

*TRANSLATION AND POWER INDEPENDENCE
FOR BERNOULLI CONVOLUTIONS*

BY

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1. Introduction. We prove that for certain symmetric Bernoulli convolutions μ on the circle there is a countable subgroup $D(\mu)$ such that translates of (not necessarily distinct) powers of μ are mutually singular unless the translating elements are equivalent modulo $D(\mu)$. As a simple special case we mention the Cantor measure

$$\mu_C = \bigotimes_{n=1}^{\infty} \frac{1}{2} (\delta(0) + \delta(3^{-n})),$$

which has the property that $\delta(x) * \mu_C^n \perp \delta(y) * \mu_C^m$ unless $m = n$ and $x - y$ is a triadic rational. In general, $D(\mu)$ will be the subgroup generated by the successive mesh divisions in the construction of μ .

The problem arose as an offshoot of a detailed study [2] of L -subalgebras of $M(T)$ associated with Bernoulli convolutions μ , where, in many cases, we required to establish the mutual power independence of μ and $\tilde{\mu}$. ($\tilde{\mu}$ denotes the involute defined by $\tilde{\mu}(E) = \overline{\mu(-E)}$ for Borel sets E , and we say that $\mu, \tilde{\mu}$ have *mutually independent powers* when $\mu^p \tilde{\mu}^q \text{ non } \perp \mu^r \tilde{\mu}^s$ implies $p = r, q = s$.) This property appears as a corollary of our methods and is also established for some measures which were introduced by Kaufman in [5].

Our results are very elementary to state and the proofs are correspondingly simple — but indirect and non-elementary.

2. Generalities. Let G be an arbitrary LCA group with measure algebra $M(G)$ having a maximal ideal space $\Delta(M(G))$. We regard elements of $\Delta(M(G))$ as generalized characters

$$\chi = (\chi_\mu) \in \prod_{\mu \in M(G)} L^\infty(\mu)$$

which satisfy

- (i) $\mu \ll \nu \Rightarrow \chi_\mu = \chi_\nu$ (μ a.e.),
- (ii) $\chi_{\mu * \nu}(x + y) = \chi_\mu(x) \chi_\nu(y)$ ($\mu \times \nu$ a.e.), and

(iii) $\sup \|\chi_\mu\|_\infty = 1$.

We write $\Delta(\mu)$ for the image of the projection $\chi \rightsquigarrow \chi_\mu$ and give $\Delta(\mu)$ the $\sigma(L^\infty(\mu), L^1(\mu))$ -topology — noting that the continuity of these projections now defines the Gelfand topology on $\Delta(M(G))$.

The terminology which follows is modelled on that of Williamson in [6]. Since the latter author includes a spectral norm condition, there is a minor discrepancy which will be made irrelevant by the fact that all measures discussed in Section 3 will be positive.

Definition. Let $\mu \in M(G)$ and suppose that D is a subgroup of G .

(i) μ is said to be *independent power* (i.p.) if, for all positive integers n, m ,

$$\mu^n \text{ non } \perp \mu^m \Rightarrow n = m.$$

(ii) μ is said to be i.p.A. (A for Atomic) if, for all positive integers n, m and for all $x, y \in G$,

$$\delta(x) * \mu^n \text{ non } \perp \delta(y) * \mu^m \Rightarrow n = m, x = y.$$

(iii) μ is said to be i.p.A. mod D if, for all positive integers n, m and for all $x, y \in G$,

$$\delta(x) * \mu^n \text{ non } \perp \delta(y) * \mu^m \Rightarrow n = m, x - y \in D.$$

Evidently, μ is i.p.A. if and only if μ is i.p.A. mod 0. Also, μ is i.p. whenever μ is i.p.A. mod G . However, we have

PROPOSITION 1. *Every μ in $M(G)$ which is i.p. is also i.p.A. mod G if and only if G is a torsion group.*

Proof. If G has an element x of infinite order, then the atom $\delta(x)$ is i.p. but not i.p.A. mod G . (If this suggests an obvious modification of the statement, then consider, for non-discrete G , $\delta(x) * \mu + \mu^2$, where μ is a continuous measure supported by a perfect independent set.)

