \in U. \tag{2.4}$$

First we will prove that

$$\alpha_i(p, p) = 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{2.5}$$

Indeed, because of $\dim \operatorname{Dir}_p U = n$ there exist linearly independent directions $l_1, \ldots$ $\ldots, l_n$ at p of the set U. Thus there exist sequences $p_{j1}, p_{j2}, \ldots$ and $p'_{j1}, p'_{j2}, \ldots$, $j = 1, 2, \ldots, n$ of points of the set U such that $p_{jr} \neq p'_{jr}$, $r = 1, 2, \ldots$, $l_{jr} \to l_j$, as $r \to \infty$, where

$$l_{jr} = (p_{jr} - p'_{jr})/|p_{jr} - p'_{jr}|, \quad r = 1, 2, \ldots, \quad j = 1, \ldots, n.$$

Setting $s_{jr} = |p_{jr} - p'_{jr}|$ we have $p_{jr} = p'_{jr} + s_{jr} l_{jr}$, $s_{jr} > 0$. This yields $\pi_i(p_{jr}) - \pi_i(p'_{jr}) = s_{jr}\pi_i(l_{jr})$. Hence, by (2.4), it follows that

$$\sum_i \alpha_i(p_{jr}, p'_{jr})\pi_i(l_{jr}) = 0.$$

According to Proposition 2.2, from the continuity of α_i we get

$$\sum_{i=1}^{n} \alpha_i(p, p)\pi_i(l_j) = 0, \quad j = 1, \ldots, n.$$

Thus we have (2.5). Now, taking any $t = (t_1, \ldots, t_n) \in \boldsymbol{R}^n$ and $\alpha \in A(p)$, by the P-d.p., we can adopt the correct definition of the number $t_p(\alpha)$ by the following formula:

$$t_p(\alpha) = \sum_{i=1}^{n} \alpha_i(p, p)\,t_i, \tag{2.6}$$

where $\alpha_1, \ldots, \alpha_n \in (A \times A)(p, p)$ satisfy (2.1). It is evident that $t_p(\alpha+\beta) = t_p(\alpha) + t_p(\beta)$ and $t_p(c\alpha) = ct_p(\alpha)$ for $\alpha, \beta \in A(p)$ and $c \in \boldsymbol{R}$. To check that t_p is a vector of $\operatorname{Der}A(p)$ we take $\beta \in A(p)$. We may assume that

$$\beta(r) - \beta(q) = \sum_i \beta_i(r, q)\,(\pi_i(r) - \pi_i(q)) \tag{2.7}$$

for $r, q \in U$, where $U \subset D_\beta$. Thus, for $r, q \in U$

$$(\alpha\beta)(r) - (\alpha\beta)(q) = \sum_i \big(\alpha_i(r, q)\beta(r) + \beta_i(r, q)\alpha(q)\big)\big(\pi_i(r) - \pi_i(q)\big).$$

Hence

$$t_p(\alpha\beta) = \sum_i \big(\alpha_i(p,p)\beta(p)+\beta_i(p,p)\alpha(p)\big)t_i = \beta(p)t_p(\alpha)+\alpha(p)t_p(\beta).$$

Assuming $t_p = 0$, we get $t_p(\pi_j) = 0$, $j = 1, \dots, n$. On the other hand,

$$\pi_j(r)-\pi_j(q) = \sum_i \delta_{ij}\big(\pi_i(r)-\pi_i(q)\big).$$

Hence it follows that

$$0 = t_p(\pi_j) = \sum_i \delta_{ij}t_i = t_j, \quad j = 1, \dots, n.$$

Hence $t = 0$. Thus we obtain the monomorphism

$$(2.8) \qquad t \mapsto t_p\colon \boldsymbol{R}^n \to \operatorname{Der} A(p).$$

To prove that this monomorphism is an isomorphism let us take any vector v $\operatorname{Der} A(p)$. Setting $t = (v(\pi_1), \dots, v(\pi_n))$, we get

$$t_p(\alpha) = \sum_i \alpha_i(p,p)v(\pi_i) \quad \text{for} \quad \alpha \in A(p).$$

On the other hand,

$$\alpha(q)-\alpha(p) = \sum_i \alpha_i(q,p)\big(\pi_i(q)-\pi_i(p)\big) \quad \text{for} \quad q \in U.$$

