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LOCAL POLYNOMIAL FUNCTIONS ON LATTICES AND UNIVERSAL ALGEBRAS

BY

D. DORNINGER AND W. NÖBAUER (WIEN)

1. Introduction. Let $F_k(A)$ be the set of all k-place functions on an algebra A. If we define pointwise operations on $F_k(A)$ corresponding to the operations of A, then $F_k(A)$ becomes an algebra of the same type as A. The elements of the subalgebra $P_k(A)$ of $F_k(A)$ generated by the projections and the constant functions will be called *k-place polynomial functions* on A (cf. [9]). If a function $f \in F_k(A)$ has the property that for every subset $M \subseteq A^k$ consisting of at most t elements ($t \in N$) there exists a polynomial function $p \in P_k(A)$ such that f and p coincide on M, then f is called a *t-local polynomial function*. A function which is t-local for any integer $t \geqslant 1$ will be called a *local polynomial function*. It is easy to see that, for a given t, the set of all t-local polynomial functions and also the set of all local polynomial functions is a subalgebra of $F_k(A)$. Denoting these algebras by $L_tP_k(A)$ and $LP_k(A)$, respectively, and denoting by $C_k(A)$ the algebra of those functions of $F_k(A)$ which are compatible with all congruences on A, we obtain the following chain of subalgebras of $F_k(A)$, called the *chain of* (*k-place*) *local polynomial functions* on A:

$$F_k(A) \supseteq C_k(A) \supseteq L_2P_k(A) \supseteq L_3P_k(A) \supseteq \ldots \supseteq LP_k(A) \supseteq P_k(A).$$

Now one can ask which members of this chain coincide for a given algebra A.

If all members coincide, i.e., if $F_k(A) = P_k(A)$, then A is called *k-polynomially complete*; if, however, $F_k(A) = LP_k(A)$, then A is called *k-locally polynomially complete*. If $C_k(A) = P_k(A)$, then A is called *k-affine complete* (cf. [12]), and if $C_k(A) = LP_k(A)$, then A is called *k-locally affine complete*.

The problem of equality of certain members in the chain of local polynomial functions has been treated in several papers. In particular, polynomially complete and affine complete algebras and interpolation of functions on algebras by polynomial functions have been studied, e.g., by Rédei and Szele, Foster, Pixley, Grätzer, Werner and other authors (see, for example, the Remarks and Comments to Chapter I in [9]). Recently, the chain of local polynomial functions on Abelian groups has been investigated extensively (cf. [7] and [8]).

In this paper we study first some properties of the chain of local polynomial functions on an arbitrary algebra A. Then we consider the chain of local polynomial functions for the case where A is a lattice. In this case $L_2P_k(A) = LP_k(A)$, and we show that there exist (unbounded) 1-affine complete lattices (1) and lattices which are 1-locally affine complete but not affine complete. Moreover, we provide examples of lattices for which, on the one hand, the equality holds at the remaining positions in the chain of local polynomial functions and it does not hold on the other hand.

2. Local polynomial functions on universal algebras. Since, clearly, $F_k(A) = C_k(A)$ iff A is simple, we first ask for conditions under which $C_k(A)$ equals $L_2P_k(A)$.

As Werner has shown in [11], a variety $\mathfrak{V}$ has permutable congruences iff for any $A \in \mathfrak{V}$ the diagonal subalgebras (in the sense of Baker and Pixley [1]) of $A \times A$ are just the congruences of A. Since $L_2P_k(A)$, by [8], is the set of those functions which are compatible with all compatible and reflexive 2-place relations, i.e., the set of functions which are compatible with all diagonal subalgebras of $A \times A$, we can conclude

THEOREM 1. *If a variety $\mathfrak{V}$ has permutable congruences, then $C_k(A) = L_2P_k(A)$ for all $A \in \mathfrak{V}$.*

It is an open question how far this result can be sharpened. The strongest version would be: $C_k(A) = L_2P_k(A)$ iff A has permutable congruences. That this fails to be true in either direction we will see later on by Theorems 8 and 9.

