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ON PROBLEMS RELATED TO CHARACTERISTIC VERTICES OF GRAPHS

BY

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In this paper we consider the Steiner problem for graphs and its generalizations. The notion of a maximal geodetic branch for a simple graph is introduced. The notion of a centroid for the class of simple graphs is generalized. A simple necessary condition for a vertex x to be in the σ-centre of a graph is obtained. For a generalized Steiner problem a necessary condition for an optimal partition is presented. Some open problems on the sets of characteristic vertices of a graph, in particular on the peripheral vertices defined in the paper, are formulated.

Introduction. Let $G = \langle X, U, \varphi \rangle$ be a simple, connected graph and $\varrho(x, y)$ a natural metric of the graph, i.e. the number of edges of a geodetic chain linking vertices $x, y \in X$. Let us denote by $\mathscr{B}$ the class of all trees.

Let us take $G \in \mathscr{B}$. Every vertex y of G cuts the tree G into connected subgraphs. We denote by

$$\mathscr{R}(y) = [X_1, X_2, \ldots, X_m], \quad X_i \cap X_j = \emptyset, \ \bigcup X_i = X \setminus \{y\},$$

the corresponding partition of the set $X \setminus \{y\}$. We shall call it the *natural partition* of X determined by the vertex y. In the theory of trees, the following functions defined on the set of vertices and related to the metric ϱ are considered:

$$e(y) = \max_{x \in X} \varrho(x, y), \tag{1}$$

$$d(y) = \max_{i=1,\ldots,m} |X_i|, \quad X_i \in \mathscr{R}(y), \tag{2}$$

$$\sigma(y) = \sum_{x \in X} \varrho(x, y), \tag{3}$$

$$d_E(y) = \max_{i=1,\ldots,m} |X_i \cap E|, \tag{4}$$

where $E \subset X$ is a set of end vertices of a tree,

$$\sigma_E(y) = \sum_{x \in E} \varrho(x, y). \tag{5}$$

It is easy to see that only functions (1) and (3) are meaningful in general case where G belongs to the class $\mathscr{G}$ of all simple graphs. The sets of characteristic vertices of a tree are closely related to functions (1)-(5) and they play an important role in practical applications in various branches of science and technology.

For the sake of simplicity, $\{y \in X: f(y) - \min\}$ will stand for

$$\{y \in X: f(y) = \min_{x \in X} f(x)\}$$

and, analogously, for max.

The set $Z = \{y \in X: e(y) - \min\}$ of vertices is called the *centre* of a graph. For $G \in \mathscr{B}$ we have the classical Jordon-Sylvester Theorem:

THEOREM 1. *Every tree has a centre composed of one or two adjacent vertices.*

The proof of Theorem 1 (see for example [2]), based on elimination of end vertices from a tree, leads to a simple procedure (linear in the number of vertices) for obtaining the set Z.

The set $P = \{y \in X: e(y) - \max\}$ is called a *set of Z-peripheral vertices* of a graph. For $G \in \mathscr{B}$ there is, evidently, $P \subset E$.

The set $W = \{y \in X: d(y) - \min\}$ is called a *centroid* of a tree. Here we have an interesting

LEMMA 1. *$y \in W$ if and only if $|X_i| \leqslant \frac{1}{2}|X|$ for $i = 1, 2, \ldots, m$.*

Recall also the classical Jordan Theorem:

THEOREM 2. *Every tree has a centroid composed of one or two adjacent vertices.*

Using Lemma 1 we can obtain a simple procedure for finding vertices of the set W: starting from an arbitrary vertex of a tree we arrive at an adjacent vertex belonging to that branch X_i for which $|X_i| > \frac{1}{2}|X|$ (it is easy to see that there exists at most one such a branch). Repeating this procedure we obtain the vertices of W.

The set $\{y \in X: d(y) - \max\}$ determines a set of peripheral vertices which, as is easy to see, consists of all terminal vertices of a tree (is identical with the set E).