Thus

$$v(\alpha) = \sum_i \alpha_i(p,p)v(\pi_i) = t_p(\alpha).$$

Hence $v = t_p$. Thus we have the mapping

$$v \mapsto \big(v(\pi_1), \dots, v(\pi_n)\big)\colon \operatorname{Der} A(p) \to \boldsymbol{R}^n,$$

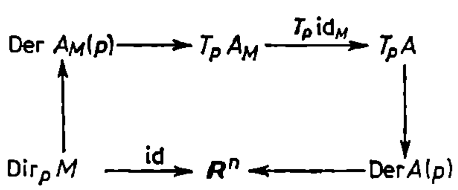
which is inverse to (2.8). This ends the proof.

PROPOSITION 2.4. *If A is a quasi-algebraic structure in $\boldsymbol{R}^n$, the topology induced in M by* top$A$ *coincides with the usual topology of M, $p \in M \subset \boldsymbol{R}^n$, and so we have a canonical monomorphism*

$$\operatorname{Dir}_p M \to \operatorname{Der} A_M(p)$$

such that the diagram

(2.9)



is commutative.

Proof. Let l be any direction of M at p. Then there exist two sequences, $p_1, p_2, \ldots$ and $p'_1, p'_2, \ldots$, of points of M tending to p, $p_k \neq p'_k$, $k = 1, 2, \ldots$, such that $l_k = (p_k - p'_k)/|p_k - p'_k| \to l$ as $k \to \infty$. Setting $s_k = |p_k - p'_k|$ we have $p_k = p'_k + s_k l_k$, where $0 < s_k \to 0$, as $k \to \infty$. Let $\alpha \in A_M(p)$. Then there exist a function $\beta \in A$ and a set U open in topology induced in M by top$A$, satisfying the conditions $p \in U \subset D_\alpha \cap D_\beta$ and $\alpha|U = \beta|U$. Every open set in this topology is, by hypothesis, open in the usual topology of M treated as a subspace of $\boldsymbol{R}^n$. So we may assume that $p_k, p'_k \in U$, $k = 1, 2, \ldots$ Therefore $\alpha(p_k) = \beta(p_k)$ and $\alpha(p'_k) = \beta(p'_k)$ for all k. By P-d.p. we have (2.7) for $q, r \in V$, where $p \in V \in \mathrm{top}A$. We may assume that $M \cap V \subset U$. Thus

$$(\alpha(p_k) - \alpha(p'_k))/|p_k - p'_k| = \sum_i \beta_i(p_k, p'_k)\pi_i(l_k) \to \sum_i \beta_i(p, p)\pi_i(l)$$

as $k \to \infty$. We then have the correct definition of $\hat{l}_p(\alpha)$ by the equality $\hat{l}_p(\alpha) = l_p(\beta)$. It is easy to check that $\hat{l}_p$ is a vector of $\mathrm{Der}A(p)$. We remark that, if $l_1, \ldots, l_r, l$ are directions of M at p, $c_1, \ldots, c_r \in \boldsymbol{R}$ and $l = c_1 l_1 + \ldots + c_r l_r$, then for any $\alpha \in A_M(p)$ we have

$$\hat{l}_p(\alpha) = l_p(\beta) = \sum_i \beta_i(p, p)\pi_i(l),$$

$$\hat{l}_{hp}(\alpha) = l_{hp}(\beta) = \sum_i \beta_i(p, p)\pi_i(l_h),$$

and

$$\Big(\sum_h c_h \hat{l}_{hp}\Big)(\alpha) = \sum_h c_h \sum_i \beta_i(p, p)\pi_i(l_h)$$

$$= \sum_i \beta_i(p, p)\pi_i\Big(\sum_h c_h l_h\Big) = \sum_i \beta_i(p, p)\pi_i(l) = \hat{l}_p(\alpha).$$

