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On a class of initial-boundary value problems in a domain with boundary containing isolated points

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Abstract. Let us consider a strongly elliptic linear differential operator $P(x, D)$ of order $2m$ defined in the domain $\Omega = \tilde{\Omega}\setminus\{0\}$, where $\tilde{\Omega}$ is a bounded domain of R_n containing the origin. The paper deals with the initial-boundary value problem for the equation $P(x, D)u+ +D_t^2 u = f(x, t)$ with initial conditions $u|_{t=0} = \varphi_0$, $D_t u|_{t=0} = \varphi_1$ and boundary condition $u(\cdot, t) \in H_m(\Omega)$ for $t > 0$. Here f, φ_1 are in L_2 and φ_0 satisfies some regularity assumptions together with the boundary condition. It is proved that the latter yields an asymptotic formula for $x \to 0$, which generalizes in a way the well-known Sobolev inequality. The problem is solved by the Fourier method, which gives a distributional solution satisfying the energy inequality.

The present paper is devoted to the study of the initial-boundary value problem for the linear equation of the form

$$(1) \qquad P(x, D)u+D_t^2 u = f(x, t),$$

where P is a strongly elliptic operator of order $2m$ with variable coefficients. Equation (1) is considered in the cylindrical domain $\Omega_\infty = \Omega\times(0, \infty)$. Here $\Omega = \tilde{\Omega}\setminus\{0\}$, where $\tilde{\Omega}$ denotes a bounded domain of R_n containing the origin. As the boundary $\partial\Omega$ has a portion of codimension $n > 1$, we are dealing with an example of the boundary value problem of the so called Sobolev type. A problem of this kind has been first introduced by S. L. Sobolev (see [9]), who considered the polyharmonic equation in a domain of R_n, whose boundary contained a finite number of disjoint surfaces of codimension greater than one. Later on B. Yu. Sternin [10] considered similar problems for general elliptic and parabolic (in the sense of Petrovskiĭ) operators. The present paper is a first step towards considering problems of Sobolev type for equations of form (1). This class contains, in particular, some equations occurring in the classical problems of physics, as the wave equation or the equation of a vibrating plate.

Our method is different from those used by Sternin [7]. Namely, we solve the initial-boundary value problem by means of the Fourier method.

Therefore we are dealing at first with the corresponding elliptic boundary value problem, which will be posed in terms of the Dirichlet bilinear form, in the so-called variational formulation (see [1], [2], [4]). The boundary condition is defined by the requirement that the solution has to belong to the closure $\mathring{H}_m(\Omega)$ of $C_0^\infty(\Omega)$ in the Sobolev space $H_m(\Omega)$. It is well known that in the case of a domain with smooth boundary the assumption $w \in \mathring{H}_m(\Omega)$ is equivalent to that of vanishing of all $D^\alpha w|_{\partial\Omega}$ with $|\alpha| < m$, where the boundary value is understood in the sense of trace (see [5]–[7]). In our case the situation is quite different, because the boundary of the domain under consideration is not a smooth manifold. It will be proved in Section 3 that the condition $w \in \mathring{H}_m(\Omega)$ yields an asymptotic formula for the function w as $x \to 0$. Obviously, our results may be easily extended to the case where the irregular part of $\partial\Omega$ consists of a finite number of isolated points.

1. The spectral properties of the elliptic boundary-value problem. In this section we give an outline of the properties of elliptic boundary value problems, which will be used in the sequel. To begin with, let Ω be an arbitrary domain of R_n, V a closed subspace of $H_m(\Omega)$ satisfying the condition

$$C_0^\infty(\Omega) \subset V \subset H_m(\Omega) \tag{2}$$

and B a bilinear form over V continuous in the topology of $H_m(\Omega)$. We shall use in the sequel the notation

$$B_\lambda(\cdot,\cdot) = B(\cdot,\cdot) + \lambda(\cdot,\cdot)$$

for arbitrary complex number λ; $(\cdot,\cdot)$ will allways denote the scalar product in $L_2(\Omega)$. By $\mathring{H}_m(\Omega)$ we shall denote the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$. If X, Y are two Hilbert spaces, then $\mathscr{L}(X, Y)$ will denote the set of all linear and continuous operators from X into Y. The derivation will be understood in the weak (distributional) sense. We say that (see [1], [7])

1° B is coercive in V if there exist constants $\lambda_0 \geqslant 0$ and $c > 0$ such that

$$\operatorname{Re} B(v, v) + \lambda_0(v, v) \geqslant c\,\|v\|_m^2 \quad (x \in V);$$

2° a function $w \in V$ is a solution of the problem $(V_{\lambda,g})$ with given $g \in L_2(\Omega)$ and complex number λ, if

$$B_\lambda(w, v) = (g, v)$$

holds identically for $v \in V$;

3° λ is an eigenvalue of $(V_{\lambda,g})$ if the problem $(V_{\lambda,0})$ has non-vanishing solutions; these solutions are called *eigenfunctions* of $(V_{\lambda,g})$ corresponding to the eigenvalue λ.

Our further considerations will be based on the following well-known

THEOREM A. *Suppose that*

(a) *the form B is coercive and hermitian in V;*

(b) *the set of all solutions of the problem $(V_{\lambda_0,g})$, where g runs over $L_2(\Omega)$ is dense in $L_2(\Omega)$;*

(c) *the embedding $V \subset L_2(\Omega)$ is completely continuous.*

Then

1° *there exists a solution operator $S \in \mathscr{L}(L_2(\Omega), V)$ such that*

$$B_{\lambda_0}(Sg, v) = (g, v) \quad (v \in V) \tag{3}$$

for arbitrary given $g \in L_2(\Omega)$;

2° *the problem $(V_{\lambda,g})$ has a contable set of real eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfying the conditions*

$$\lambda_k < \lambda_0 \quad (k = 1, 2, \dots) \tag{4}$$

and

$$\lim_{k\to\infty} \lambda_k = -\infty; \tag{5}$$

3° *there exists a sequence of eigenfunctions $\{w_k\}_{k=1}^{\infty}$ which forms an orthonormal basis in $L_2(\Omega)$.*