1. Centroid and σ-centre of a graph. The set $Z_\sigma = \{y \in X: \sigma(y) - \min\}$ is called a *σ-centre* (*median*) of a graph. The set Z_σ is a solution of Steiner's problem (see [3] and references in that paper) for graphs. Let $G \in \mathscr{B}$ with $|X| = n$. Let $y \in X$, $\mathscr{R}(y) = [X_1, \ldots, X_m]$. For every vertex $x \in X_j$ $(j = 1, 2, \ldots, m)$, which is adjacent to vertex y, we have

$$(*) \qquad \varrho(x, z) = \begin{cases} \varrho(y, z) + 1 & \text{for } z \notin X_j, \\ \varrho(y, z) - 1 & \text{for } z \in X_j. \end{cases}$$

Let $|X_j| = k$. Let $Y = (X \setminus X_j) \setminus \{y\}$. Hence $|Y| = n-k-1$. By (*) we have

$$\sum_{z\in X} \varrho(x, z) = \sum_{u\in Y} \varrho(y, u) + (n-k-1) + \varrho(x, y) + \sum_{v\in X_j\setminus\{x\}} \varrho(y, v) - (k-1)$$

$$= \sum_{z\in X} \varrho(y, z) + n - 2k.$$

Let $|X_j| > \frac{1}{2}|X|$. Hence $n-2k = |X| - 2|X_j| < 0$, so that

$$\sigma(y) = \sum_{z\in X} \varrho(y, z) > \sigma(x).$$

Thus we have

LEMMA 2. *If* $y \in Z_\sigma$, *then* $|X_i| \leqslant \frac{1}{2}|X|$ *for* $i = 1, 2, \ldots, m$.

Let $|X_j| < \frac{1}{2}|X|$. Then, for $x \in X_j$ adjacent to the vertex y, we have $n-2k > 0$ and $\sigma(x) > \sigma(y)$. This property is hereditary for every $x_p \in X_j$ adjacent to x_{p-1}. Hence we have

LEMMA 3. *If* $|X_j| < \frac{1}{2}|X|$ *for* $j = 1, 2, \ldots, m$, *then* $Z_\sigma = \{y\}$.

Now let us assume that there exists a set X_j belonging to the partition $\mathscr{R}(y)$ such that $|X_j| = \frac{1}{2}|X|$. There is only one set with that property. For $x \in X_j$, incident with y, we have $\sigma(x) = \sigma(y) + n - 2k = \sigma(y)$. Thus, by Lemmas 2 and 3, we have

LEMMA 4. $y \in Z_\sigma$ *if and only if* $|X_i| \leqslant \frac{1}{2}|X|$ *for* $i = 1, 2, \ldots, m$.

Hence, for $G \in \mathscr{B}$, we obtain

THEOREM 3. $W = Z_\sigma$.

Thus we have obtained another definition of the centroid in the class of trees which makes also sense for an arbitrary simple graph G. By Lemma 4 we have also obtained a known simple procedure for the determination of the set Z_σ for an arbitrary tree.

Till now there is no practical procedure for the determination of the set Z_σ for a simple graph. As it was shown in [3], the problem posed for the class $\mathscr{G}$ of simple graphs may be reduced to the determination of Z_σ in a bi-connected graph. Similarly, the problem of the determination of a centre Z for an arbitrary graph is solved by the determination of a centre of a bi-connected graph [1]. The set $P_\sigma = \{y \in X: \sigma(y) - \max\}$ is called a *set of σ-peripheral vertices* of a graph. Evidently, for a tree we have $P_\sigma \subset E$.

Until now nothing is known about the position of P and P_σ in a tree or in an arbitrary graph. It is easy to give an example of a tree with disjoint sets Z and $Z_\sigma = W$.

PROBLEM 1. For what class of graphs $Z_\sigma = Z$? For what class of graphs $P_\sigma = P$? (**P 1149**)

The notion of an end vertex of a tree may be generalized in various ways to an arbitrary simple graph. One should notice that σ-peripheral vertices can be defined as end vertices and it is possible to introduce the notion of pseudo-end vertices of a graph as the set of vertices which are ends of simple maximal paths of a graph. It is easy to prove that if a vertex cuts a graph into two or more non-empty parts, then it cannot be a pseudo-end vertex. If a pair of vertices cuts a graph into three or more non-empty parts, then no vertex is a pseudo-end vertex.

For an arbitrary graph there are no theorems on sets of σ-peripheral or pseudo-end vertices and on the position of these sets and the set of Z-peripheral vertices.

The problem of constructing a simple procedure for the determination of the set Z_σ for an arbitrary bi-connected graph is a hard one. Even minor theoretical results play a significant role. It seems important to study a subclass of the class of spanning trees of graph, namely the class of geodetic trees. The properties of the whole class of partial trees are studied in the matroid theory.

A geodetic tree $B_G(x)$ of a vertex x in a graph G is a partial tree of the graph G rooted at vertex x and such that

$$\varrho_G(x, y) = \varrho_B(x, y) \quad \text{for every } y \in X.$$

A *geodetic tree of vertex* x is the tree of all the shortest paths from x to all other vertices in G.