Thus $\hat{l}_p = \sum_h c_h \hat{l}_{hp}$. Then there exists exactly one linear extension of the function $l \mapsto \hat{l}_p$ to $\mathrm{Dir}_p M$. This extension will be written in the form

$$l \mapsto l_p \colon \mathrm{Dir}_p M \to \mathrm{Der}A_M(p). \tag{2.10}$$

Assuming $\hat{l}_p = 0$ where l is in $\mathrm{Dir}_p M$, we have $l = \sum_{h=1}^{m} c_h l_h$ where $l_1, \ldots, l_m$ are directions of M at p and form a basis of $\mathrm{Dir}_p M$. Thus

$$0 = \hat{l}_p(\pi_i) = \sum_h c_h \hat{l}_{hp}(\pi_i) \quad \text{and} \quad \hat{l}_{hp}(\pi_i) = \sum_j \delta_{ij}\pi_j(l_h) = \pi_i(l_h).$$

Hence

$$0 = \sum_h c_h \pi_i(l_h) = \pi_i\Big(\sum_h c_h l_h\Big), \quad i = 1, \ldots, n.$$

Hence it follows that $l = 0$. And we have monomorphism (2.10).

To check that diagram (2.9) is commutative we take any direction l of M at p. Then $\hat{l}_p$ is a vector of $\mathrm{Der}A_M(p)$. Hence there exists an u in T_pA_M for which $\hat{l}_p = u \circ A^{(2)}_{Mp} \circ A^0_{Mp}$. According to Proposition 1.1 we have

$$v = u \circ A^{(2)}_{Mp} \circ A^0_{Mp} \circ \mathrm{id}^*_M = \hat{l}_p \circ \mathrm{id}^*_M, \quad \text{where} \quad v = \overline{(T_p\mathrm{id}_M)(u)}.$$

Applying, by Proposition 2.3, isomorphism (3), we get

$$\big(v(\pi_1), \ldots, v(\pi_n)\big) = \big(\hat{l}_p(\mathrm{id}^*_M(\pi_1)), \ldots, \hat{l}_p(\mathrm{id}^*_M(\pi_n))\big) = \big(l_p(\pi_1), \ldots, l_p(\pi_n)\big) = l.$$

Hence it follows that diagram (2.9) is commutative. This ends the proof.

3. (A, m)-smooth representability of sets

Let A be a quasi-algebraic structure in $\boldsymbol{R}^n$. A subset M of $\boldsymbol{R}^n$ will be called (A, m)-*smooth representable* iff for any $p \in M$ there exist functions $F_1, \ldots, F_n \in A \times A$ and a neighbourhood B of p open in $\mathrm{top}A_M$ such that for exactly one function f we have

(a) $D_f \subset \boldsymbol{R}^m$ and $f[D_f] = B$;

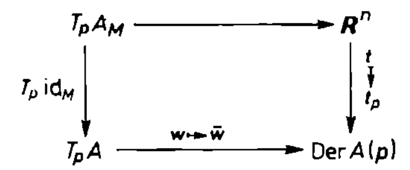
(b) for any $u \in D_f$, $\big(\iota(u), f(u)\big) \subset D_{F_1} \cap \ldots \cap D_{F_n}$ and

$$F\big(\iota(u), f(u)\big) = f(u),$$

where $\iota(u_1, \ldots, u_m) = (u_1, \ldots, u_m, 0, \ldots, 0)$ for $(u_1, \ldots, u_m) \in \boldsymbol{R}^m$ and $F(t) = \big(F_1(t), \ldots, F_n(t)\big)$.

THEOREM. *If A is a quasi-algebraic structure in $\boldsymbol{R}^n$, $\mathrm{top}A$ coincides with the usual topology of $\boldsymbol{R}^n$ and M is such a subset of $\boldsymbol{R}^n$ that $\dim T_pA_M = m$ at any point p of M, then M is (A, m)-smooth representable.*

Proof. Let $p \in M$ and φ denote the composition of the monomorphism $T_p\mathrm{id}_M\colon T_pA_M \to T_pA$ and, by Proposition 2.3, of isomorphism (2.3). Then we have the commutative square