For the convenience of the reader we give an outline of the proof. It follows from (a) that the form B_{λ_0} is a scalar product over V inducing a norm equivalent to the usual Sobolev norm $\| \ \|_m$. According to the Riesz theorem, the linear functional

$$V \ni v \to (q, v)$$

may be represented by means of B_{λ_0}. This yields assertion 1°. According to assumption (c), the operator S considered as a member of $\mathscr{L}(L_2(\Omega), L_2(\Omega))$ is completely continuous. It can be shown without difficulty that if is self-adjoint in $L_2(\Omega)$ and that the problem $(V_{\lambda,0})$ is equivalent to the equation

$$w + \mu S w = 0 \tag{6}$$

with $\mu = \lambda - \lambda_0$. Assertions 2° and 3° now follow from the well-known properties of a completely continuous operator in a Hilbert space.

According to Theorem A every function $g \in L_2(\Omega)$ can be developed into its Fourier series

$$g = \sum_{k=1}^{\infty} (g, w_k) w_k, \tag{7}$$

convergence being meant in the $L_2(\Omega)$-norm.

We now prove

THEOREM 1. *Suppose that the assumptions of Theorem* A *are fulfilled and that* $g \in \operatorname{Im} S$. *Then the development* (7) *remains true with convergence in* $H_m(\Omega)$.

Proof. It is easy to verify that S is invertible, because $Sg_1 = Sg_2$ yields

$$(g_1, v) = (g_2, v)$$

identically for $v \in C_0^\infty(\Omega)$. Thus S^{-1} is well defined on $\operatorname{Im} S$. For any $g_1, g_2 \in \operatorname{Im} S$ let us put $h_j = S^{-1} g_j$ $(j = 1, 2)$. Then

$$B_{\lambda_0}(g_1, v) = (h_1, v) \quad (v \in V) \tag{8}$$

and

$$B_{\lambda_0}(v, g_2) = (v, h_2) \quad (v \in V). \tag{9}$$

Putting $v = g_2$ in (8) and $v = g_1$ in (9) we get

$$(S^{-1}g_1, g_2) = (g_1, S^{-1}g_2). \tag{10}$$

We have proved that S^{-1} is symmetric.

According to (7) we have

$$(S^{-1}g, g) = \sum_{k=1}^{\infty} (S^{-1}g, w_k)\overline{(g, w_k)};$$

hence it follows from (10) and (6) that

$$(S^{-1}g, g) = -\sum_{k=1}^{\infty} \mu_k |(g, w_k)|^2 \quad (\mu_k = \lambda_k - \lambda_0) \tag{11}$$

for each $g \in \operatorname{Im} S$. Let us denote

$$r_p = g - \sum_{j=1}^{p} (g, w_j) w_j.$$

We have to prove that $\|r_p\|_m \to 0$ as $p \to \infty$. Obviously $r_p \in \operatorname{Im} S$; Thus (10) with g_1, g_2 replaced by r_p yields

$$(S^{-1} r_p, r_p) = -\sum_{k=1}^{\infty} \mu_k |(r_p, w_k)|^2. \tag{12}$$

But a simple calculation shows that

$$(r_p, w_k) = \begin{cases} 0 & \text{for } k = 1, \ldots, p, \\ (g, w_k) & \text{for } k = p+1, \ldots, \end{cases}$$

and so (12) gives

$$(S^{-1} r_p, r_p) = -\sum_{k=p+1}^{\infty} \mu_k |(g, w_k)|^2. \tag{13}$$

As the series on the right of (11) converges, the right-hand side of (13) tends to zero as $p \to \infty$ and so does $(S^{-1}r_p, r_p)$. But, according to the definition of the operator S and the coercivity of B, we obtain

$$(S^{-1}r_p, r_p) \geqslant c\,\|r_p\|_m^2$$

and this completes the proof.

From now on we shall suppose that the elliptic part of (1) is given in the divergent form

$$P(x, D) = \sum_{|\alpha|,|\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta}(x) D^\beta \quad (x \in \Omega). \tag{14}$$

Till the end of this section B will be the corresponding Dirichlet bilinear form:

$$B(w, v) = \sum_{|\alpha|,|\beta| \leqslant m} (a_{\alpha\beta} D^\beta w, D^\alpha v) \quad (w, v \in H_m(\Omega))$$

and the subspace V will be equal to $\mathring{H}_m(\Omega)$.

We now prove the following

THEOREM 2. *Suppose that:*

(a_1) $\Omega = \tilde{\Omega}\setminus\{0\}$, *where* $\tilde{\Omega}$ *is a bounded domain of* R_n *having the segment property and containing the origin;*

(a_2) P *is strongly elliptic in* $\bar{\Omega}$ *and formally selfadjoint;*

(a_3) $a_{\alpha\beta}$(1) *are continuous in* $\bar{\Omega}$ *for* $|\alpha| = |\beta| = m$ *and bounded for* $|\alpha|+|\beta| < 2m$;

(a_4) $P(x, D)\varphi \in L_2(\Omega)$ *for* $\varphi \in C_0^\infty(\Omega)$.

Then the assertions of Theorem A *are satisfied.*

Proof. It suffices to show that the assumptions of Theorem A are satisfied in our case.

It remains to prove (b). Using (a_3), (a_4) we obtain for $\varphi, v \in C_0^\infty(\Omega)$ yields the coercivity of B. It follows also from (a_2) that $a_{\alpha\beta} = \bar{a}_{\beta\alpha}$; therefore B is hermitian and so assumption (a) is verified. The space $\mathring{H}_m(\Omega)$ is contained in $H_m(\tilde{\Omega})$; therefore, according to Rellich's theorem (see [1]), we have (c).