Now we consider direct products of algebras. For this purpose we use a mapping described in [10]. Let $A = B \times C$ be the direct product of the algebras B and C. Then, for any $\varphi \in C_k(A)$, there exist uniquely determined $\varrho \in C_k(B)$ and $\sigma \in C_k(C)$ such that

$$\varphi\big((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k)\big) = \big(\varrho(b_1, b_2, \ldots, b_k), \sigma(c_1, c_2, \ldots, c_k)\big)$$

for all $\big((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k)\big) \in A^k$. Moreover, $\mu: \varphi \to (\varrho, \sigma)$ is a monomorphism of $C_k(A)$ into $C_k(B) \times C_k(C)$.

THEOREM 2. *Let B and C be algebras and let $A = B \times C$. If the monomorphism $\mu: C_k(A) \to C_k(B) \times C_k(C)$ induces an isomorphism of $P_k(A)$ onto $P_k(B) \times P_k(C)$, then it also induces an isomorphism of $L_tP_k(A)$ onto*

(1) Examples of bounded affine complete lattices have been already given by Grätzer [5].

$L_tP_k(B) \times L_tP_k(C)$ for any $t \geqslant 2$ and an isomorphism of $LP_k(A)$ onto $LP_k(B) \times LP_k(C)$.

Proof. Let $\varphi \in L_tP_k(A)$ and $t \geqslant 2$; then $\varphi \in C_k(A)$. Given t elements

$$\big((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)})\big) \in A^k \qquad (i = 1, 2, \ldots, t),$$

there exists a $p \in P_k(A)$ such that

$$\varphi\big((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)})\big) = p\big((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)})\big)$$

for $i = 1, 2, \ldots, t$. Consequently,

$$\big(\varrho(b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}), \sigma(c_1^{(i)}, c_2^{(i)}, \ldots, c_k^{(i)})\big)$$
$$= \big(p_1(b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}), p_2(c_1^{(i)}, c_2^{(i)}, \ldots, c_k^{(i)})\big),$$

where $(p_1, p_2) \in P_k(B) \times P_k(C)$. Hence

$$\mu\varphi = (\varrho, \sigma) \in L_tP_k(B) \times L_tP_k(C).$$

Conversely, let $(\varphi_1, \varphi_2) \in L_tP_k(B) \times L_tP_k(C)$. We define $\varphi \in F_k(A)$ by

$$\varphi\big((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k)\big) = \big(\varphi_1(b_1, b_2, \ldots, b_k), \varphi_2(c_1, c_2, \ldots, c_k)\big)$$

for all $\big((b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k)\big) \in A^k$. Since μ is a surjective mapping of $P_k(A)$ onto $P_k(B) \times P_k(C)$, it is easy to see that for any subset $M \subseteq A^k$ consisting of t elements there always exists a $p \in P_k(A)$ such that p equals φ on M. Clearly, $\mu\varphi = (\varphi_1, \varphi_2)$.

If $\varphi \in LP_k(A)$ and $\mu\varphi = (\varrho, \sigma)$, then $\varphi \in L_tP_k(A)$ for any $t \geqslant 2$ and, therefore, $(\varrho, \sigma) \in L_tP_k(B) \times L_tP_k(C)$ for all $t \geqslant 2$; this implies that $\varrho \in LP_k(B)$ and $\sigma \in LP_k(C)$. Conversely, if $(\varrho, \sigma) \in LP_k(B) \times LP_k(C)$, then $(\varrho, \sigma) \in L_tP_k(B) \times L_tP_k(C)$ for all $t \geqslant 2$. From this we conclude that $\mu^{-1}(\varrho, \sigma) \in L_tP_k(A)$ for all $t \geqslant 2$, and that means $\mu^{-1}(\varrho, \sigma) \in LP_k(A)$.

We denote the chain of k-place local polynomial functions on an algebra A by $\Omega_k(A)$. The members $L_tP_k(A)$, $LP_k(A)$ and $P_k(A)$ of $\Omega_k(A)$ will be called *P-elements.*

COROLLARY 1. *If μ induces an isomorphism of $P_k(A)$ onto $P_k(B) \times P_k(C)$, then two P-elements of $\Omega_k(A)$ coincide iff the corresponding P-elements of $\Omega_k(B)$ and $\Omega_k(C)$ coincide.*

Proof. If the P-elements $\Lambda A \supseteq MA$ of $\Omega_k(A)$ coincide and ΛB, MB, ΛC, MC are the corresponding P-elements of $\Omega_k(B)$ and $\Omega_k(C)$, then $\Lambda B \times \Lambda C = MB \times MC$ and, therefore, $\Lambda B = MB$ and $\Lambda C = MC$. Conversely, if $\Lambda B = MB$ and $\Lambda C = MC$, then $\Lambda B \times \Lambda C = MB \times MC$, whence $\Lambda A = MA$.