Suppose, conversely, that μ is i.p. Then, by (3.3) of [1], for each constant $e^{i\theta}$ of modulus one, there exists $\chi^\theta \in \Delta(M(G))$ such that $\chi_\mu^\theta = e^{i\theta}$ (μ a.e.). Thus $\delta(x) * \mu^n \text{ non } \perp \delta(y) * \mu^m$ implies $\chi_{\delta(x-y)}^\theta(x-y) = e^{i(m-n)\theta}$. Hence, if $m \neq n$, $x-y$ has infinite order. (The map $x \rightsquigarrow \chi_{\delta(x)}^\theta(x) = \delta(x) \wedge (\chi^\theta)$ is a (not necessarily continuous) character of G .) This completes the proof.

Before specializing to our main objective in the next section we give another simple proposition. Let us write $G(\mu)$ for the L -subalgebra of $M(T)$ generated by μ and the atoms $\delta(x)$ for $x \in G$ so that $G(\mu)$ consists of the measures absolutely continuous with respect to sums of the form

$$\sum_{n=0}^{\infty} (2\|\mu\|)^{-n} (\delta(x_n) * |\mu|^n), \quad \text{with } |\mu|^0 = \delta(0).$$

We note that generalized characters in $\Delta(G(\mu))$ satisfy the same formal relations as those written down for $\Delta(M(T))$ and we denote by $\Delta^G(\mu)$ the collection of μ -coordinates of generalized characters of $\Delta(G(\mu))$. Then $\Delta(\mu) \subseteq \Delta^G(\mu)$ and we have

PROPOSITION 2. Let $\mu \in M(G)$. Then the following are equivalent:

- (i) μ is i.p.A. mod G ;
- (ii) $\Delta^G(\mu)$ contains a non-zero constant with modulus strictly less than one;
- (iii) $\Delta^G(\mu)$ contains all constants with modulus not greater than one.

COROLLARY. If $\Delta(\mu)$ contains a non-zero constant with modulus strictly less than one, then μ is i.p.A. mod G .

Proof. For $\zeta \in \mathbb{C}$, $|\zeta| \leq 1$, define $\varphi(\delta(x) * \mu^n) = \zeta^n$, $n = 0, 1, 2, \dots$. A glance at the defining relations for generalized characters shows that (i) guarantees the consistency of the obvious extension of φ to a generalized character of $G(\mu)$. Thus (i) implies (iii).

It is trivial that (iii) implies (ii); so let us assume that there is $\chi \in \Delta(G(\mu))$ with $\chi_\mu = \zeta$ (μ a.e.), where $0 < |\zeta| < 1$. If $\delta(x) * \mu^m \perp \delta(y) * \mu^n$, then $\chi_{\delta(x-y)}(x-y) = \zeta^{m-n}$. Since $|\chi_{\delta(x-y)}(x-y)| = 1$, this gives $m = n$. Thus (ii) implies (i), and the proof is complete.

The corollary is immediate. It seems reasonable to conjecture that its converse is false but the question does not appear to be easy (P 845).

3. Bernoulli convolutions. Throughout this section μ denotes a symmetric Bernoulli convolution

$$\mu = \bigotimes_{n=1}^{\infty} \left(\frac{1}{2} \delta(0) + \frac{1}{2} \delta(d_n) \right)$$

in $M(T)$, where $c = \sum_{n=1}^{\infty} d_n < 1$. (The limit is in the $\sigma(M(T), C(T))$ -topology.) Thus $\tilde{\mu} = \delta(-c) * \mu$.

In fact, we shall be concerned with two special classes of convolutions.

We say that μ belongs to A if every constant of modulus not greater than one belongs to the closure of the (restrictions of the) continuous characters of T in $\Delta(\mu)$.

We say that μ belongs to B if there is a sequence (a_n) of integers (≥ 2) such that

$$d_n^{-1} = p_n = \prod_{r=1}^n a_r \quad \text{for each } n.$$

Hewitt and Kakutani proved in [3] that if $\mu \in B$ satisfies $\sum_{n=1}^{\infty} 1/a_n < \infty$, then $\mu \in A$. We show in [2] that given μ in B , then $\mu \in A$ if and only if $\sup_n a_n = \infty$, but this stronger result will not be needed here. Kaufman [5] extended the results of Hewitt and Kakutani in a different way. He exhibited measures μ , with much less arithmetical constraint than membership of B implies but with strong lacunarity properties, which belong

to **A**. The same author (loc. cit.) gave a direct probabilistic argument that certain μ are i.p.A. mod \mathbf{T} , and hence that $\mu + \tilde{\mu} = \mu + \delta(-c) * \mu$ has independent powers.