It remains to prove (b). Using (a_3), (a_4) we obtain for $\varphi, v \in C_0^\infty(\Omega)$

$$\big(P(x, D)\varphi, v\big) = B(\varphi, v)$$

and passing to the limit in the Sobolev norm $\|\ \|_m$ we obtain the last equality for any $v \in \mathring{H}_m(\Omega)$. Therefore

$$B_{\lambda_0}(\varphi, v) = (h, v) \quad (v \in \mathring{H}_m(\Omega)), \tag{15}$$

where $h = P(x, D)\varphi + \lambda_0\varphi$. This means that each $\varphi \in C_0^\infty(\Omega)$ is a solution of $(V_{\lambda_0,h})$ with suitably defined $h \in L_2(\Omega)$, and so condition (b) is proved.

As a corollary we obtain

(1) Obviously, $a_{\alpha\beta} \in H_{\max(|\alpha|,|\beta|)}(\Omega)$ is sufficient for (a_4) to hold.

THEOREM 3. *Assume* (a_1)–(a_4) *and suppose that a function* $g \in \mathring{H}_m(\Omega)$ *satisfies regularity condition*

$$P(x, D)g \in L_2(\Omega). \tag{16}$$

Then the development (7) *holds true with convergence in* $H_m(\Omega)$.

The proof follows immediately from the two precedings theorems if we note that (15) holds with φ replaced by g and therefore $g \in \operatorname{Im} S$.

2. A generalization of Sobolev's inequality. Let Ξ be a domain in the unit sphere of R_n. For fixed $\mathring{x} \in R_n$ and real numbers $0 < r < \delta$ we introduce the following notation:

$$\Gamma_\delta(\mathring{x}, \Xi) = \{x \in R_n \colon x = \mathring{x} + s\xi, \text{ where } \xi \in \Xi, 0 < s < \delta\},$$

$$\Gamma_{r,\delta}(\mathring{x}, \Xi) = \{x \in \Gamma_\delta(\mathring{x}, \Xi) \colon s > r\}.$$

For an arbitrary domain Ω in R_n we shall now denote by $\|\ \|_{m,\Omega}$ the norm in the Sobolev space $H_m(\Omega)$. By $d\sigma_\xi$ we shall mean the surface measure on the unit sphere. We say that a point $\mathring{x} \in \bar{\Omega}$ has the cone property if there exist δ and Ξ such that $\Gamma_\delta(\mathring{x}, \Xi) \subset \Omega$. For simplicity we shall write in the sequel Γ_δ or $\Gamma_{r,\delta}$, omitting the arguments $\mathring{x}$, Ξ.

PROPOSITION 1. *Let* Ω *be a domain in* R_n *and let* $\mathring{x} \in \bar{\Omega}$ *be a point having the cone property. Let* $u \in C^m(\Gamma_\delta) \cap H_m(\Omega)$, *where*

$$1 \leqslant m \leqslant \tfrac{1}{2}n. \tag{17}$$

Then there exist positive constants c_δ, r_δ (*not depending on* u) *such that*

$$r^n \int_\Xi |u(\mathring{x} + r\xi)|^2 d\sigma_\xi \leqslant c_\delta \|u\|^2_{m,\Gamma_{r,\delta}} \tag{18}$$

for $0 < r < r_\delta$.

Proof. We use the Taylor formula in the form

$$f(r) = \sum_{j=0}^{m-1} \frac{(r-s)^j}{j!} f^{(j)}(s) + \frac{1}{(m-1)!} \int_s^r (r-\tau)^{m-1} f^{(m)}(\tau)\, d\tau \qquad (0 < r < s < \delta) \tag{19}$$

with $f(r) = g(r)u(\mathring{x} + r\xi)$, where $g \in C^m(0, \delta)$. Applying the Schwarz inequalities we get from (19)

$$|g(r)u(\mathring{x} + r\xi)|^2 \leqslant c_1\Big(\sum_{j=0}^{m-1} \delta^{2j} \sum_{k=0}^{j} |g^{(k)}(s)|^2 \sum_{|\alpha| = j-k} |D^\alpha u(\mathring{x} + s\xi)|^2 + \sum_{k=0}^{m} d_k(r) \sum_{|\alpha| = m-k} \int_r^\delta |D^\alpha u(\mathring{x} + \tau\xi)|^2 \tau^{n-1}\, d\tau\Big), \tag{20}$$

where

$$d_k(r) = \int_r^\delta \tau^{2m-n-1} |g^{(k)}(\tau)|^2\, d\tau. \tag{21}$$

For fixed $p \in (\frac{1}{2}n-1, \frac{1}{2}n)$ let us write

$$g(r) = r^p; \tag{22}$$

then

$$d_k(r) = \frac{[p(p-1)\dots(p-k+1)]^2}{2(m+p-k)-n}\,(\delta^{2(m+p-k)-n} - r^{2(m+p-k)-n}). \tag{23}$$

It follows from (17) that $p > m-1$; thus all the derivatives $g^{(k)}$ $(k = 0, 1, \dots, m-1)$ are bounded in the interval $(0, \delta)$. Moreover, we have the estimates

$$0 < d_k(r) \leqslant \frac{[p(p-1)\dots(p-k+1)]^2}{2(m+p-k)-n}\,\delta^{2(m+p-k)-n} \quad (k = 0, 1, \dots, m-1)$$

and

$$0 < d_m(r) \leqslant \frac{[p(p-1)\dots(p-m+1)]^2}{n-2p}\,r^{2p-n},$$

which together with (20) yield

$$r^{2p}|u(\mathring{x}+r\xi)|^2 \leqslant c_2\Big(\sum_{|\alpha|\leqslant m} |D^\alpha u(\mathring{x}+s\xi)|^2 + \sum_{1\leqslant|\alpha|\leqslant m} I_\alpha(r, \xi) + r^{2p-n} I_0(r, \xi)\Big), \tag{24}$$

where

$$I_\alpha(r, \xi) = \int_r^\delta |D^\alpha u(\mathring{x}+\tau\xi)|^2 \tau^{n-1}\,d\tau \quad (0 \leqslant |\alpha| \leqslant m).$$