Remark. The hypothesis of Corollary 1 is satisfied especially for all direct products of rings with identity and for all direct products of groups of relatively prime orders.

COROLLARY 2. *If A is an algebra of a variety which has permutable congruences and μ is an isomorphism of $P_k(A)$ onto $P_k(B)\times P_k(C)$, then A is k- (locally) affine complete iff both B and C are k- (locally) affine complete.*

THEOREM 3. *If two members of a chain $\Omega_k(A)$ coincide, then so do also the corresponding members of the chain $\Omega_{k-1}(A)$ $(k \geqslant 2)$.*

Proof. Let $\Lambda_k \supseteq M_k$ be two elements of $\Omega_k(A)$ which coincide, let Λ_{k-1}, M_{k-1} be the corresponding elements of $\Omega_{k-1}(A)$, and let $\varphi \in \Lambda_{k-1}$.

We define $\psi \in F_k(A)$ by

$$\psi(x_1, x_2, \ldots, x_k) = \varphi(x_1, x_2, \ldots, x_{k-1})$$

for all $(x_1, x_2, \ldots, x_k) \in A^k$.

As one can easily see by checking all possible cases for Λ_{k-1}, ψ belongs to Λ_k. Therefore $\psi \in M_k$. Since

$$\varphi(x_1, x_2, \ldots, x_{k-1}) = \psi(x_1, x_2, \ldots, x_{k-1}, a),$$

a discussion of all possible cases for M_k shows immediately that $\varphi \in M_{k-1}$.

COROLLARY 3. *If for any algebra A two members of $\Omega_k(A)$ are distinct, so are also the corresponding elements in any chain $\Omega_r(A)$ such that $r \geqslant k$.*

THEOREM 4. *If ϑ is an epimorphism of an algebra A onto an algebra B, then*

$$\big(\varphi(\vartheta)f\big)(\vartheta a_1, \vartheta a_2, \ldots, \vartheta a_k) = \vartheta f(a_1, a_2, \ldots, a_k)$$

defines a homomorphism $\varphi(\vartheta)$: $C_k(A) \to C_k(B)$. This homomorphism maps $P_k(A)$ onto $P_k(B)$ and any other P-element of $\Omega_k(A)$ into the corresponding P-element of $\Omega_k(B)$.

Proof. $\varphi(\vartheta)$ is well defined. Indeed, let $\vartheta a_i = \vartheta b_i$ for $i = 1, 2, \ldots, k$. Then $a_i \equiv b_i$ (mod Ker ϑ), whence

$$f(a_1, a_2, \ldots, a_k) \equiv f(b_1, b_2, \ldots, b_k) \ (\mathrm{mod\,Ker}\,\vartheta),$$

which implies

$$\vartheta f(a_1, a_2, \ldots, a_k) = \vartheta f(b_1, b_2, \ldots, b_k).$$

$\varphi(\vartheta)$ is compatible. Indeed, let Θ be a congruence on B and let

$$\alpha_i \equiv \beta_i \ (\mathrm{mod}\,\Theta), \quad \alpha_i = \vartheta a_i \quad \text{and} \quad \beta_i = \vartheta b_i.$$

Then $a_i \equiv b_i$ (mod $\vartheta^{-1}\Theta$), whence

$$f(a_1, a_2, \ldots, a_k) \equiv f(b_1, b_2, \ldots, b_k) \ (\mathrm{mod}\,\vartheta^{-1}\Theta).$$

Therefore

$$\vartheta f(a_1, a_2, \ldots, a_k) \equiv \vartheta f(b_1, b_2, \ldots, b_k) \ (\mathrm{mod}\,\Theta),$$

which shows that

$$(\varphi(\vartheta)f)(a_1, a_2, \ldots, a_k) \equiv (\varphi(\vartheta)f)(\beta_1, \beta_2, \ldots, \beta_k) \pmod{\Theta}.$$

That $\varphi(\vartheta)$ is a homomorphism one can easily see.