We now prove

PROPOSITION 3. *If μ is i.p.A. mod \mathbf{T} and the constant $re^{i\theta}$, where $0 < r \leq 1$ and θ/π is irrational, belongs to the closure of the continuous characters in $\Delta(\mu)$, then $\mu, \tilde{\mu}$ have mutually independent powers.*

COROLLARY. *For every μ in **A**, $\mu, \tilde{\mu}$ have mutually independent powers.*

Proof. Note first that the hypothesis that μ is i.p.A. mod \mathbf{T} is redundant if $r \neq 1$. (Use the corollary to Proposition 2.) This shows that the present corollary follows from the proposition.

To prove the proposition assume that μ, r and θ satisfy the given conditions and that $\mu^p \tilde{\mu}^q \text{non } \perp \mu^s \tilde{\mu}^t$. For convenience, write $\chi^{(n)}$ for the generalized character corresponding to z^n (i.e., for each $v \in M(\mathbf{T})$, $\chi_v^{(n)}(x) = \exp(2\pi inx) = z^n(x)$) and suppose $z^{n(k)} \rightarrow re^{i\theta}$ in $\Delta(\mu)$. By extracting a subsequence, if necessary, we can suppose that $\chi_{\mu+\tilde{\mu}}^{(n(k))} \rightarrow \chi_{\mu+\tilde{\mu}}$ in $\Delta(\mu+\tilde{\mu})$ for some $\chi \in \Delta(M(\mathbf{T}))$. Then we have $\chi_\mu = re^{i\theta}$ (a.e. μ), while $\chi_{\tilde{\mu}}$ is the $\sigma(L^\infty(\tilde{\mu}), L^1(\tilde{\mu}))$ -limit of $\chi_\mu^{(n(k))} = z^{n(k)}$, and hence equals $re^{-i\theta}$ (a.e. $\tilde{\mu}$). We now have

$$r^{p+q} e^{i(p-q)\theta} = r^{s+t} e^{i(s-t)\theta},$$

which gives that $p - q = s - t$. But

$$\delta(-qd) * \mu^{p+q} \text{non } \perp \delta(-td) * \mu^{s+t},$$

gives that $p + q = s + t$. It follows that $p = s, q = t$. Thus $\mu, \tilde{\mu}$ have mutually independent powers, and the proof is complete.

From now on we concentrate on the class **B** and the proof of our main result which will require some preliminary lemmas. For $\mu \in \mathbf{B}$ we write

$$D_n = \left\{ \sum_{i=1}^n \varepsilon_i d_i : \varepsilon_i = 0 \text{ or } 1 \right\}$$

and denote by D ($D = D(\mu)$) the subgroup generated by $\{d_n : n = 1, 2, \dots\}$. We write also

$$\mu_n = \underset{*}{\times}_{k=n+1}^{\infty} \frac{1}{2} (\delta(0) + \delta(d_k))$$

and make the obvious but important remark that

$$\mu = 2^{-n} \sum_{d \in D_n} \delta(d) * \mu_n.$$

This leads quickly to the fact that, in the case where $a_n = 2$ for all n greater than some n_0 , μ is a weighted sum of translates of the Lebesgue measure restricted to some interval. Accordingly, we rule out these measures and restrict attention to the class \mathbf{B}' for which infinitely many a_n do not equal 2. It is also clear that the measures $\{\delta(d) * \mu_n: d \in D_n, n = 1, 2, \dots\}$ span $L^1(\mu)$. This gives the following criterion due to Johnson [4]:

LEMMA 1. Let $\mu \in \mathbf{B}'$. Then $z^{n(k)} \rightarrow re^{i\theta}$ in $\Delta(\mu)$ if and only if $\hat{\mu}(n(k)) \rightarrow re^{i\theta}$ and $z^{n(k)} \rightarrow 1$ pointwise on D .