Let us integrate both sides of (24) over the cone $\Gamma_{r,\delta}$. We have

$$\int_{\Gamma_{r,\delta}} I_\alpha = \int_{\Xi}\int_r^\delta s^{n-1} \int_r^\delta |D^\alpha u(\mathring{x}+\tau\xi)|^2 \tau^{n-1}\,d\tau\,ds\,d\sigma_\xi$$

and this identity yields, after interchanging the integrals,

$$\int_{\Gamma_{r,\delta}} I_\alpha \leqslant \delta^n \int_{\Gamma_{r,\delta}} |D^\alpha u|^2. \tag{25}$$

Moreover,

$$\int_{\Gamma_{r,\delta}} |u(\mathring{x}+r\xi)|^2 = \frac{\delta^n - r^n}{n} \int_{\Xi} |u(\mathring{x}+r\xi)|^2\,d\sigma_\xi, \tag{26}$$

and for $0 \leqslant r \leqslant r_\delta = \delta(2)^{-1/n}$ we have

$$\frac{1}{n}(\delta^n - r^n) \geqslant \frac{\delta^n}{2n}; \tag{27}$$

therefore integrating (24) over $\Gamma_{r,\delta}$ and using (25)–(27) we obtain the estimate

$$r^{2p}\int_{\Xi} |u(\mathring{x}+r\xi)|^2\,d\sigma_\xi \leqslant c_3(\|u\|^2_{m,\Gamma_{r,\delta}} + r^{2p-n}\|u\|^2_{0,\Gamma_{r,\delta}}) \quad (0 < r \leqslant r_\delta),$$

which yields (18).

Remark. If $u \in C^m(\bar{\Gamma}_\delta)$ and $m > \frac{1}{2}n$, then (18) follows immediately from the well-known inequality due to Sobolev (see [1], [9]):

$$\sup_{x \in \Gamma_\delta} |u(x)| \leqslant \text{const } \|u\|_{m,\Gamma_\delta}. \tag{28}$$

PROPOSITION 2. *Under the suppositions of Proposition* 1 *the inequality*

$$\sup_{r_0 \leqslant r \leqslant r_\delta} \Big(r^n \int_\Xi |u(\mathring{x}+r\xi)|^2 d\sigma_\xi\Big)^{1/2} \leqslant c_\delta \|u\|_{m,\Gamma_{r_0,\delta}} \tag{29}$$

holds for every $r_0 > 0$ *and* $u \in H_m(\Omega)$ *(with* c_δ *not depending on* u, r_0).

Proof. As the cone $\Gamma_{r_0,\delta}$ has obviously the segment property, there exists a sequence $\{u_\nu\} \subset C^\infty(R_n)$ tending to u in $H_m(\Gamma_{r_0,\delta})$. The left-hand side of (29) defines a norm in the space of continuous functions in Γ_{r_0,r_δ}, which we denote by $|\ |_{r_0,r_\delta}$. For smooth u inequality (29) follows immediately from (18); therefore

$$|u_\nu|_{r_0,r_\delta} \leqslant c\, \|u_\nu\|_{m,\Gamma_{r_0,\delta}}. \tag{30}$$

It remains to prove that one may pass to the limit in (30).

It follows from (29) applied to the function $u_\nu - u_\mu$ that $\{u_\nu\}$ is a Cauchy sequence in the norm $|\ |_{r_0,r_\delta}$, and so it has a limit $\tilde{u}$ belonging to the completion of the set $C^\infty(\overline{\Gamma_{r_0,r_\delta}})$ in the norm $|\ |_{r_0,r_\delta}$. To identify $\tilde{u}$ and $u|_{\Gamma_{r_0,r_\delta}}$, it sufficies to show that the two norms under consideration are compatible on Γ_{r_0,r_δ}; in other words: if $|u_\nu|_{r_0,r_\delta} \to 0$ and $\{u_\nu\}$ is a Cauchy sequence in the norm $\|\ \|_{m,\Gamma_{r_0,\delta}}$, then $\|u_\nu\|_{m,\Gamma_{r_0,r_\delta}} \to 0$ as $\nu \to \infty$. The last statement follows immediately from the following

LEMMA 1. *Convergence in the norm* $|\ |_{r_0,r_\delta}$ *is stronger than convergence in the space* $\mathscr{D}'(\Gamma_{r_0,r_\delta})$.

Proof of the lemma. After introducing the spherical coordinates we have for fixed $\varphi \in \mathscr{D}(\Gamma_{r_0,r_\delta})$ and continuous u

$$\Big|\int_{\Gamma_{r_0,r_\delta}} \varphi(x)u(x)dx\Big| \leqslant (r_\delta - r_0) \sup_{r_0 \leqslant r \leqslant r_\delta} r^{n-1} \Big|\int_\Xi \varphi(\mathring{x}+r\xi)u(\mathring{x}+r\xi)d\sigma_\xi\Big|. \tag{31}$$

Applying to the right-hand side of (31) the Schwarz inequality we get

$$\Big|\int_{\Gamma_{r_0,r_\delta}} \varphi(x)u(x)dx\Big| \leqslant r_\delta r_0^{-1} |\varphi|_{r_0,r_\delta} |u|_{r_0,r_\delta}$$

and this ends the proof.