If $f \in P_k(A)$, then there exists a word $w(c_j, x_1, x_2, \ldots, x_k)$ in the constants c_j and the indeterminates $x_1, x_2, \ldots, x_k$ such that

$$f(a_1, a_2, \ldots, a_k) = w(c_j, a_1, a_2, \ldots, a_k).$$

Hence

$$\vartheta f(a_1, a_2, \ldots, a_k) = w(\vartheta c_j, \vartheta a_1, \vartheta a_2, \ldots, \vartheta a_k),$$

thus

$$(\varphi(\vartheta)f)(\vartheta a_1, \vartheta a_2, \ldots, \vartheta a_k) = w(\vartheta c_j, \vartheta a_1, \vartheta a_2, \ldots, \vartheta a_k),$$

which implies $\varphi(\vartheta)f \in P_k(B)$.

If, on the other hand, we have $\psi \in P_k(B)$, then there exists a word $\bar{w}(d_l, x_1, x_2, \ldots, x_k)$ such that

$$\psi(b_1, b_2, \ldots, b_k) = \bar{w}(d_l, b_1, b_2, \ldots, b_k).$$

Replacing the constants d_l in $\bar{w}$ by inverse images under ϑ we obtain a word representing a function of $P_k(A)$, which is mapped on ψ by $\varphi(\vartheta)$.

Now let $f \in L_t P_k(A)$ and $\vartheta\mathfrak{a}_1, \vartheta\mathfrak{a}_2, \ldots, \vartheta\mathfrak{a}_t \in B^k$. (For short we denote the k-tuples $(a_1, a_2, \ldots, a_k)$ by single vector symbols.) $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_t$ are elements of A^k, thus $f(\mathfrak{a}_j) = p(\mathfrak{a}_j)$, where $p \in P_k(A)$ and $j = 1, 2, \ldots, t$. Hence

$$(\varphi(\vartheta)f)(\vartheta\mathfrak{a}_j) = \vartheta f(\mathfrak{a}_j) = \vartheta p(\mathfrak{a}_j) = (\varphi(\vartheta)p)(\vartheta\mathfrak{a}_j),$$

where $\varphi(\vartheta)p \in P_k(B)$, thus $\varphi(\vartheta)f \in L_t P_k(B)$.

Finally, if $f \in LP_k(A)$, then $f \in L_t P_k(A)$ for all t, thus $\varphi(\vartheta)f \in L_t P_k(B)$ for all t, and hence $\varphi(\vartheta)f \in LP_k(B)$.

Remark. $\varphi(\vartheta)$ is an epimorphism in the case of P_k, but for the other P-elements of $\Omega_k(A)$ this is not true in general.

Counterexamples. 1. Take $A = Z_p \times Z_p$, $B = Z_p$ and ϑ a projection of A (Z_n — a cyclic group of order n, p — a prime). By Theorem 1, $C_k(A) = L_2 P_k(A)$ and $C_k(B) = L_2 P_k(B)$. Moreover, as proved in [10], $C_k(A) = P_k(A)$. Since $C_k(B) = F_k(B)$ and (apart from $p = 2$ and $k = 1$) $F_k(B) \supset P_k(B)$ (cf., e.g., [9]), we obtain

$$\varphi(\vartheta)C_k(A) = \varphi(\vartheta)P_k(A) = P_k(B) \subset C_k(B),$$

which shows that $\varphi(\vartheta)$, in general, is not surjective for C_k and $L_2 P_k$.

2. Take

$$A = (Z_p \times Z_{p^2} \times Z_{p^3} \times \ldots) \times Z_{p^\infty}, \quad B = (Z_p \times Z_{p^2} \times Z_{p^3} \times \ldots)$$

and ϑ the projection of A onto B. By Theorem 2 in [8], $LP_k(A) = P_k(A)$ and $LP_k(B) \neq P_k(B)$, whence

$$\varphi(\vartheta)LP_k(A) = \varphi(\vartheta)P_k(A) = P_k(B) \subset LP_k(B).$$

Thus $\varphi(\vartheta)$ is, in general, not surjective for LP_k.