Proof.

$$\int z^{n(k)} d(\delta(d) * \mu_n) = z^{n(k)}(d) \int z^{n(k)} d\mu_n,$$

$$2^n \hat{\mu}(n(k)) = \left(\sum_{d \in D_n} z^{n(k)}(d) \right) \int z^{n(k)} d\mu_n \quad \text{for } n = 1, 2, \dots$$

LEMMA 2. Let $\mu \in \mathbf{B}'$. Then there exist sequences $(n(k))$ and (q_k) of positive integers such that $q_k < a_{n(k)}$, $k = 1, 2, \dots$, and $\alpha_k = q_k/a_{n(k)} \rightarrow \alpha$ with $0 \leq \alpha < \frac{1}{2}$. For any such choice, $z^{q_k p_{n(k)} - 1} \rightarrow 1$ pointwise on D , and

$$\cos \pi \alpha_k \geq |\hat{\mu}(q_k p_{n(k)} - 1)| \geq \sin 2\pi \alpha_k / 2\pi \alpha_k, \quad k = 1, 2, \dots$$

COROLLARY. Every $\mu \in \mathbf{B}'$ is i.p.A. mod \mathbf{T} , in particular, μ is singular and $\mu + \tilde{\mu}$ has independent powers.

Proof. The existence of the sequences is obvious. Any $d \in D$ is a finite sum of the form $\sum_{m=1}^{m'} b_m d_m$, where b_m are integers, and hence

$$q_k p_{n(k)} - 1 d \equiv 0 \pmod{1} \quad \text{for } n(k) > m' + 1.$$

Thus $z^{q_k p_{n(k)} - 1} \rightarrow 1$ pointwise on D .

Observe next that

$$\begin{aligned} & \hat{\mu}(q_k p_{n(k)} - 1) \\ &= \frac{1}{2} (1 + \exp(2\pi i \alpha_k)) \prod_{m=1}^{\infty} \left(\frac{1}{2} + \frac{1}{2} \exp(2\pi i \alpha_k / a_{n(k)+1} a_{n(k)+2} \dots a_{n(k)+m}) \right). \end{aligned}$$

Therefore,

$$\cos \pi \alpha_k \geq |\hat{\mu}(q_k p_{n(k)} - 1)| \geq \prod_{m=0}^{\infty} \cos(\pi \alpha_k / 2^m) = \sin 2\pi \alpha_k / 2\pi \alpha_k.$$

By passing to a subsequence, if necessary, we can assume that $\hat{\mu}(q_k p_{n(k)} - 1)$ converges. Then, by Lemma 1 and the corollary to Proposition 2, the corollary follows.

Note. Of course, neither the lemma nor the corollary is in any sense a best possible result. The lemma is what we need and the corollary is stated here to show how much can be obtained with the minimum of effort.

LEMMA 3. Let (s_k) be a sequence of real numbers such that $as_k \rightarrow 0 \pmod{1}$ for a in a set of the Lebesgue measure $\frac{1}{2}$ in $[0, \frac{1}{2}]$. Then $s_k \rightarrow 0$ (no mods).

Proof. We see that $\exp(2\pi is_k y) \rightarrow 1$ pointwise except on a set of y 's of the Lebesgue measure zero in \mathbf{R} . By the Dominated Convergence Theorem, it follows that the characters of \mathbf{R} , determined by the $2\pi s_k$, converge to 1 in the $\sigma(L^\infty(m), L^1(m))$ -topology of Haar measure m , i.e. they converge in the Gelfand topology of the maximal ideal space of $L^1(\mathbf{R})$, and hence in the usual topology of \mathbf{R} . It follows that $s_k \rightarrow 0$ and the lemma is proved.

LEMMA 4. Suppose that μ belongs to \mathbf{B}' , that $\delta(x) * \mu^n \text{ non } \perp \mu^m$, and that $z^{n(k)} \rightarrow \zeta \neq 0$ (a constant) in $\Delta(\mu)$. Then there is a sequence $(k(j))$ such that $n(k(j))x \rightarrow 0 \pmod{1}$.

Proof. Observe first that $n = m$ by the previous corollary. We write again $\chi^{(n)}$ for the generalized character induced by z^n on each measure in $M(\mathbf{T})$ and we denote the measure $\delta(x) + \sum_{n=1}^{\infty} 2^{-n} \mu^n$ by ν . Since $z^{n(k)} \rightarrow \zeta$ in $\Delta(\mu)$, there exists a sequence $(k(j))$ and a generalized character χ of $\Delta(M(\mathbf{T}))$ such that $\chi_{\nu}^{n(k(j))} \rightarrow \chi_{\nu}$ in $\Delta(\nu)$. Then

$$\chi_{\delta(x)}(x) \chi_{\mu^n}(t) = \chi_{\mu^m}(t) = \chi_{\mu^n}(t)$$

for some set of t with positive measure relative to μ^n . This gives the equation $\chi_{\delta(x)}(x) \zeta^n = \zeta^n$, which implies $z^{n(k(j))} \rightarrow 1$ and the required result follows.

