Logarithmic derivatives of solutions of disconjugate differential equations*

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Abstract. Let $P_n(t, \lambda) = \lambda^n + a_{n-1}(t)\lambda^{n-1} + \ldots + a_0(t)$ be a polynomial with continuous real-valued coefficients on $0 \leqslant t < \omega$ ($\leqslant \infty$) such that the differential equation (*) $P_n(t, D)y = 0$ is disconjugate on $0 \leqslant t < \omega$, where $D = d/dt$. The main theorems imply that (i) if a is a constant, n is even, and $P_n(t, a) \leqslant 0$ (e.g., $a = 0$ and $a_0(t) \leqslant 0$), then (*) has solutions $x(t)$, $z(t)$ satisfying $x'/x \leqslant a \leqslant z'/z$ for t near ω; (ii) if a is a constant, n is odd, and $P_n(t, a) \geqslant 0$ [or $P_n(t, a) \leqslant 0$], then (*) has a solution $x(t)$ satisfying $x'/x \leqslant a$ [or a solution $z(t)$ satisfying $a < z'/z$] for t near ω; and (iii) if $a < \beta$ are constants, $P_n(t, a)$ and $P_n(t, \beta)$ do not change signs, neither $y = e^{at}$ nor $y = e^{\beta t}$ are solutions of (*) for t near ω, and $P_n(t, a)P_n(t, \beta) \leqslant 0$, then (*) has a solution $y(t)$ satisfying $a < y'/y < \beta$ for t near ω.

1. Introduction. We shall assume that coefficients in the linear differential equation below, for example, in

$$(1.1) \quad P_n(t, D)y \equiv D^n y + \sum_{k=0}^{n} a_k(t) D^k y = 0, \quad \text{where } Dy = dy/dt = y',$$

are real-valued and continuous on a specified interval I. Equation (1.1) is said to be *disconjugate* on I if every solution ($\not\equiv 0$) has at most $n-1$ zeros on I (counting multiplicities).

Section 3 deals with the derivatives of solutions of a disconjugate differential equation on $0 \leqslant t < \omega$. The main results of the paper are given in Section 4. Theorem 4.1 implies that if (1.1) is disconjugate on $0 \leqslant t < \omega$, n is even, and a is a constant satisfying $P_n(t, a) \equiv a^n + \sum a_k(t)a^k \leqslant 0$ (e.g., $a = 0$ and $a_0(t) \leqslant 0$), then (1.1) has a pair of solutions $x(t)$, $z(t)$ satisfying $x'/x \leqslant a < z'/z$ for t near ω. While if n is odd and $P_n(t, a) \geqslant 0$ [or $P_n(t, a) \leqslant 0$], then (1.1) has a solution $x(t)$ satisfying $x'/x \leqslant a$ [or a solution $z(t)$ satisfying $a < z'/z$] for t near ω. Also, Theorem 4.2 implies that if there exist constants $a < \beta$ such that $P_n(t, a)$, $P_n(t, \beta)$ do not change signs, neither $y = e^{at}$ nor $y = e^{\beta t}$ is a solution of (1.1) for t near ω, and $P_n(t, a)P_n t, \beta) \leqslant 0$, then (1.1) has a solution $y(t)$ satisfying $a < y'/y < \beta$

for t near ω. For these particular cases of Theorems 4.1 and 4.2 when $n = 2$, see Olech [7] who uses the method of Ważewski [10].

Our methods depend on the paper Pólya [8] and on results obtained by Hartman [3], [4] and by Levin [6]; cf. Coppel [1]. (Page references to Levin [6] refer to the English translation.)

An Appendix deals with counter examples for Theorem 4.1 when $n = 3$ and the assumption that (1.1) is disconjugate is omitted.

2. Notation and preliminaries. In this section, we introduce some notation and recall some basic facts about solutions of disconjugate equations.

We denote the Wronskian determinant of the functions $u_1, \ldots, u_k$ by $W(u_1, \ldots, u_k) = \det(D^{i-1} u_j)$, $i, j = 1, \ldots, k$. An ordered set of functions $(u_1, \ldots, u_m)$ of class C^{m-1} on an interval is said to *satisfy* (Pólya's [8]) *condition* W there if the Wronskians $u_1 = W(u_1)$, $W(u_1, u_2)$, $\ldots$, $W(u_1, \ldots, u_m)$ do not vanish.