3. Local polynomial functions on lattices.

THEOREM 5. *Let L be an arbitrary lattice. Then $L_tP_k(L) = LP_k(L)$ for all $t \geqslant 2$.*

Proof (cf. [1], the proof of the equivalence theorem). Let $f \in L_tP_k(L)$ and $\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_{t+1} \in L^k$. Then there exist polynomial functions $p, q, r \in P_k(L)$ such that

$$f(\mathfrak{x}_i) = p(\mathfrak{x}_i) \quad (i = 1, 2, \ldots, t),$$

$$f(\mathfrak{x}_i) = q(\mathfrak{x}_i) \quad (i = 2, 3, \ldots, t+1)$$

and

$$f(\mathfrak{x}_i) = r(\mathfrak{x}_i) \quad (i = 1, 3, 4, \ldots, t+1).$$

We define a polynomial function $g \in P_k(L)$ by

$$g(\mathfrak{x}) = \big((p(\mathfrak{x}) \cup q(\mathfrak{x})) \cap r(\mathfrak{x})\big) \cup \big(p(\mathfrak{x}) \cap q(\mathfrak{x})\big).$$

As one can immediately see, $g(\mathfrak{x}_i) = f(\mathfrak{x}_i)$ for $i = 1, 2, \ldots, t+1$. Thus $f \in L_{t+1}P_k(L)$.

Remark. As Grätzer has proved in [4], $C_k(L) = P_k(L)$ for any Boolean algebra L.

THEOREM 6. *Let $A = B \times C$. Then, for arbitrary lattices B and C, the monomorphism μ defined in Theorem 2 is an isomorphism of $C_k(A)$ onto $C_k(B) \times C_k(C)$ and induces an isomorphism of $LP_k(A)$ onto $LP_k(B) \times \times LP_k(C)$.*

If B and C are distributive lattices both having more than one element, then $\mu P_k(A) = P_k(B) \times P_k(C)$ if and only if both B and C are bounded.

COROLLARY 4. *The lattice $A = B \times C$ is locally affine complete if and only if B and C are; $LP_k(A) = P_k(A)$ holds only if $LP_k(B) = P_k(B)$ and $LP_k(C) = P_k(C)$, hence A is k-affine complete only if B and C are.*

Proof. The first statement is obvious, the second one follows immediately if we observe that

$$\mu P_k(A) \subseteq P_k(B) \times P_k(C)$$

(cf. [9], Proposition 3.41).

Proof of Theorem 6. Let B and C be arbitrary lattices. Then, by [3], every congruence of A is the direct product of a congruence of B and a congruence of C. By [10] this implies that μ is an isomorphism of $C_k(A)$ onto $C_k(B) \times C_k(C)$. This proves the first statement of the theorem.

To prove the second statement we must only show that

$$\mu(L_2P_k(A)) = L_2P_k(B) \times L_2P_k(C).$$

By the proof of Theorem 2, μ is a monomorphism of $L_2P_k(A)$ into $L_2P_k(B) \times L_2P_k(C)$, thus we have to show that μ is surjective.

Let

$$(\varphi_1, \varphi_2) \in L_2P_k(B) \times L_2P_k(C) \quad \text{and} \quad \varphi = \mu^{-1}(\varphi_1, \varphi_2).$$

Moreover, let $((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)}))$ for $i = 1, 2$ be arbitrary elements of A^k. If we put

$$(b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}) = \mathfrak{b}^{(i)} \quad \text{and} \quad (c_1^{(i)}, c_2^{(i)}, \ldots, c_k^{(i)}) = \mathfrak{c}^{(i)},$$

then

$$\varphi((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)})) = (\varphi_1(\mathfrak{b}^{(i)}), \varphi_2(\mathfrak{c}^{(i)})) = (p_1(\mathfrak{b}^{(i)}), p_2(\mathfrak{c}^{(i)})),$$

where $(p_1, p_2) \in P_k(B) \times P_k(C)$.

Let

$$\alpha_1 = p_1(\mathfrak{b}^{(1)}) \cup p_1(\mathfrak{b}^{(2)}), \qquad \beta_1 = p_1(\mathfrak{b}^{(1)}) \cap p_1(\mathfrak{b}^{(2)}),$$
$$\alpha_2 = p_2(\mathfrak{c}^{(1)}) \cup p_2(\mathfrak{c}^{(2)}), \qquad \beta_2 = p_2(\mathfrak{c}^{(1)}) \cap p_2(\mathfrak{c}^{(2)})$$

and let (b, c) be an arbitrary element of $B \times C$. Moreover, let $\mathfrak{x}$ denote the vector $(x_1, x_2, \ldots, x_k)$ and let $w_1(d_r, \mathfrak{x})$ and $w_2(f_s, \mathfrak{x})$ be words in the constants $d_r \in B$ and $f_s \in C$, which represent the polynomial functions $p_1(\mathfrak{x})$ and $p_2(\mathfrak{x})$, respectively.