THEOREM. Let

$$\mu = \ast_{n=1}^{\infty} \left(\frac{1}{2} \delta(0) + \frac{1}{2} \delta(d_n) \right),$$

where $d_n = (a_1 a_2 \dots a_n)^{-1}$, be a measure in \mathbf{B} . Let $D(\mu)$ denote the countable subgroup of \mathbf{T} generated by $\{d_n: n = 1, 2, \dots\}$. Then μ is i.p.A. mod $D(\mu)$ if and only if the sequence (a_n) does not contain arbitrarily long blocks of consecutive 2's.

COROLLARY 1. Suppose $\mu \in \mathbf{B}$ and (a_n) does not contain arbitrarily long blocks of consecutive 2's. Then $\mu, \tilde{\mu}$ have mutually independent powers if and only if

$$c = \sum_{n=1}^{\infty} d_n$$

is irrational.

COROLLARY 2. Let μ belong to \mathbf{B} and suppose that

$$\sup_n a_n = \infty.$$

Then $\mu, \tilde{\mu}$ have mutually independent powers.

Remark. The class of measures for which μ fails to be i.p.A. mod $D(\mu)$ is, of course, much larger than the complement of B' . An interesting example of a measure μ which is not i.p.A. mod $D(\mu)$ but is nevertheless such that $\mu, \tilde{\mu}$ have mutually independent powers is obtained by defining $a_n = 2$, unless n is prime, in which case $a_n = 2^n$. (The assertion is justified by Corollary 2.) On the other hand, the Cantor measure

$$\mu = \times_{n=1}^{\infty} \frac{1}{2} (\delta(0) + \delta(3^{-n}))$$

is i.p.A. mod $D(\mu)$ but $\mu^2 = \tilde{\mu}^2$.

Proof of the theorem. Suppose first that (a_n) does contain arbitrarily long blocks of consecutive 2's and choose a sequence $(n(k))$ of positive integers such that, for $k = 2, 3, \dots$,

$$(1) \quad n(k) - n(k-1) > k \quad \text{and} \quad a_{n(k)} = a_{n(k)-1} = \dots = a_{n(k)-k} = 2.$$

Write

$$y = \sum_{k=2}^{\infty} d_{n(k)},$$

and, for each $m = 2, 3, \dots$,

$$y(m) = \sum_{k=2}^m d_{n(k)}.$$

Clearly, $y \notin D(\mu)$. Since μ is fixed throughout the proof, let us agree to write D for $D(\mu)$. For $p < q$, we write also

$$D_{a,p} = \left\{ \sum_{i=p+1}^q \varepsilon_i d_i : \varepsilon_i = 0 \text{ or } 1 \right\};$$

in particular, $D_{0,q}$ is the set we denote also by D_q . We prove next that, for $m = 2, 3, \dots$,

$$(2) \quad \|\delta(y(m)) * \mu - \mu\| \leq 2 \left(1 - \prod_{k=2}^m (1 - 2^{-k}) \right).$$

Fix m and recall that

$$(3) \quad \mu = 2^{-n(m)} \sum_{d \in D_{n(m)}} \delta(d) * \mu_{n(m)},$$

where the measures being summed are mutually singular positive measures of unit norm. Now, given $d \in D_{n(k-1), n(k)}$ ($k = 2, \dots, m$), then $d + d_{n(k)} \in D_{n(k-1), n(k)}$ except possibly when $d = d' + \sum d_n$, where $d' \in D_{n(k-1), n(k)-k}$ and the summation is over all values of n from $n(k) - k + 1$ to $n(k)$. We see in this way that, for at least $2^{n(m)} \prod_{k=2}^m (1 - 2^{-k})$ choices of $d \in D_{n(m)}$, it follows that $y(m) + d$ belongs to $D_{n(m)}$. But in such a case $\delta(y(m)) * \delta(d) * \mu_{n(m)}$

is one of the measures appearing in the summation on the right-hand side of (3). The resulting cancellation gives estimate (2).

Observe next that $\delta(y(m)) * \mu \rightarrow \delta(y) * \mu$ in the $\sigma(M(T), C(T))$ -topology as $m \rightarrow \infty$. Hence, from (2) we deduce

$$(4) \quad \|\delta(y) * \mu - \mu\| < 2.$$

But (4) shows that $\delta(y) * \mu, \mu$ fail to be mutually singular, and hence that μ is not i.p.A. mod D .