Let $B_G(x)$ be an arbitrary geodetic tree of vertex x and let x be not in the σ-centre of the tree. Then for y in the σ-centre of tree $B_G(x)$ we have $\sigma_G(y) < \sigma_G(x)$. Hence

LEMMA 5. *If x belongs to the σ-centre of a graph G, then x is in the σ-centre of its every geodetic tree.*

There is a simple method for obtaining all geodetic trees of a vertex x in a graph G. To the vertex x there corresponds exactly one subgraph (a partial graph) of the graph G obtained by omitting the edges which are of no use in evaluation of $\varrho(x, y)$ for $y \in X$. Next we divide the set of vertices of the graph G into subsets $N_0(x), N_1(x), \ldots, N_p(x)$ as follows:

$$N_0(x) = \{x\}, \quad N_1(x) = \Gamma_x,$$

$$N_{k+1}(x) = \Big(\bigcup_{y_j \in N_k(x)} \Gamma_{y_j} \setminus N_k(x)\Big) \setminus N_{k-1}(x).$$

From every N_k we eliminate the edges $[y_i, y_j]$ for $y_i, y_j \in N_k(x)$. The partial graph of the graph G, obtained in this way, is called a *geodetic subgraph* of G with respect to the vertex x. It is easy to see that, for every $y \in N_i(x)$, by omitting all but one edges linking y with the vertices of $N_{i-1}(x)$, we shall have an arbitrary geodetic tree of vertex x.

Now let us form a family of subsets of a set of geodetic subgraph vertices of G with respect to the vertex x. We do it as follows: let $\Gamma_x = \{y_1, \ldots, y_s\}$ and for every $y_i \in \Gamma_x$ let us form subsets $D(y_i)$ of X such that

$$y_i \in D(y_i), \qquad \Gamma_{y_i} \setminus \{x\} \subset D(y_i),$$
$$z \in D(y_i) \cap N_k(x) \Rightarrow \Gamma_x \cap N_{k+1}(x) \subset D(y_i).$$

The set $D(y_i)$ is called a *maximal geodetic branch* of the vertex x. The family of maximal geodetic branches $D(y_1), D(y_2), \ldots, D(y_s)$ of the vertex x need not be a partition of the set $X \setminus \{x\}$.

To every geodetic branch there corresponds at least one geodetic tree of x for which $D(y_i)$ is an element of the partition $\mathscr{R}(x)$.

Let us consider the function:

$$d(x) = \max_{i=1,\ldots,s} |D(y_i)|. \tag{2'}$$

Since for a class of trees the family $D(y_1), \ldots, D(y_s)$ corresponds to the elements of partition $\mathscr{R}(x)$: $X_1, X_2, \ldots, X_s$, the function in (2′) is identical with that in (2).

The set $W = \{x \in X: d(x) - \min\}$ is called the *centroid* of a simple graph G. The set $\{x \in X: d(x) - \max\}$ determines a set of vertices which we call *W-peripheral.*

Since for trees the latter set is identical with the set E, it is natural to call these vertices the *end vertices* of a simple graph.

PROBLEM 2. In a bi-connected graph, determine the mutual position of the sets of Z-, σ- and W-peripheral vertices and of pseudo-end vertices. Characterize the classes of graphs for which these sets are equal. (**P 1150**)

By Lemmas 2 and 5 we get

THEOREM 4. *If $x \in Z_\sigma$ in a graph G, then for every geodetic branch $D(y_i)$ of G we have $|D(y_i)| \leqslant \frac{1}{2}|X|$.*

Thus we have obtained a necessary condition for a vertex x to be in the σ-centre of a simple graph.

In general case, for an arbitrary graph, one can introduce the notion of a characteristic set $\overline{W}$ of those vertices $x \in X$ for which the inequality $|D(y_i)| < \frac{1}{2}|X|$ is valid for every geodetic branch $D(y_i)$.

Hence Theorem 4 is reduced to $Z_\delta \subset \overline{W}$. It is easy to see that $W \subset \overline{W}$. For the class of trees we have $Z_\sigma = W = \overline{W}$. In Fig. 1 we present an

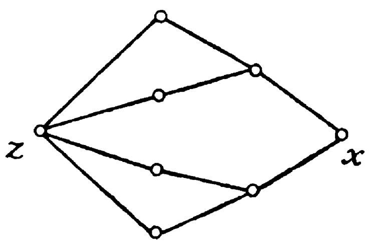

Fig. 1

example of a graph for which $Z_\sigma = W \neq \overline{W}$. Here for the vertex x we have $\sigma(x) = 13$, $d(x) = 4$, $x \notin Z_\sigma$, $x \notin W$, $x \in \overline{W}$, and for the vertex z we have $\sigma(z) = 11$, $d(z) = 3$, $z \in Z_\sigma$, $z \in W$, $z \in \overline{W}$.