Now we replace every d_r in w_1 by the pair (d_r, c) and every f_s in w_2 by the pair (b, f_s) (b and c being fixed). We consider the polynomial function $p \in P_k(A)$ which is represented by the word

$$(w_1((d_r, c), \mathfrak{x}) \cap (\alpha_1, \beta_2)) \cup (w_2((b, f_s), \mathfrak{x}) \cap (\beta_1, \alpha_2)).$$

This polynomial function satisfies

$$p((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)}))$$
$$= ((p_1(\mathfrak{b}^{(i)}) \cap \alpha_1) \cup (\gamma_i \cap \beta_1), (\delta_i \cap \beta_2) \cup (p_2(\mathfrak{c}^{(i)}) \cap \alpha_2))$$
$$= (p_1(\mathfrak{b}^{(i)}), p_2(\mathfrak{c}^{(i)})) \quad (i = 1, 2),$$

where γ_i, δ_i are elements of B, C, respectively. Thus there exists a $p \in P_k(A)$ such that

$$\varphi((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)}))$$
$$= p((b_1^{(i)}, c_1^{(i)}), (b_2^{(i)}, c_2^{(i)}), \ldots, (b_k^{(i)}, c_k^{(i)})) \quad (i = 1, 2).$$

Therefore $\varphi \in L_2P_k(A)$.

Now, let B and C be distributive lattices of order greater than 1. Without loss of generality we assume that C does not have a lower bound. Moreover, let b be a constant function of $P_1(B)$ such that $b \neq 0$ if B has a lower bound, and let x be the identity mapping. Suppose that there exists a function $r \in P_1(A)$ such that $\mu r = (b, x)$. Then r is x or takes the form

$$\big((b_1, c_1)\cap x\big)\cup(b_2, c_2), \quad \text{where } (b_1, c_1) \geqslant (b_2, c_2),\ (b_2, c_2)\cup x,\ (b_1, c_1)\cap x$$

(cf. [6]). Since C does not have a lower bound, r cannot be of the first or of the second form, thus $b_1 \cap x = b$ or $x = b$ for all $x \in B$, which is obviously a contradiction. Therefore, μ does not induce an isomorphism of $P_1(A)$ onto $P_1(B)\times P_1(C)$.

To show that μ also does not induce an isomorphism of $P_k(A)$ onto $P_k(B)\times P_k(C)$ for $k > 1$, we define a mapping τ_L of $P_k(L)$ into $P_1(L)$ (for an arbitrary lattice L) by

$$(\tau_L \varphi)(x_1) = \varphi(x_1, x_1, \ldots, x_1).$$

Denoting the monomorphism μ of $C_k(A)$ into $C_k(B)\times C_k(C)$ by μ_k, we consider the following diagram:

$$\begin{array}{ccc} P_k(B\times C) & \xrightarrow{\mu_k} & P_k(B)\times P_k(C) \\ \downarrow{\scriptstyle \tau_{B\times C}} & & \downarrow{\scriptstyle \tau_B\times\tau_C} \\ P_1(B\times C) & \xrightarrow{\mu_1} & P_1(B)\times P_1(C) \end{array}$$

Since τ_L is always an epimorphism and the diagram turns out to be commutative, surjectivity of μ_k would imply that μ_1 is surjective.

Conversely, if the distributive lattices B and C are both bounded, then μ is always an isomorphism of $P_k(A)$ onto $P_k(B)\times P_k(C)$, since, by [2], this is true for an arbitrary bounded lattice.

THEOREM 7. *Let L be a bounded distributive lattice. Then $LP_k(L) = P_k(L)$ for any k. There also exist unbounded distributive lattices L such that $LP_1(L) = P_1(L)$.*

Proof. Let L be a distributive lattice with 0 and 1. As the normal form system for the polynomial functions of $P_k(L)$ in [9] shows, any $p \in P_k(L)$ is uniquely determined by its values at the 2^k places $(i_1, i_2, \ldots, i_k)$, where $i_\nu = 0$ or 1. Hence we conclude that $LP_k(L) = P_k(L)$. The second statement follows from the