Suppose now that $\mu \in B$, (a_n) does not contain arbitrarily long blocks of 2's, and that $\delta(x) * \mu^n \text{ non } \perp \mu^m$. Since we have already seen that $n = m$ in this situation, it will suffice to assume that $x \notin D$ and obtain a contradiction. Write

$$x = \sum_{n=1}^{\infty} x_n d_n \quad \text{with } 0 \leq x_n < a_n.$$

(i) Suppose $0 < x_{n(k)} < a_{n(k)} - 1$ for infinitely many k with

$$a = \sup_k a_{n(k)} < \infty.$$

Then

$$p_{n(k)-1} x \equiv r_k \pmod{1},$$

where

$$r_k = (x_{n(k)}/a_{n(k)}) + (x_{n(k)+1}/a_{n(k)} a_{n(k)+1}) + \dots$$

Then $a^{-1} \leq r_k \leq 1 - a^{-1}$ but, by Lemmas 2 and 4, some subsequence of (r_k) tends to zero (mod 1). This is the desired contradiction.

(ii) Suppose $0 < x_{n(k)} < a_{n(k)} - 1$ for infinitely many k with $a_{n(k)} \rightarrow \infty$.

It is important to note first that

$$(5) \quad q_k p_{n(k)-1} x \rightarrow 0 \pmod{1}$$

for every sequence of integers $0 < q_k < a_{n(k)}$ such that $q_k/a_{n(k)} = \alpha_k$ converges to a with $0 \leq a < \frac{1}{2}$. This is also justified by reductio ad absurdum. For, in the contrary case, the sequence of fractional parts $q_k p_{n(k)-1} x - [q_k p_{n(k)-1} x]$ has a subsequence converging to, say, $\beta \neq 0$ or 1. Hence the corresponding subsequence of

$$z^{q_k p_{n(k)-1}}(x) = \exp(2\pi i q_k p_{n(k)-1} x) \rightarrow \exp(2\pi i \beta) \neq 1.$$

In view of Lemma 2, we can find a subsequence of this such that the corresponding subsequence of $\hat{\mu}(q_k p_{n(k)-1})$ converges and hence, by Lemma 1, the corresponding subsequence of $z^{q_k p_{n(k)-1}}$ converges to a non-zero constant in $\Delta(\mu)$. Taking yet another subsequence of this, as guaranteed by Lemma 4, we find a subsequence (of the first subsequence chosen) so that the corresponding powers of z evaluated at x tend to 1. This is a contradiction and (1) has been established.

We now write

$$s_k = x_{n(k)} + (x_{n(k)+1}/a_{n(k)+1}) + (x_{n(k)+2}/a_{n(k)+1}a_{n(k)+2}) + \dots,$$

make the estimation $1 < s_k < a_{n(k)} - 1$, and replace s_k by $t_k = s_k$ or $a_{n(k)} - s_k$ whichever has smaller modulus. This gives that

(6) t_k does not converge to zero.

Using (5), we see that, for allowable a_k ,

(7)
$$a_k t_k \rightarrow 0 \pmod{1}.$$

Making the special choice $q_k \equiv 1$, we have $t_k/a_{n(k)} \rightarrow 0 \pmod{1}$, but by the choice $|t_k/a_{n(k)}| \leq \frac{1}{2}$ so that

(8)
$$t_k/a_{n(k)} \rightarrow 0 \quad (\text{no mods}).$$

Now, for $0 < a < \frac{1}{2}$, choose q_k as the best approximants in the formula

(9)
$$|(q_k/a_{n(k)}) - a| \leq 1/a_{n(k)}.$$

Then, by (7)-(9),

(10)
$$a t_k = a_k t_k + (a - a_k) t_k \rightarrow 0 \pmod{1}.$$

By Lemma 3 it follows that

(11)
$$t_k \rightarrow 0 \quad (\text{no mods}).$$

The combination of (6) and (11) gives a contradiction.