Remark 1. If the coefficients of (1.1) are continuous on $I : 0 \leqslant t < \omega$ ($\leqslant \infty$), then a necessary (Pólya [8]) and a sufficient (Sherman [9]; cf. [3], p. 313) condition for (1.1) to be disconjugate on I is that there exists an ordered set of solutions $u_1(t), \ldots, u_{n-1}(t)$ satisfying condition W on $0 < t < \omega$. In this case, if $y = \zeta_k(t)$ is a solution of (1.1) satisfying

$$(2.1) \qquad \zeta_k = \ldots = D^{n-k-1}\zeta_k = 0, \qquad (-1)^{k-1} D^{n-k}\zeta_k > 0 \qquad \text{at } t = \gamma,$$

$0 \leqslant \gamma < \omega$, then $(\zeta_1, \ldots, \zeta_{n-1})$ satisfies condition W on $\gamma < t < \omega$, in fact, $W(\zeta_1, \ldots, \zeta_k) > 0$ for $\gamma < t < \omega$, $1 \leqslant k \leqslant n$; Pólya [8]. They also satisfy $W(\zeta_1', \ldots, \zeta_{n-1}') > 0$ at $t = \gamma$.

Remark 2. If (1.1) is disconjugate on $0 \leqslant t < \omega$, then it has solutions $u_1(t), \ldots, u_{n-1}(t)$ with the property that if $y(t)$ is any solution linearly independent of them, then

$$(2.2) \qquad u_{n-1} = o(y) \qquad \text{and} \qquad u_k = o(u_{k+1}) \qquad \text{as } t \to \omega, k = 1, \ldots, n-1;$$

furthermore, if (2.2) holds for an ordered set of solutions, then $(u_1, \ldots$ $\ldots, u_{n-1}, y)$ satisfies condition W on $0 < t < \omega$; Theorem 7.2_n, p. 331–334 or Appendix A, p. 352–358, in [3]. In this case, u_1 is called a *first principal solution* of (1.1) (at $t = \omega$), u_2 a *second principal solution*, etc. Here, in contrast to [3], we also call y an *n-th principal solution*. Clearly, u_1 is unique up to constant factor. Also, $u_1 \neq 0$ for $0 < t < \omega$; Theorem 7.1_n, [3], p. 329–330.

Remark 3. If (1.1) is disconjugate on $0 \leqslant t < \omega$ and if $0 < \gamma < \omega$, then there exists an ordered set of principal solutions $\eta_1, \ldots, \eta_n$ satisfying

$$(2.3) \qquad \eta_k = \ldots = D^{k-2}\eta_k = 0, \qquad D^{k-1}\eta_k > 0 \qquad \text{at } t = \gamma$$

(hence $W(\eta_2', \ldots, \eta_n') > 0$ at $t = \gamma$), and the conditions

(2.4) $W(\eta_i, \eta_{i+1}, \ldots, \eta_j) > 0$ on $\gamma < t < \omega, 1 \leqslant i \leqslant j < n$;

Theorem 7.2_n, [3]. In particular, $\eta_j > 0$ on $\gamma < t < \omega$, by the case $i = j$. Note that $\eta_2, \ldots, \eta_n$ span the linear manifold of solutions of (1.1) vanishing at $t = \gamma$ (since $\eta_1 > 0$ on $0 < t < \omega$). The solutions $\eta_1, \ldots, \eta_n$ are called *special principal solutions* (depending on γ, $0 < \gamma < \omega$).

3. First derivatives of solutions. In this section, we state results about solutions of (1.1) and their first derivatives. Generally, we shall make the following assumption.

HYPOTHESIS (H). (1.1) *is disconjugate on* $0 \leqslant t < \omega$. *The function* $w(t) \in C^n[0, \omega)$ *is such that* $P_n(t, D)w$ *does not change signs* (e.g., $w \equiv 1$ *and* $a_0(t)$ *does not change signs*).

THEOREM 3.1. *Assume hypothesis* (H), $0 \leqslant j < n$, *and let* $u_1, \ldots, u_j$ *be j linearly independent solutions of* (1.1). (i) *Then*

(3.1) $W(u_1, \ldots, u_j, w)$ *does not change signs for t near ω*

and if $W(u_1, \ldots, u_j, w) \neq 0$ *for t near ω, then*

(3.2) $W(u_1, \ldots, u_j, y)/W(u_1, \ldots, u_j, w)$ *is monotone for t near ω*

for any solution $y(t)$ of (1.1).