PROBLEM 3. Is it true that $Z_\sigma = W$ for an arbitrary graph? (**P 1151**)

PROBLEM 4. Find a simple method for verifying that a given partial tree is in the class of geodetic trees of a graph. Characterize the class of graphs for which every spanning tree is a geodetic one. (**P 1152**)

Remark 1. Let us consider a hypergraph $H = \langle X, \mathscr{B}\rangle$, $\mathscr{B} = \{B_i\}_{i\in I}$ of all spanning trees of a simple graph $G = \langle X, U, \varphi\rangle$. Let y_i be a σ-centre of a tree $B_i \in \mathscr{B}$. One can define the function $\sigma(B_i) = \sigma(y_i)$ on the family $\mathscr{B}$. It is easy to see that if y_i is a σ-centre of G, then the family of geodetic trees of y_i is contained in the set $\{B_i \in \mathscr{B}: \sigma(B_i) - \min\}$. The values

$$\max_{i\in I} \sigma(B_i) \quad \text{and} \quad \min_{i\in I} \sigma(B_i)$$

yield an interesting characterization of a given simple graph in the class of graphs with the same number of vertices $|X| = n$. The following problem, related to Problem 4, arises:

PROBLEM 5. Is it possible by means of the function $\sigma(B_i)$ to determine a subset of geodetic trees in the class $\mathscr{B}$? (**P 1153**)

Similarly, the function $e(B_i)$ may be defined on the family of partial trees and the set $\{B_i \in \mathscr{B}: e(B_i) - \min\}$ may be determined.

Remark 2. A graph is geodetic if it is geodetic with respect to every of its vertices. It is easy to prove that the class of geodetic graphs coincides with the class of all bi-chromatic graphs (i.e., with no cycles of odd length).

2. Generalization of Steiner's problem. Steiner's problem for graphs may be formulated in a more general way: *find a vertex of a graph which has the property that the sum of distances from a given subset $Y \subset X$ is minimal.*

Let us consider the case where Y is the set of end vertices of a tree. The set

$$Z_{\sigma_E} = \{y \in X: \sigma_E(y) - \min\}$$

is called the *σ_E-centre* of a tree.

Let u belong to a σ_E-centre. Let an element X_i of partition $\mathscr{R}(y)$ of the set $X\setminus\{u\}$ contain $|E_i|$ end vertices of a tree and let $|E_i| > \frac{1}{2}|E|$. Denote by x a vertex incident with u in X_i. We have

$$\sigma_E(x) = \sigma_E(u) + |E\setminus E_i| - |E_i| < \sigma_E(u),$$

a contradiction. Hence $|E_i| \leqslant \frac{1}{2}|E|$, thus $\sigma_E(x) = \sigma_E(y)$ and x is also in the σ_E-centre. Similarly as in the previous section, for Z_σ we obtain

LEMMA 6. *$y \in Z_{\sigma_E}$ if and only if* $|E_i| \leqslant \frac{1}{2}|E|$.

It is easy to notice that, in a contrast to Z_σ, Z_{σ_E} may be arbitrarily large. Nevertheless, we prove

THEOREM 5. *The set Z_{σ_E} determines a subgraph of G which is a simple path. Every inner vertex of the path has the degree equal to 2 in G.*

Proof. Let two non-adjacent vertices x and z belong to a σ_E-centre and let $[x, u_1, \ldots, u_n, z]$ be a path linking these vertices. Assume that $u_1, u_2, \ldots, u_m$ do not belong to the σ_E-centre. Then for an element E_i of the partition, determined in the tree by the vertex z, containing vertices x and u_m, we have $|E_j| < \frac{1}{2}|E|$. For the element of vertex x and for that of u_m containing u_m and z, respectively, we have $|E_j| \geqslant \frac{1}{2}|E|$. Since x belongs to the σ_E-centre, $|E_j| = \frac{1}{2}|E|$. It follows that the vertex u_1 also belongs to the σ_E-centre and we get a contradiction. If vertices x, u and z of the path $[x, \ldots, u, \ldots, z]$ belong to the σ_E-centre, then the elements of partition determined by u and containing x or z have exactly $\frac{1}{2}|E|$ end vertices. Since every element of the partition contains at least one end vertex of G, the rank vertex u is equal to 2.