(iii) The remaining case is where $x_n = 0$ or $a_n - 1$ for all but finitely many n . Since $x \notin D$, there must be a subsequence $(n(k))$ such that $x_{n(k)} = a_{n(k)} - 1$ and $x_{n(k)+1} = 0$ for all k . If $a_{n(k)} \rightarrow \infty$, then the estimate $a_{n(k)} - 1 \leq s_k \leq a_{n(k)} - \frac{1}{2}$ ensures (6) and a contradiction as before. If

$$\limsup a_{n(k)} = a < \infty \quad \text{and} \quad a \neq 2,$$

then, by passing to a subsequence, if necessary, we can arrange $a_{n(k)} \equiv a$ and use the estimate $\frac{1}{2} \leq r_k \leq 1 - (2a)^{-1}$. Thus, the problem is reduced to the case $a_{n(k)} \equiv 2$. This is stage at which we are forced to use the hypothesis that the length L of the longest block of consecutive 2's in (a_n) is finite. In fact, we assume $n'(k)$ to be the greatest integer less than $n(k)$ such that $a_{n'(k)} \neq 2$. Denoting by r'_k, s'_k the quantities associated with $(n'(k))$ and corresponding to r_k, s_k , we obtain the estimates

$$2^{-L} \leq s'_k \leq a_{n'(k)} - 2^{-(L+1)}, \quad a^{-1} 2^{-L} \leq r'_k \leq 1 - 2^{-(L+1)} a^{-1},$$

where $a = \sup a_{n'(k)}$.

The methods of (i) and (ii) remain available, and the proof is complete.

Proof of Corollary 1. Rationality of c gives a relation of the form $\mu^a = \tilde{\mu}^a$, whereas if c is irrational, then $qc \notin D$ for every integer q .

Proof of Corollary 2. Suppose that $\sup a_n = \infty$, and, for any positive integer q , write

$$qc = x = \sum_{n=1}^{\infty} x_n d_n$$

as in the proof of the theorem. Noting that, for any positive integer m ,

$$q \sum_{n=m+1}^{\infty} d_n < qd_m,$$

we have a sequence $(n(k))$ such that $a_{n(k)} \rightarrow \infty$ and $q < x_{n(k)} < 2q$. By part (ii) of the proof of the theorem, we deduce the required result.

The corollaries are not best possible and it seems to be difficult to determine necessary and sufficient conditions for $\mu, \tilde{\mu}$ to be mutually power independent. We give two examples to indicate the possibilities.

Example 1. There exists $\mu \in \mathbf{B}$ such that $\mu + \tilde{\mu}$ has independent powers and μ is not i.p.A. mod $D(\mu)$.

In fact, we can find such a μ by defining

$$\begin{aligned} a_n = 3 \Leftrightarrow n+1 = \frac{1}{2}k(k+3) & \quad \text{for a positive integer } k, \\ a_n = 2 & \quad \text{otherwise.} \end{aligned}$$

We see after some calculation that

$$-3c \equiv \sum_{k=2}^{\infty} 3^{1-k} 2^{-(k+2)(k-1)/2} \pmod{1} \quad \text{for } c = \sum_{n=1}^{\infty} d_n.$$

The infinite sum can be rewritten as $\sum_{k=2}^{\infty} d_{n(k)}$, where $n(k) - n(k-1) > k$ and $a_{n(k)} = a_{n(k)-1} = \dots = a_{n(k)-k+1} = 2$. Therefore, the method of the first part of the proof of the theorem shows that $\delta(-3c) * \mu \text{ non } \perp \mu$. It follows that

$$\tilde{\mu}^3 = \delta(-3c) * \mu * \mu^2 \text{ non } \perp \mu^3.$$

We have now shown that μ is not i.p.A. mod $D(\mu)$ and that $\mu, \tilde{\mu}$ fail to have mutually independent powers. On the other hand, $\mu \in \mathbf{B}'$ so, by the corollary to Lemma 2, $\mu + \tilde{\mu}$ has independent powers.

Example 2. There exists $\mu \in \mathbf{B}$ with $\sup a_n < \infty$ such that μ is not i.p.A. mod $D(\mu)$ but $\mu, \tilde{\mu}$ do have mutually independent powers.