LEMMA. *Let K be an arbitrary chain. Then $LP_1(K) = P_1(K)$.*

Proof. Let $\varphi \in LP_1(K)$ and $\varphi \neq x$; then there exists an element $a \in K$ such that $\varphi(a) = b \neq a$. Without loss of generality we assume that $a < b$. Suppose that $y \in K$ and $y \leqslant b$; then, by $\varphi \in LP_1(K)$, there exists a $p \in P_1(K)$ such that $p(x) = \varphi(x)$ for $x = a$ and $x = y$. By [6], p is of the form $(\alpha\cap x)\cup\beta$ $(\alpha\geqslant\beta)$, or $\alpha\cap x$, or $\beta\cup x$, or x. From $p(a)=b$ one can easily see that either $p=(\alpha\cap x)\cup b$ $(\alpha\geqslant b)$ or $p=x\cup b$, whence $\varphi(y)=p(y)=b$ for any $y\leqslant b$. If now $\varphi(x)=x$ for all $x>b$, then $\varphi=x\cup b\in P_1(K)$. Otherwise, there exists $c>b$ such that $\varphi(c)=d\neq c$. Suppose that $c<d$. Then, by the preceding argument, $\varphi(x)=d$ for all $x\leqslant d$, thus $\varphi(b)=d$, a contradiction. Thus $c>d$. Dualizing the preceding argument we see that $\varphi(x)=d$ for any $x\geqslant d$. If $d\leqslant b$, then $d=\varphi(d)=b$, thus $\varphi(x)=b$ for any x, and hence $\varphi=b\in P_1(K)$. If, however, $d>b$ and $b<y<d$, then, by the preceding argument, $\varphi(y)\neq y$ would either imply $\varphi(b)\neq b$ or $\varphi(d)\neq d$, a contradiction. Thus $\varphi(x)=b$ if $x\leqslant b$, $\varphi(x)=x$ if $b\leqslant x\leqslant d$, and $\varphi(x)=d$ if $x\geqslant d$. Therefore

$$\varphi=(d\cap x)\cup b\in P_1(K).$$

By the Lemma and Theorem 6 we can easily construct examples of lattices L such that $LP_k(L)\neq P_k(L)$ for all k. Let L be the direct product of two unbounded chains K_1 and K_2. Then

$$\mu\big(LP_1(K_1\times K_2)\big)=LP_1(K_1)\times LP_1(K_2)=P_1(K_1)\times P_1(K_2),$$

but, on the other hand,

$$\mu\big(P_1(K_1\times K_2)\big)\neq P_1(K_1)\times P_2(K_2).$$

Thus $LP_1(K_1\times K_2)\neq P_1(K_1\times K_2)$ and, by Corollary 3,

$$LP_k(K_1\times K_2)\neq P_k(K_1\times K_2)\quad\text{for all } k\geqslant 1.$$

THEOREM 8. *If a lattice L has an atom, then $C_k(L)\neq LP_k(L)$.*

Proof. By Corollary 3 we have to prove the theorem only if $k=1$.

Let a be an atom of L and let 0 denote the zero of L. We define a function $f\in F_1(L)$ by $f(x)=0$ for $x\geqslant a$ and by $f(x)=a$ otherwise.

If Θ is a congruence of L such that $(0,a)\in\Theta$, then, obviously, $(x,y)\in\Theta$ implies $\big(f(x),f(y)\big)\in\Theta$. Let, therefore, Θ be a congruence such that $(0,a)\notin\Theta$.

Let $(0,x)\in\Theta$. Then $x\geqslant a$ would imply $(0,a)\in\Theta$, whence $f(x)=a$ and, therefore, $\big(f(0),f(x)\big)=(a,a)\in\Theta$.

Now let $(a,x)\in\Theta$ and $x\neq 0$. If a were incomparable with x, then we would have $a\cap x=0$, which would imply that a,x and 0 belong to the same Θ-class, a contradiction. Thus $x\geqslant a$ and, consequently, $\big(f(a),f(x)\big)=(0,0)\in\Theta$.

Finally, let $(x,y)\in\Theta$ and $x,y\neq 0$, $x,y\neq a$. We consider two cases. If $x\geqslant a$, then also $y\geqslant a$, since $(x,y)\in\Theta$ implies

$$(x\cap a,y\cap a)=(a,y\cap a)\in\Theta;$$

thus $\big(f(x),f(y)\big)=(0,0)\in\Theta$. If, however, x is incomparable with a, so is y and, therefore, $\big(f(x),f(y)\big)=(a,a)\in\Theta$.