Let $z_1, \ldots, z_m$ be an orthonormal basis of $\mathrm{Im}\varphi$. We take vectors $z'_1, \ldots, z'_m$ such that $\varphi(z'_h) = z_h$, $h = 1, \ldots, m$. They are linearly independent. We complete the basis $z_1, \ldots, z_m$ to the orthonormal basis $z_1, \ldots, z_n$ of $\boldsymbol{R}^n$, and we set $\gamma_k(q) = z_k(q-p)$ for $q \in \boldsymbol{R}^n$. From the fact that the canonical projections $\pi_i\colon \boldsymbol{R}^n \to \boldsymbol{R}$ belong to A, $i = 1, \ldots, n$, it follows that $\gamma_1, \ldots, \gamma_n \in A$. We have $\gamma_k(p) = 0$. We set $\xi_k = A^0_{Mp}(\gamma_k \circ \mathrm{id}_M)$ and $\omega_k = A^{(2)}_{Mp}(\xi_k)$, $k = 1, \ldots, n$. Thus, see diagram (1.3),

$$\begin{aligned} z'_h(\omega_k) &= z'_h\big(A^{(2)}_{Mp}(A^0_{Mp}(\mathrm{id}^*_M\gamma_k))\big) = z'_h\Big(T^*_p\mathrm{id}_M\big(A^{(2)}_p(A^0_p(\gamma_k))\big)\Big) \\ &= (z'_h \circ T^*_p\mathrm{id}_M)\big(A^2_p(A^0_p(\gamma_k))\big) = (T_p\mathrm{id}_M)(z'_h)\big(A^{(2)}_p(A^0_p(\gamma_k))\big) \\ &= \overline{\big((T_p\mathrm{id}_M)(z'_h)\big)}(\gamma_k) = \big(\varphi(z'_h)\big)_p(\gamma_k) = z_{hp}(\gamma_k), \end{aligned}$$

where for any t in $\boldsymbol{R}^n$ the vector t_p of $\operatorname{Der} A(p)$ has been defined by formula (2.6). We have

$$\gamma_k(r)-\gamma_k(q) = z_k(r-q) = \sum_i \big(\pi_i(r)-\pi_i(q)\big)(z_k e_i) \quad \text{for} \quad q, r \in \boldsymbol{R}^n,$$

where $e_i = (\delta_{i1}, \ldots, \delta_{in})$. Thus

$$z_{hp}(\gamma_k) = \sum_i (z_k e_i)(z_h e_i) = \delta_{kh}.$$

Hence $z_h'(\omega_k) = \delta_{kh}$, $h = 1, \ldots, m$; $k = 1, \ldots, n$. We then have linearly independent vectors $\omega_1, \ldots, \omega_m$ of $T_p^* A_M$. On the other hand, $\dim T_p^* A_M = \dim T_p A_M = m$. Therefore $\omega_1, \ldots, \omega_m$ is a basis of $T_p^* A_M$. Thus, for some real numbers a_{hj}, we have

$$\xi_j = \sum_{h=1}^{m} a_{hj}\omega_h, \quad j = m+1, \ldots, n.$$

This yields

$$A_{Mp}^{(2)}\xi_j = \sum_{h=1}^{m} a_{hj} A_{Mp}^{(2)}\xi_h = A_{Mp}^{(2)}\Big(\sum_{h=1}^{m} a_{hj}\xi_h\Big).$$

Hence there exist $\mu_{1j}, \nu_{1j}, \ldots, \mu_{sj}, \nu_{sj}$ in $\tilde{A}_M^0(p)$ such that

$$\xi_j = \sum_{h=1}^{m} a_{hj}\xi_h + \sum_{t=1}^{s} \mu_{tj}\nu_{tj}, \quad j = m+1, \ldots, n.$$

Hence it follows that

$$A_{Mp}^0(\gamma_j|M) = \sum_{h=1}^{m} a_{hj} A_{Mp}^0(\gamma_h|M) + \sum_{t=1}^{s} A_{Mp}^0(\alpha_{tj}') A_{Mp}^0(\beta_{tj}'),$$

where $\alpha_{tj}', \beta_{tj}' \in A_M^0(p)$. Hence

$$A_{Mp}^0(\gamma_j|M) = A_{Mp}^0\Big(\sum_{h=1}^{m} a_{hj}\gamma_h|M + \sum_{t=1}^{s} \alpha_{tj}'\beta_{tj}'\Big).$$

Thus there exist functions $\alpha_{tj}, \beta_{tj} \in A^0(p)$ and $U \in \operatorname{top} A_M$ such that $p \in U \subset \bigcap_{t,j} (D_{\alpha_{tj}} \cap D_{\beta_{tj}})$ and

$$\gamma_j|U = \Big(\sum_{h=1}^{m} a_{hj}\gamma_h + \sum_{t=1}^{s} \alpha_{tj}\beta_{tj}\Big)\Big| U, \quad j = m+1, \ldots, n. \tag{3.1}$$