For given $\mathring{x}, \Xi$ let us denote by $M_r(u; \mathring{x})$ the mean value of the function $|u|^2$ over the intersection of the sphere with centre at $\mathring{x}$ and radius r with the cone $\Gamma_\delta(\mathring{x}, \Xi)$. Obviously,

$$M_r(u; \mathring{x}) = |\Xi|^{-1} \int_\Xi |u(\mathring{x}+r\xi)|^2 d\sigma_\xi,$$

where $|\Xi|$ is the measure of the set Ξ. We now prove

PROPOSITION 3. *Suppose that*

(a) $\mathring{x} \in \bar{\Omega}$ *has the cone property*;

(b) $u \in H_m(\Omega)$ *with* $m \leqslant \frac{1}{2}n$;

(c) *there exists a sequence* $\{u_\nu\} \subset H_m(\Omega)$ *such that for each* ν *the function* u_ν *vanishes in a neighbourhood of* $\mathring{x}$ *and*

$$\lim_{\nu \to \infty} \|u - u_\nu\|_{m,\Omega} = 0.$$

Then

$$\lim_{r \to 0+} r^n M_r(u;\ \mathring{x}) = 0. \tag{32}$$

Proof. It follows from (29) applied to the difference $u - u_\nu$ that for $\nu \geqslant \nu(\varepsilon)$

$$\sup_{0 < r \leqslant r_\delta} \Big(r^n \int_{\Xi} |u(\mathring{x} + r\xi) - u_\nu(\mathring{x} + r\xi)|^2 \, d\sigma_\xi\Big)^{1/2} \leqslant \sqrt{\varepsilon}, \tag{33}$$

where ε is an arbitrary given positive number. Let us choose $\eta(\varepsilon)$ so small that the function $u_{\nu(\varepsilon)}(x)$ vanishes if $|x - \mathring{x}| \leqslant \eta(\varepsilon)$, $x \in \Gamma_\delta$. Then (33) gives

$$r^n \int_{\Xi} |u(\mathring{x} + r\xi)|^2 \, d\sigma_\xi \leqslant \varepsilon \quad \text{for } r \leqslant \eta(\varepsilon).$$

Let us now suppose that Ω satisfies assumption (a_1) of Theorem 2. On the smooth parties of $\partial\tilde{\Omega}$ every function $u \in \mathring{H}_m(\Omega)$ vanishes together with all the derivatives $D^\alpha u$ ($|\alpha| \leqslant m-1$), if the boundary value is understood in the sense of trace. As we have supposed that $\tilde{\Omega}$ has only the segment property, in general case we have the following

THEOREM 4. *Let us suppose that the point* $\mathring{x} \in \partial\Omega$ *has the cone property (in particular, this assumption is satisfied by the origin if* Ξ *is the whole unit sphere). Then for every* $u \in \mathring{H}_m(\Omega)$ *we have*

$$\lim_{\substack{x \to \mathring{x} \\ x \in \Gamma_\delta}} D^\alpha u(x) = 0 \quad (|\alpha| < m - \tfrac{1}{2}n) \tag{34}$$

and

$$\lim_{r \to 0+} r^n M_r(D^\alpha u;\ \mathring{x}) = 0 \quad (\max(0, m - \tfrac{1}{2}n) \leqslant |\alpha| \leqslant m-1). \tag{35}$$

The proof is evident in view of the Sobolev inequality (28) and our Proposition 3. Let us note that, according to Sobolev's lemma, all the derivatives occurring in (34) are continuous in a neighbourhood of the origin and in the whole of Ω, if $\tilde{\Omega}$ has the cone property.

3. Energy inequality. In this section Ω will be an arbitrary domain of R_n, V a subspace of the Sobolev space $H_m(\Omega)$ satisfying (2) and B a continuous bilinear form over V. Let X be an arbitrary linear normed space and g, h two X-valued functions of the real variable, defined in an interval (α, β). We say that

1° h is the strong X-derivative of g at the point $t_0 \in (\alpha, \beta)$ if

$$\lim_{\tau \to 0} \left\| \frac{g(t_0+\tau)-g(t_0)}{\tau} - h(t_0) \right\|_X = 0; \tag{36}$$

2° g is of class $C^r((\alpha, \beta); X)$ $(0 \leqslant r < \infty)$ if it has at each point $t_0 \in (\alpha, \beta)$ the strong X-derivatives up to order r, which are continuous X-valued functions.

Obviously, we may replace in 1°, 2° an open interval (α, β) by a half-closed $[\alpha, \beta)$ — in such a case, for $t_0 = \alpha$ we consider only the right-hand derivative ($\tau > 0$ in formula (36)).

We will prove the following

THEOREM 5. *Let u be a V-valued function of class $C^1([0, \infty); V) \cap C^2([0, \infty); L_2(\Omega))$ satisfying the identity*

$$B(u(t), v)+(D_t^2 u(t), v) = (f(t), v) \quad (v \in V, t \in [0, \infty)), \tag{37}$$

where f is a continuous $L_2(\Omega)$-valued function. Suppose that B is hermitian and coercive in V. Then for every $T > 0$ there exists a positive constant c_T (not depending on u) such that

$$\sup_{0 \leqslant t \leqslant T} (\|u(t)\|_m^2 + \|D_t u(t)\|_0^2) \leqslant c_T (\|u(0)\|_m^2 + \|D_t u(0)\|_0^2 + \int_0^T \|f(t)\|_0^2 dt). \tag{38}$$

The proof runs in a standard way (see [11] for $m = 2$). At first we give some lemmas, which will be needed in the proof.

LEMMA 2. *Let b be a bilinear form continuous over X and let g_j $(j = 1, 2)$ be of class $C^1([\alpha, \beta); X)$. Then the function*

$$[\alpha, \beta) \ni t \to b(g_1(t), g_2(t))$$

is of class C^1 and

$$D_t b(g_1, g_2) = b(D_t g_1, g_2)+b(g_1, D_t g_2). \tag{39}$$

The proof follows immediately from the following decomposition of the difference quotient:

$$\frac{1}{h}(b(g_1, g_2)|_{t+h} - b(g_1, g_2)|_t) \tag{40}$$
$$= b\left(\frac{g_1(t+h)-g_1(t)}{h}, g_2(t+h)\right)+b\left(g_1(t), \frac{g_2(t+h)-g_2(t)}{h}\right).$$

Passing to the limit with $h \to 0$ we obtain (39). Identity (39) yields the following formula of "integration by parts", valid under the assumptions of Lemma 4:

(41) $$\int_a^c b(D_t g_1, g_2)\,dt = -\int_a^c b(g_1, D_t g_2)\,dt + b(g_1, g_2)|_{t=a}^{t=c}.$$

LEMMA 3. *If* $w \in X$ *and* $z \in C^r([\alpha, \beta))$, *then the function*

$$[\alpha, \beta) \ni t \mapsto z(t)w$$

is of class $C^r([\alpha, \beta); X)$ *and*

$$D_t^j(zw) = z^{(j)}w \quad (j = 1, \ldots, r).$$

LEMMA 4. $C^r([\alpha, \beta); X)$ *is a linear space over the field of complex numbers.*

The proofs may be left to the reader.