(ii) *If* $W(u_1, \ldots, u_j, w) \neq 0$ *for t near ω, then there exists a T, $0 \leqslant T < \omega$, such that $u_1, \ldots, u_j$, w are solutions of a non-singular, disconjugate $(j+1)$-st order linear differential equation on $T \leqslant t < \omega$. Hence* (Levin, [6], p. 93) $\lim u_k/w$ *exists (possibly $\pm\infty$) as $t \to \infty$ for $1 \leqslant k \leqslant j$.*

(iii) *If* $W(u_1, \ldots, u_j, w) \neq 0$ *for t near ω, then there exists a T', $T \leqslant T' < \omega$, such that $(u_1/w)', \ldots, (u_j/w)'$ are solutions of a j-th order, non-singular, disconjugate linear differential equation on $T' \leqslant t < \omega$. In particular,* $\lim (u_i/w)'/(u_k/w)'$ *exists (possibly $\pm\infty$) as $t \to \omega$ for $1 \leqslant i$, $k \leqslant j$.*

In (3.1), we make the convention $W(u_1, \ldots, u_j, w) = w$ if $j = 0$. The proof of Theorem 3.1 gives

COROLLARY 3.1. *Assume hypothesis* (H) *and* $P_n(t, D)w \not\equiv 0$ *for t near ω. Let* $1 \leqslant j < n$ *and* $u_1, \ldots, u_j$ *be linearly independent solutions of* (1.1). *Then* $W(u_1, \ldots, u_j, w) \neq 0$ *for t near ω (so that* (ii) *and* (iii) *in Theorem 3.1 are valid).*

In some cases, we can determine the values of T, T' in Theorem 3.1. This is the situation in the following assertions.

THEOREM 3.2. *Assume hypothesis* (H) *and*

(3.3) $P_n(t, D)w \neq 0$ *on* $0 \leqslant t < \omega$.

Then the $(n+1)$-dimensional linear manifold of functions spanned by $w(t)$ and the solutions of (1.1) is disconjugate on $0 \leqslant t < \omega$, i.e., a function $(\not\equiv 0)$ in this manifold has at most n zeros (counting multiplicities) on $0 \leqslant t < \omega$.

Remark 1. Condition (3.3) cannot be omitted. For if $n = 1$, $P_1(t, D)y \equiv y' - a(t)y$ and $w > 0$ is a solution of $w' - b(t)w = 0$, so that $P_1(t, D)w = (b-a)w \geqslant 0$ if $b(t) - a(t) \geqslant 0$. But if $b(t_0) - a(t_0) = 0$ and $y(t_0) = w(t_0)$, then $y(t) - w(t)$ has two zeros at $t = t_0$.

Theorem 3.2 has the following consequence.

COROLLARY 3.2. *Assume the conditions of Theorem 3.2 and $w > 0$. Let d, $0 \leqslant d \leqslant n$, be the dimension of the linear manifold of solutions $y(t)$ of (1.1) satisfying $y(t) = O\big(w(t)\big)$ as $t \to \omega$. Let $(u_1, \ldots, u_n)$ be a set of n principal solutions.*

(i) If $d \leqslant 1$, then $\big((u_1/w)', \ldots, (u_n/w)'\big)$ satisfies condition W on $0 < t < \omega$; so that if $y(t) \neq 0$ is a solution of (1.1), then $(y/w)'$ has at most $n-1$ zeros (counting multiplicities) on $0 \leqslant t < \omega$.

(ii) If $d = 2$ and u_1, u_2 are positive on $\gamma < t < \omega$, then $(u_1/w)'$, $(u_2/w)'$ have at most one (necessarily simple) zero on $\gamma < t < \omega$. If either $(u_1/w)' \neq 0$ or $(u_2/w)' \neq 0$ on $\gamma \leqslant T' < t < \omega$, then correspondingly, $\big((u_1/w)', \ldots, (u_n/w)'\big)$ or $\big((u_2/w)', (u_1/w)', (u_3/w)', \ldots, (u_n/w)'\big)$ satisfies condition W on $T' < t < \omega$.

Despite the Remark 1 following Theorem 3.2, we can obtain some analogous results if condition (3.3) is replaced by

$$(3.4) \qquad\qquad w > 0 \quad \text{and} \quad (-1)^{n-1}P(t, D)w \geqslant 0,$$

and $(u_1, \ldots, u_n)$ is replaced by $(u_1, \ldots, u_{n-1})$.