Remark 3. Taking into account the distance from the set of end vertices we can consider functions

$$\tilde{e}_E(y) = \min_{x \in E} \varrho(x, y) \tag{6}$$

and the set $\{y \in X: \tilde{e}_E(y) - \max\}$. There are simple examples where that set is disjoint with Z. The definition can be generalized to an arbitrary simple graph where the set E contains generalized end vertices.

Now let us take up the generalized Steiner problem for graphs related to various practical applications. We shall formulate the problem for a class of trees.

A partition of the set X of vertices of $G = \langle X, R \rangle$ into subsets $Y_1, Y_2, \ldots, Y_k$,

$$Y_i \cap Y_j = \emptyset, \quad i \neq j, \quad \bigcup_{i=1}^{k} Y_i = X,$$

such that a subgraph $G_{Y_i} = \langle Y_i, R_{X_i} \rangle$ is a tree, is called a *proper k-partition* of G.

Let $y_i \in Y_i$ be one of the σ-centre vertices of G_{Y_i}. Consider the sum

$$\sum_{i=1}^{k} \sigma(y_i). \tag{**}$$

The partition for which number (**) is minimal is called a *$\sigma^{(k)}$-partition* of G, and the set $\{y_1, \ldots, y_k\}$ is called a *$\sigma^{(k)}$-centre* of G.

We have no theorems on $\sigma^{(k)}$-partitions of trees which would help us to construct a simple procedure for the solution of the problem even in case of $k = 2$.

It is possible to obtain a necessary condition which allows a reduction of the procedure of determination of the $\sigma^{(2)}$-partition. Let $[Y_1, Y_2]$ be a $\sigma^{(2)}$-partition of the tree. We denote by (u, v) the edge linking the sets Y_1 and Y_2, where $u \in Y_1$, $v \in Y_2$. Let $\varrho(y_1, u) > \varrho(y_2, u)$, where y_1 and y_2 are vertices belonging to the $\sigma^{(2)}$-centre of the tree. Let us denote by V the set of vertices belonging to the components of G obtained by omitting the vertex u and a vertex which does not contain y_1 and y_2, $V \subset Y_1$. Let $|V| = p$. Combining the sets $\{u\} \cup V$ and Y_2 we obtain a new proper 2-partition. The sum (**) is increased by at most

$$p\varrho(y_2, u) + \sum_{z \in V} \varrho(u, z)$$

and decreased not less than by

$$p\varrho(y_1, u) + \sum_{z \in V} \varrho(u, z),$$

hence it decreased by not less than $p[\varrho(y_1, u) - \varrho(y_2, u)]$. Thus we have

THEOREM 6. *If y_1 and y_2 belong to the $\sigma^{(2)}$-centre of a tree G and u, v is an edge of partition, then $\varrho(y_1, u) = \varrho(y_2, v)$ or $|\varrho(y_1, u) - \varrho(y_2, v)| = 1$.*

Evidently, we also obtain

THEOREM 7. *If a set of edges $\overline{U}$, which are not hanging edges of a tree, is not empty, then there exists an edge belonging to $\overline{U}$ determining a $\sigma^{(2)}$-partition.*

Application of both necessary conditions for the $\sigma^{(2)}$-partition may be considered in the most simple case of a simple path. Let $[x_1, \ldots, x_k]$ $(k > 3)$ be a simple path. The partition of the path is determined by the edge $[x_i, x_{i+1}]$. With no loss of generality we can assume that each part of the partition contains an odd number of vertices. Thus the vertices $x_{(i+1)/2}$ and $x_{(i+k+1)/2}$ are the σ-centres of each of the parts, respectively, and the necessary condition (see Theorem 6) for the edge $[x_i, x_{i+1}]$ to determine the $\sigma^{(2)}$-partition is here reduced to the equality

$$i - \frac{1+i}{2} = \frac{i+k+1}{2} - (i+1),$$

hence $i = k/2$.

It is easy to see that the problem of $\sigma^{(k)}$-partition may be generalized to an arbitrary graph. In that case a proper partition of a graph into connected subgraphs will be determined by a certain cut of the graph. The problem of obtaining even simple necessary conditions is certainly a hard one.

In some practical considerations an additional condition $e(y_i) \leqslant C$ for the parts of $\sigma^{(k)}$-partition may be introduced, where C is a fixed constant.

PROBLEM 6. It seems that a classification of trees with the fixed $|X| = n$ with respect to a minimal value of $\sigma(y)$, $y \in X$, should be performed in considering the problems of $\sigma^{(k)}$-partitions of trees. **(P 1154)**

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