We need also the following elementary

LEMMA 5. *Let* F *be a real-valued function of class* $C^1(0, T)$ *satisfying the inequality*

(42) $$F'(t) \leqslant \mu F(t) + \gamma \quad (0 < t < T;\ \mu, \gamma > 0)$$

and the initial condition

(43) $$F(0) = 0.$$

Then

(44) $$\sup_{0<t<T} F'(t) \leqslant \gamma(e^{\mu T} + 1).$$

To prove this lemma, we introduce the function

$$G(t) = F(t)e^{-\mu t}.$$

Then (42) yields

(45) $$G'(t) \leqslant \gamma e^{-\mu t} \quad (0 < t < T).$$

Integrating (40) we obtain

$$G(t) \leqslant \gamma/\mu$$

and thus

(46) $$F(t) \leqslant \gamma e^{\mu t}/\mu \quad (0 < t < T).$$

Inequalities (42) and (46) give (44).

Proof of Theorem 5. Putting $v = D_t u$ in identity (37), we get for fixed $T > 0$

(47) $$\int_0^T B_{\lambda_0}(u, D_t u)\,dt - \lambda_0 \int_0^T (u, D_t u)\,dt + \int_0^T (D_t^2 u, D_t u)\,dt = \int_0^T (f(t), D_t u)\,dt,$$

where

$$B_\lambda(w, v) = B(w, v)+\lambda(w, v).$$

Using formula (41), we obtain

$$\int_0^T (u, D_t u)\,dt = -\int_0^T (D_t u, u)\,dt+(u, u)|_{t=0}^{t=T};$$

thus

(48) $$\mathrm{Re}\int_0^T (u, D_t u)\,dt = \tfrac{1}{2}\,\|u(t)\|_0^2|_{t=0}^{t=T}.$$

Similarly,

(49) $$\mathrm{Re}\int_0^T (D_t^2 u, D_t u)\,dt = \tfrac{1}{2}\,\|D_t u(t)\|_0^2|_{t=0}^{t=T}.$$

Using formula (41) once more, we have

$$\int_0^T B_{\lambda_0}(u, D_t u)\,dt = -\int_0^T B_{\lambda_0}(D_t u, u)\,dt+B_{\lambda_0}(u, u)|_{t=0}^{t=T}$$

and thus

(50) $$\mathrm{Re}\int_0^T B_{\lambda_0}(u, D_t u)\,dt = \tfrac{1}{2}\,B_{\lambda_0}(u, u)|_{t=0}^{t=T}.$$

According to (48), (49), (50), we get from (47) the identity

(51) $$[B_{\lambda_0}(u, u)-\lambda_0\,\|u\|_0^2+\|D_t u\|_0^2]_{t=0}^{t=T} = 2\,\mathrm{Re}\int_0^T (f(t), D_t u)\,dt.$$

We now introduce the energy integral

$$J^2(t) = B_{\lambda_0}(u(t), u(t))+\|D_t u(t)\|_0^2.$$

Then (51) may be written, with T replaced by t, in the form

(52) $$J^2(t)-J^2(0) = \lambda_0\,\|u(t)\|_0^2-\lambda_0\,\|u(0)\|_0^2+2\,\mathrm{Re}\int_0^t (f(\tau), D_\tau u)\,d\tau.$$

According to Lemma 2, both sides of (52) are continuously differentiable with respect to t, and so we get

(53) $$J(t)J'(t) = \lambda_0\,\mathrm{Re}\,(u, D_t u)+\mathrm{Re}\,(f, D_t u).$$

We apply to the right-hand side of (53) the Schwarz inequality. Supposing for a moment that $J(t) \neq 0$ and using the obvious estimate

$$\|D_t u(t)\|_0 \leqslant |J(t)|,$$

we get from (53)

(54) $$|J'(t)| \leqslant \lambda_0\,\|u(t)\|_0+\|f(t)\|_0.$$

As

$$|J(t)| \leqslant \int_0^t |J'(\tau)|\, d\tau + |J(0)|,$$

we obtain from (60)

$$(55) \qquad |J(t)| \leqslant \lambda_0 \int_0^t \|u(\tau)\|_0\, d\tau + \int_0^t \|f(\tau)\|_0\, d\tau + |J(0)|;$$

evidently, (55) is satisfied in the case $J(t) = 0$, as well.

It follows from the coercivity of the form B that the function J may be estimated from below:

$$(56) \qquad J^2(t) \geqslant c\, \|u(t)\|_m^2 + \|D_t u(t)\|_0^2.$$

Suppose now that $t \in [0, T]$, where T is a fixed positive number. Then estimates (56) and (55) in a slightly different form (after applying the Schwarz inequalities for sums and for integrals and using the continuity of B) yield (42), where

$$F(t) = \int_0^t \left(c\, \|u(\tau)\|_m^2 + \|D_\tau u(\tau)\|_0^2\right) d\tau$$

and

$$\gamma = 3T \int_0^T \|f(t)\|_0^2\, dt + c_1\, \|u(0)\|_m^2 + 3\, \|D_t u(0)\|_0^2$$

(here c_1 is a positive constant not depending on u and $\mu = 3T\lambda_0^2 c^{-1}$). Our theorem follows now immediately from Lemma 5.

4. Initial-boundary value problem. From now on we put $V = \mathring{H}_m(\Omega)$ and $\Omega_T = \Omega x(0, T)$ with Ω being a domain of R_n and T a positive number. $B(\cdot, \cdot)$ will be the bilinear Dirichlet form corresponding to $P(x, D)$ given in the divergent form (14).