In fact, a suitable choice of μ follows from the definition

$$a_n = \begin{cases} 2, & k(k-1) < n \leq k^2, \\ 3, & k^2 < n \leq k(k+1), \end{cases}$$

where $k = 1, 2, 3, \dots$

It is an immediate consequence of the main theorem that μ is not i.p.A. mod $D(\mu)$. Now, we show that if s, m are positive integers, then $\delta(m\alpha) * \mu^s \text{ non } \perp \mu^s$ leads to a contradiction. In fact, we write

$$m\alpha \equiv \sum_{n=1}^{\infty} x_n d_n \quad \text{with } 0 \leq x_n < a_n$$

and show that the proof of the theorem can be adapted. Observe at the outset that the only case which can cause trouble is where $x_n = 0$ or $a_n - 1$ for all but finitely many n . From the equation

$$\sum_{n=k(k-1)}^{\infty} d_n = d_{k(k-1)}(1 + 2^{-1} + \dots + 2^{-k}) + \sum_{n=k^2+1}^{\infty} d_n,$$

together with the inequality

$$2m > m(1 + 2^{-1} + \dots + 2^{-k}) > (2m - 1) + \frac{1}{2},$$

which holds for k sufficiently large, we can write

$$m \sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} z_n d_n,$$

where, for k sufficiently large,

$$(12) \quad \begin{cases} z_n = m, & k^2 < n < k(k+1), \\ z_n = 0 \text{ or } 1, & k(k+1) < n \leq k^2, \\ z_{k(k+1)} = 2m - 1, & z_{k(k+1)+1} = 1. \end{cases}$$

For notational convenience let us now fix k (in fact to be chosen suitably large depending on m) and write $q = k(k+1)$. Then the inequality

$$m \sum_{n=q+1}^{\infty} d_n < m d_q$$

shows that

$$(13) \quad \sum_{n=q+1}^{\infty} z_n d_n < d_q.$$

Since

$$d_{q-1}^{-1} \sum_{n=1}^{\infty} z_n d_n = d_{q-1}^{-1} \sum_{n=1}^{\infty} x_n d_n \pmod{1},$$

we see from (12) that

$$\frac{1}{3} (2m - 1) - \frac{1}{3} x_q \equiv d_{q-1}^{-1} \left(\sum_{n=q+1}^{\infty} x_n d_n \right) - d_{q-1}^{-1} \left(\sum_{n=q+1}^{\infty} z_n d_n \right) \pmod{1}.$$

In view of (13), we deduce that

$$(14) \quad x_q \equiv 2m - 1 \pmod{3}.$$

The residue must be 0 or 2. To rule out the first possibility free k for the moment and consider the sequence $(n(k))$, where $n(k) = k(k+1)$, together with the estimate $\frac{1}{6} < r_k < \frac{2}{3}$ (in the notation of the proof of the theorem). This argument forces $x_{k(k+1)} = 2$ for large k and, feeding this information back into (14), we have $x_q = 2$ for the original q .

Let us write $2m - 1 = 3p_1 + 2$, where p_1 is a positive integer. Now, the congruence,

$$d_{q-2}^{-1} \left(\sum_{n=1}^{\infty} z_n d_n \right) \equiv d_{q-2}^{-1} \left(\sum_{n=1}^{\infty} x_n d_n \right) \pmod{1},$$

gives that

$$\frac{1}{3}(m + p_1) + \frac{2}{9} + (9d_q)^{-1} \sum_{n=q+1}^{\infty} z_n d_n \equiv \frac{1}{3}x_{q-1} + \frac{2}{9} + (9d_q)^{-1} \sum_{n=q+1}^{\infty} x_n d_n \pmod{1},$$

from which we deduce that

$$(15) \quad x_{q-1} \equiv m + p_1 \pmod{3}.$$

This time the zero residue is ruled out by consideration of the sequence $(n(k))$, where $n(k) = k(k+1) - 1$, together with the estimate $2/9 < r_k < 4/9$. It is now clear that proceeding in this way we obtain

$$x_q = x_{q-1} = \dots = x_{q-k+1} = 2$$

together with k positive integers p_1, p_2, \dots, p_k , satisfying the relation

$$(16) \quad m + p_j = 3p_{j+1} + 2, \quad j = 1, \dots, k-1.$$

But (16) leads to

$$(17) \quad p_k - p_{k-1} = 3^{-k+2}(p_2 - p_1) < 3^{-k+1}m.$$

This shows that, for $k > 1 + (\log m / \log 3)$, the p_j are all equal to some constant p . But then

$$2m = 3(p+1) = 2(p+1) = 0,$$

and we have obtained a contradiction. This completes the proof that $\mu, \tilde{\mu}$ have mutually independent powers.

It is clear that the above techniques can be used to obtain many more sufficient conditions on a measure $\mu \in \mathcal{B}$ for $\mu, \tilde{\mu}$ to have mutually independent powers.

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Reçu par la Rédaction le 7. 2. 1972