Thus $(x, y) \in \Theta$ for any congruence Θ implies $(f(x), f(y)) \in \Theta$, which means that $f \in C_1(A)$. Since f is no order endomorphism, $f \notin L_2P_1(L)$.

Theorem 8 gives us a large class of lattices L such that $C_k(L) \neq LP_k(L)$. Another class of lattices of this kind is the class of subdirectly irreducible lattices L (of order greater than 1). Such a lattice always contains two elements u, v such that $u \neq v$ and $(u, v) \in \Theta$ for all congruences Θ greater than the 0-congruence. Without loss of generality take $u > v$. We define a function $f \in F_1(L)$ by $f(u) = v$, $f(v) = u$ and $f(x) = x$ for $x \neq u, v$. Then $f \in C_1(L)$ but $f \notin OF_1(L)$, which implies that $f \notin LP_1(L)$.

Corollary 4 provides now some more lattices that are not (locally) affine complete.

THEOREM 9. *A chain K is* 1-*affine complete if and only if it does not contain a prime interval* [2].

Proof. Suppose that K contains a prime interval $[a, b]$. Then the mapping defined by $f(x) = a$ for all $x \geqslant b$ and by $f(x) = b$ for all $x \leqslant a$ is in $C_1(K)$, which can easily be proved by the argument of Theorem 8. Since f is not order-preserving, $f \notin P_1(K)$.

Now, let us assume that K does not contain a prime interval and that $\varphi \in C_1(K)$. First we show that $b < \varphi(b)$ implies $\varphi(x) = \varphi(b)$ for all $x < \varphi(b)$ and $\varphi(x) \leqslant x$ for all $x \geqslant \varphi(b)$. By the way of contradiction suppose that $b < \varphi(b)$ and $x < \varphi(b)$, but $\varphi(x) \neq \varphi(b)$. Then we can find a congruence of K such that x and b are congruent, but $\varphi(x)$ and $\varphi(b)$ are not. Similarly we obtain a contradiction from $b < \varphi(b)$, $x \geqslant \varphi(b)$ and $\varphi(x) > x$. Dualizing this result we infer that $a > \varphi(a)$ implies $\varphi(x) = \varphi(a)$ for all $x > \varphi(a)$ and $\varphi(x) \geqslant x$ for all $x \leqslant \varphi(a)$.

Suppose now that φ is not the identity mapping. Then without loss of generality we can assume that there exists an element $b \in K$ such that $b < \varphi(b)$. If $\varphi(x) = x$ for all $x \geqslant \varphi(b)$, then

$$\varphi = x \cup \varphi(b) \in P_1(K).$$

Otherwise, there exists an $a \geqslant \varphi(b)$ such that $\varphi(a) \neq a$, whence $\varphi(a) < a$. Thus $\varphi(x) = \varphi(a)$ for all $x > \varphi(a)$ and $\varphi(x) \geqslant x$ for all $x \leqslant \varphi(a)$. If $\varphi(a) < \varphi(b)$, then, choosing c such that $\varphi(a) < c < \varphi(b)$, we obtain $\varphi(c) = \varphi(a) = \varphi(b)$, a contradiction. If, however, $\varphi(a) \geqslant \varphi(b)$ and $\varphi(b) \leqslant x \leqslant \varphi(a)$, then $\varphi(x) \leqslant x$ and $\varphi(x) \geqslant x$, whence $\varphi(x) = x$. Thus

$$\varphi = (x \cup \varphi(b)) \cap \varphi(a) \in P_1(K),$$

which completes the proof.

As a consequence of Theorem 9 and Corollary 4 we infer that a direct product of two unbounded chains K_1 and K_2 which do not contain prime

(2) For finite K cf. [5].

intervals is 1-locally affine complete. Since, as we have seen previously,

$$LP_1(K_1 \times K_2) \neq P_1(K_1 \times K_2),$$

we have found a class of lattices that are 1-locally affine complete but not 1-affine complete.

Another consequence of Theorem 9 is that the identity $C_1(A) = L_2P_1(A)$ for an algebra A does not imply that A has permutable congruences. On the other hand, Theorem 8 shows that, conversely, if an algebra A has permutable congruences, then this does not imply $C_k(A) = L_2P_k(A)$. Thus we see that the strongest version of Theorem 1 we have mentioned fails to be true in either direction.

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TECHNISCHE UNIVERSITÄT WIEN

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