We are now going to solve equation (1) in the domain Ω_∞ with the boundary condition

$$(57) \qquad u(\cdot, t) \in \mathring{H}_m(\Omega) \quad (0 < t < \infty)$$

and the initial conditions

$$(58) \qquad u(x, 0) = \varphi_0(x) \quad (x \in \Omega),$$

$$(59) \qquad D_t u(x, 0) = \varphi_1(x).$$

We suppose that assumptions $(a_1)-(a_4)$ of Theorem 2 are fulfilled. As regards the data, we suppose that:

(a_5) $\varphi_0 \in \mathring{H}_m(\Omega)$, $P(x, D)\varphi_0 \in L_2(\Omega)$;

(a_6) $\varphi_1 \in L_2(\Omega)$;

(a_7) f defines the continuous function $[0, \infty)\, t \mapsto f(\cdot, t) \in L_2(\Omega)$.

Let $\{w_k\}_{k=1}^{\infty}$ be the complete orthonormal system of eigenfunctions of the problem $(V_{\lambda,0})$ (according to Theorems A and 2, such a system does exist). Then for fixed t the function $f(\cdot, t)$ may be developed in $L_2(\Omega)$ into its Fourier series

$$f(x,t) = \sum_{k=1}^{\infty} w_k(x) h_k(t) \quad (0 \leqslant t < \infty) \tag{60}$$

with

$$h_k(t) = \big(w_k, f(\cdot, t)\big). \tag{61}$$

We seek a solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} w_k(x) z_k(t), \tag{62}$$

where z_k satisfy the ordinary differential equations

$$z_k'' - \lambda_k z_k = h_k(t) \quad (k = 1, 2, \ldots;\ t \in [0, \infty)) \tag{63}$$

and λ_k denote the eigenvalues corresponding to w_k.

We shall need in the sequel the following

LEMMA 6. *For any $f \in L_2(\Omega_T)$ the development* (60) *converges in $L_2(\Omega_T)$.*

Proof. Obviously, for almost all $t \geqslant 0$ we have

$$\|f(\cdot, t)\|_0^2 = \sum_{k=1}^{\infty} |h_k(t)|^2 \tag{64}$$

and the left-hand side of (64) is integrable in $(0, T)$. But

$$\Big\|f - \sum_{k=1}^{j} w_k h_k\Big\|_{L_2(\Omega_T)}^2 = \int_0^T c_j(t)\, dt, \tag{65}$$

where

$$c_j(t) = \sum_{k=j+1}^{\infty} |h_k(t)|^2.$$

According to (65), we have

$$0 \leqslant c_j(t) \leqslant \|f(\cdot, t)\|_0^2;$$

thus one may pass to the limit under the integral sign in (65). This completes the proof.

It follows from our assumptions (a_1)–(a_6) and from Theorem 3 that the initial data may be developed as follows:

$$\varphi_0 = \sum_{k=1}^{\infty} a_k w_k \tag{66}$$

(with convergence in $H_m(\Omega)$) and

(67) $$\varphi_1 = \sum_{k=1}^{\infty} b_k w_k$$

(convergence in $L_2(\Omega)$). This yields the initial conditions

(68) $$z_k(0) = a_k, \quad z_k'(0) = b_k,$$

provided we are able to differentiate series (62) term by term.

We can now prove two theorems about the initial-boundary value problem for equation (1). All the derivations are understood in the weak (distributional) sense.

THEOREM 6. *Under assumptions* (a_1)–(a_7) *the function u defined by series* (62) *satisfies* (1) *in the domain Ω_∞ and has the following properties*:

(i) *the $\mathring{H}_m(\Omega)$-valued function $t \mapsto u(\cdot, t)$ is continuous in* $[0, \infty)$;

(ii) *the $L_2(\Omega)$-valued function $t \mapsto D_t u(\cdot, t)$ is continuous in* $(0, \infty)$;

(iii) $u(\cdot, 0) = \varphi_0$;

(iv) $\lim_{t \to 0+} \|D_t u(\cdot, t) - \varphi_1\|_{L_2(\Omega)} = 0$;

(v) *for each $T > 0$ the energy inequality*

(69) $$\sup_{0 \le t \le T} (\|u(\cdot, t)\|_m^2 + \|D_t u(\cdot, t)\|_0^2) \le c_T (\|\varphi_0\|_m^2 + \|\varphi_1\|_0^2 + \|f\|_{L_2(\Omega_T)}^2)$$

holds with a positive constant c_T not depending on φ_j $(j = 1, 2)$, f, u;

(vi) *the derivatives $D_x^\alpha u$ $(0 \le |\alpha| \le m)$ and $D_t u$ may be computed by differentiating series* (48) *term by term; the series of derivatives does converge in $L_2(\Omega)$ uniformly with respect to $t \in [0, T]$ for arbitrarily fixed $T > 0$.*

Proof. Obviously, the functions h_k defined by (61) are continuous in the interval $[0, \infty]$. Therefore for each $k = 1, 2, \ldots$ the function z_k is well defined as the solution of (63) satisfying (68) and is of class $C^2([0, \infty))$. Let us denote by u_j (or $\varphi_{0,j}, \varphi_{1,j}, f_j$) the sum of the first j terms of series (62) (or (66), (67), (60), respectively). It follows from Lemmas 5 and 6 that the functions $t \mapsto u_j(\cdot, t)$ and $t \mapsto u_j(\cdot, t) - u_k(\cdot, t)$ satisfy for each j, k the assumptions of Theorem 5 with f replaced by f_j or $f_j - f_k$, respectively. According to (66)–(68), the initial conditions

(70) $$u_j(\cdot, 0) = \varphi_{0,j}$$

and

(71) $$D_t u_j(\cdot, 0) = \varphi_{1,j}$$

are satisfied. Thus it follows from the energy inequality (69) that $\{u_j\}$ is a Cauchy sequence in the norm

(72) $$u \mapsto \left(\sup_{0 \le t \le T} (\|u(\cdot, t)\|_m^2 + \|D_t u(\cdot, t)\|_0^2)\right)^{1/2}.$$

Accordingly; for arbitrarily fixed positive T there exists in $[0, T]$ a continuous $\mathring{H}_m(\Omega)$-valued function $t \mapsto u(\cdot, t)$ which is the sum of series (62) with convergence in $H_m(\Omega)$, uniform with respect to t. Similarly, the partial sums of the derivated series $D_t u_j$ converge to an $L_2(\Omega)$-valued function u_t, which is continuous in $[0, T]$, and it is easy to verify that $u_t = D_t u$ in Ω_T. This yields properties (i)-(iii) and (v), (vi).