We have $U = M \cap V$, where $V \in \operatorname{top} A$ and $D_{\alpha_{tj}}, D_{\beta_{tj}} \in \operatorname{top} A$. Thus there exists an $r_0 > 0$ such that

$$B^n(p; r_0) \subset \bigcap_{t=1}^{s} \bigcap_{j>m} (D_{\alpha_{tj}} \cap D_{\beta_{tj}}) \cap V.$$

Then $M \cap B^n(p; r_0) \subset U$. By Proposition 2.2 the functions α_{tj}, β_{tj} are locally Lipschitzian. Hence there exist $L > 0$ and $r_1 \in (0; r_0)$ such that $|\alpha_{tj}(q)-\alpha_{tj}(q_1)| \leqslant L|q-q_1|$ and $|\beta_{tj}(q)-\beta_{tj}(q_1)| \leqslant L|q-q_1|$ for $q, q_1 \in B^n(p; r_1)$. We have $\alpha_{tj}(p) = \beta_{tj}(p) = 0$.

Thus there exists an $r_2 \in (0; r_1)$ such that $|\alpha_{ij}(q)| < \varepsilon$ and $|\beta_{ij}(q)| < \varepsilon$ for $q \in B^n(p; r_2)$, where $\varepsilon = 1/(3Ls(n-m))$. Setting

$$\lambda_j(q) = \sum_{i=1}^{s} \alpha_{ij}(q)\beta_{ij}(q) \quad \text{for} \quad q \in B^n(p; r_2),$$

we get $\lambda_j \in A$ and

$$|\lambda_j(q) - \lambda_j(q_1)| \leqslant \sum_{i=1}^{s} \big(|\alpha_{ij}(q) - \alpha_{ij}(q_1)||\beta_{ij}(q)| + |\beta_{ij}(q) - \beta_{ij}(q_1)||\alpha_{ij}(q_1)|\big).$$

Hence

$$(3.2) \qquad |\lambda_j(q) - \lambda_j(q_1)| \leqslant \frac{2}{3(n-m)}|q - q_1| \quad \text{for} \quad q, q_1 \in B^n(p; r_2).$$

Let us set for $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and $q \in B^n(p; r_2)$

$$F(x, q) = p + \sum_{i=1}^{m} x_i z_i + \sum_{j=m+1}^{n} \Big(\sum_{i=1}^{m} a_{ij} x_i + \lambda_j(q)\Big) z_j.$$

From the fact that $\pi_1, \ldots, \pi_n$ belong to A and that $B^n(p; r_2) \in \operatorname{top} A$ it follows that $F_1, \ldots, F_n$ defined by the equality

$$(3.3) \quad F(x, q) = F_1(x, q)e_1 + \ldots + F_n(x, q)e_n \quad \text{for} \quad x \in \boldsymbol{R}^n \text{ and } q \in B^n(p; r_2)$$

belong to $A \times A$. Here $e_1, \ldots, e_n$ form the standard basis of $\boldsymbol{R}^n$.

We have $\operatorname{Im}\varphi \subset \boldsymbol{R}^n$. Setting $H = p + \operatorname{Im}\varphi$, by Proposition 2.4, we get the hyperplane H containing p and all points $p + l$, where l is any direction of M at p. Thus there exist $r \in (0; r_2)$ (see [7]) and a function g satisfying the following conditions:

(i) the domain D_g of g is contained in H and g is uniformly continuous in D_g;

(ii) $g[D_g] = M \cap B^n(p; r)$;

(iii) for any $q \in D_g$ the point q is the orthogonal projection of $g(q)$ onto the hyperplane H.

Now let us take the isometry h of $\boldsymbol{R}^n$ onto $\boldsymbol{R}^n$ which sends $0, e_1, \ldots, e_n$ into $p, p+z_1, \ldots, p+z_n$, respectively. Setting $f = g \circ h \circ \iota$, where $\iota\colon \boldsymbol{R}^m \to \boldsymbol{R}^n$ and $\iota(u_1, \ldots, u_m) = (u_1, \ldots, u_m, 0, \ldots, 0)$ for $(u_1, \ldots, u_m) \in \boldsymbol{R}^m$, we have the mapping f with the following properties:

(iv) $D_f \subset B^m(0; r)$ and f is uniformly continuous in D_f;

(v) $f[D_f] = M \cap B^n(p; r)$.

Consider any $u = (u_1, \ldots, u_m) \in D_f$. Then we have

$$h(\iota(u)) = p + u_1 z_1 + \ldots + u_m z_m \in D_g \quad \text{and} \quad f(u) = g(h(\iota(u))) \in U.$$

Hence, by (3.1), we have

$$\gamma_j(f(u)) = \sum_{i=1}^{m} a_{ij}\gamma_i(f(u)) + \sum_{i=1} \alpha_{ij}(f(u))\beta_{ij}(f(u)), \quad j = m+1, \ldots, n.$$

From the definition of γ_i we obtain in turn

$$\gamma_i(g(q)) = z_i(g(q)-p) = z_i(q-p) \quad \text{for} \quad q \in D_g, \quad i = 1, \ldots, m,$$
$$\gamma_i(f(u)) = z_i(h(\iota(u))-p) = u_i,$$
$$\gamma_j(f(u)) = \sum_{i=1}^{m} a_{ij}u_i + \lambda_j(f(u)), \quad j = m+1, \ldots, n,$$
$$f(u) = p + \sum_{k=1}^{n} \gamma_k(f(u)-p)z_k = p + \sum_{i=1}^{m} u_i z_i + \sum_{j=m+1}^{n} \Big(\sum_{i=1}^{m} a_{ij}u_i + \lambda_j(f(u))\Big) z_j.$$

Hence, by the definition of the function F, we get the formula

$$f(u) = F(\iota(u), f(u)) \quad \text{for} \quad u \in D_f. \tag{3.4}$$

From inequality (3.2), for $x \in \boldsymbol{R}^m$ and $q, q_1 \in B^n(p; r_2)$, we get

$$|F(x, q) - F(x, q_1)| \leqslant \sum_{j=m+1}^{n} |\lambda_j(q) - \lambda_j(q_1)| \leqslant \tfrac{2}{3}|q - q_1|.$$

Remark that $F(\iota(0), p) = p$ and for $u \in \boldsymbol{R}^m$

$$F(\iota(u), p) = p + \sum_{i=1}^{m} u_i z_i + \sum_{j=m+1}^{n} \Big(\sum_{i=1}^{m} a_{ij}u_i\Big) z_j.$$

Thus there exists an $r' \in (0; r)$ such that $|F(\iota(u), p) - p| \leqslant r/4$ for $u \in B^m(0; r)$. This yields the inequality

$$\sup\{|F(\iota(u), p) - p|;\ u \in B^m(0; r')\} < r/3.$$

Hence it follows (see [4], p. 190) that, for every $u \in B^m(0; r')$, $f(u)$ is the unique point $q \in B^n(p; r)$ such that $F(\iota(u), q) = q$. Hence f is a local (A, m)-smooth representation of the set M at the point p. This ends the proof.

As corollaries of the theorem proved above we obtain the results of papers [6] and [7]. It suffices to take as the $\boldsymbol{R}$-quasi-algebraic structures the $\boldsymbol{R}$-q.a.s.'s A_3 and A_4 from Examples 3 and 4, respectively.

References

[1] S. Łojasiewicz, *Ensembles semi-analytiques*, Institut des Hautes Etudes Scientifiques Bures-sur-Yvette (Seine-et-Dise, France, 1965.

[2] M. M. Postnikov, *Introduction to the Morse Theory*, Moskva 1971 [in Russian].

[3] R. Sikorski, *Abstract covariant derivative*, Colloq. Math. 18 (1967), 251–279.

[4] —, *Advanced Calculus. Functions of Several Variables*, PWN, Warszawa 1969.

[5] W. Waliszewski, *Quasi-algebraic spaces*, Zeszyty Naukowe Akademii Górniczo-Hutniczej w Krakowie (a volume devoted to the memory of Professor Antoni Hoborski) (to appear).

[6] —, *A characterization of a regular analytic m-dimensional hypersurfaces of the Euclidean space*, Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, Boszkowo, Poland, February 1–10, 1978, Scientific communications, Part 2, Institute of Mathematics, Polish Academy of Sciences, pp. 222–232.

[7] —, *On differential subspaces of Cartesian space*, Colloq. Math. 45, 2 (1981), 257–265.

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