Property (v) follows from (69) applied to the function u_j if we pass to the limit with $j \to \infty$.

In order to prove (iv), let us note that for each $t_0 \in (0, T]$ we have

$$(73) \qquad \|D_t u(\cdot, t_0) - \varphi_1\|_0 \leqslant \|D_t u(\cdot, t_0) - D_t u_j(\cdot, t_0)\|_0 + \\ + \|D_t u_j(\cdot; 0) - \varphi_1\|_0 + \|D_t u_j(\cdot, t_0) - D_t u_j(\cdot, 0)\|_0 .$$

Fix $j = j_\varepsilon$ such that each of the first two members on the right-hand side of (73) is not greater than $\varepsilon/3$. Then the last member does not exceed $\varepsilon/3$ if $t_0 < \delta_\varepsilon$ and this ends the proof of (iv).

Finally, we prove that u satisfies (1) in the distributional sense. Let us choose arbitrarily a $\varphi \in C_0^\infty(\Omega_\infty)$; then $\varphi \in C_0^\infty(\Omega_T)$ for some $T > 0$. As Theorem 5 is valid for the function u_j, we get from (37), after replacing (v) by $\varphi(\cdot, t)$ and integrating with respect to t,

$$(74) \qquad \int_0^T B(u_j(\cdot, t), \varphi(\cdot, t))\, dt + (u_j, D_t^2 \varphi)_{L_2(\Omega_T)} = (f, \varphi)_{L_2(\Omega_T)} .$$

It follows from our assumptions that

$$\lim_{j \to \infty} \|a_{\alpha\beta} D^\beta u_j - a_{\alpha\beta} D^\beta u\|_{L_2(\Omega_T)} = 0,$$

and so we may pass to the limit with $j \to \infty$ in (80). This completes the proof.

THEOREM 7. *Assume conditions* (a_1)-(a_6) *and suppose that*

(a_8) $f \in L_2(\Omega_T)$

for each $T > 0$. *Then the function* u *defined by* (62) *is a solution of* (1) *in the domain* Ω_∞ *and it shares properties* (i)-(v) *of Theorem* 6. *The functions* z_k *are of class* C^1 *in* $[0, \infty)$, *they satisfy initial conditions* (68) *and equation* (63) *in the distributional sense. Series* (62) *and the series of derivatives*

$$(75) \qquad D_t u(x, t) = \sum_{k=1}^{\infty} w_k(x) z_k'(t)$$

do converge in $L_2(\Omega_T)$ *for each* $T > 0$.

Proof. For arbitrarily fixed T we can approximate in $L_2(\Omega_T)$ the function f by a sequence $\{F_s\}$ of continuous functions. According to Theorem 6, applied to the mixed problem (1), (57)-(59) with f replaced by F_s, we obtain by the Fourier method a solution U_s having properties (i)–(v). The difference $U_s - U_r$ is a solution of the same problem, but with vanishing initial data and with f replaced by $F_s - F_r$. Thus it follows from (69) that $\{U_s\}$ is a Cauchy sequence in the norm (72). Its limit u satisfies equation (1) and has properties (i)–(v), as well.

It remains to prove that u is the sum of (62). As $t \mapsto u(\cdot, t)$ is a continuous $L_2(\Omega)$-valued function, it admits (for fixed t) the development in $L_2(\Omega)$:

$$u(x, t) = \sum_{k=1}^{\infty} w_k(x) z_k(t), \tag{76}$$

where

$$z_k(t) = \big(w_k, u(\cdot, t)\big)$$

are continuous in $[0, T)$. Similarly, the $L_2(\Omega)$-valued function

$$\tilde{u} = \begin{cases} D_t u & \text{in } \Omega_T, \\ \varphi_1 & \text{for } t = 0 \end{cases}$$

is continuous and may be developped in $L_2(\Omega)$ as follows:

$$\tilde{u}(x, t) = \sum_{k=1}^{\infty} w_k(x) p_k(t) \quad (0 \leqslant t \leqslant T), \tag{77}$$

where

$$p_k(t) = \begin{cases} \big(w_k, D_t u(\cdot, t)\big) & \text{for } 0 < t < T, \\ b_k & \text{for } t = 0 \end{cases}$$

are continuous, too. According to Lemma 6, the two series (76), (77) do converge in $L_2(\Omega_T)$ and it is easy to verify that $p_k = z_k'$.

To complete the proof, it remains to show that z_k satisfy (63). Let us write

$$z_{k,s}(t) = \big(w_k, U_s(\cdot, t)\big)$$

and

$$h_{k,s}(t) = \big(w_k, F_s(\cdot, t)\big).$$

Then $z_{k,s}$ is a smooth solution of the equation

$$z_{k,s}'' - \lambda_k z_{k,s} = h_{k,s}(t). \tag{78}$$

As for fixed k we have

$$\lim_{s \to \infty} \sup_{0 \leqslant t \leqslant T} |z_{k,s}(t) - z_k(t)| = 0$$

and

$$\lim_{s \to \infty} \|h_{k,s} - h_k\|_{L_2(0,T)} = 0,$$

we may pass to the limit with $s \to \infty$ in equation (78), obtaining (63).

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