THEOREM 3.3. *Assume hypothesis* (H); $0 \leqslant \gamma < \omega$; *and* (3.4). *Let* $u_1, \ldots, u_{n-1}$ *be solutions of* (1.1) *satisfying*

$$(3.5) \qquad W(u_1, \ldots, u_k) > 0 \quad \text{for } \gamma < t < \omega, \ 1 \leqslant k < n,$$

and the inequality in

$$(3.6) \quad W\big(u_1, \ldots, u_{n-1}, (-1)^{n-1}w\big) \equiv w^n W\big((u_1/w)', \ldots, (u_{n-1}/w)'\big) > 0$$

at $t = \gamma$ (e.g., let $0 < \gamma < \omega$ and $(u_1, \ldots, u_{n-1}) = (\eta_2, \ldots, \eta_n)$ be the set of the last $n-1$ special principal solutions). Then (3.6) holds for $\gamma \leqslant t < \omega$, so that $u_1, \ldots, u_{n-1}, w$ are solutions of a non-singular, disconjugate n-th order linear differential equation on $\gamma \leqslant t < \omega$ (and $(u_1/w)', \ldots, (u_{n-1}/w)'$ are linearly independent solutions of a non-singular $(n-1)$-st order equation, say,

$$(3.7) \qquad Q_{n-1}(t, D)y \equiv D^{n-1}y + \sum_{k=0}^{n-2} b_k(t)D^k y = 0$$

on $\gamma \leqslant t < \omega$).

We can obtain here an analogue of Corollary 3.2.

COROLLARY 3.3. *Assume the conditions of Theorem 3.3 and let $(\eta_1, u_1,$ $\ldots, u_{n-1})$ be a set of n principal solutions of* (1.1). *Let d, $0 \leqslant d \leqslant n$, be the dimension of the linear manifold of solutions $y(t)$ of* (1.1) *satisfying $y(t)$ $= O(w(t))$ as $t \to \omega$.*

(i) *If $d \leqslant 2$, then $((u_1/w)', \ldots, (u_{n-1}/w)')$ satisfies condition* W *on $\gamma < t < \omega$ and so* (3.7) *is disconjugate on $\gamma \leqslant t < \omega$.*

(ii) *If $d = 3$ and u_1, u_2 are positive on $\gamma < t < \omega$, then $(u_2/w)'$, $(u_3/w)'$ have at most one (necessarily simple) zero on $\gamma < t < \omega$. If either $(u_2/w)' \neq 0$ or $(u_3/w)' \neq 0$ on $(\gamma \leqslant) T' \leqslant t < \omega$, then, correspondingly, $((u_2/w)', \ldots$ $\ldots, (u_{n-1}/w)')$ or $((u_3/w)', (u_2/w)', (u_3/w)', \ldots, (u_{n-1}/w)')$ satisfies condition* W *on $T' < t < \omega$, so that* (3.7) *is disconjugate on $T' \leqslant t < \omega$.*

Remark 2. If $0 < \gamma < \omega$ and $(u_1, \ldots, u_{n-1}) = (\eta_2, \ldots, \eta_n)$, then case (i) implies that if $y(t) \not\equiv 0$ is a solution of (1.1) vanishing at $t = \gamma$, then $(y/w)'$ has at most $n-2$ zeros (counting multiplicities) on $\gamma \leqslant t < \omega$.

Theorem 3.1 and Corollary 3.1 will be proved in Section 5; Theorem 3.2 and Corollary 3.2 in Section 6; Theorem 3.3 and Corollary 3.3 in Section 7.

4. Logarithmic derivatives.

The main results of this paper are contained in the next two theorems.

THEOREM 4.1. *Assume hypothesis* (H) *and $w(t) > 0$.*

(i) *If n is even and $P_n(t, D)w \leqslant 0$, then* (1.1) *has a pair of solutions $x(t)$, $z(t)$ satisfying, for t near ω,*

$$(4.1) \qquad x > 0, \quad z > 0, \quad x'/x \leqslant w'/w < z'/z.$$

In particular, if $w(t) = e^{at}$ and $P_n(t, a) \leqslant 0$, where

$$(4.2) \qquad P_n(t, a) = a^n + \sum_{k=0}^{n-1} a_k(t) a^k$$

(e.g., $a = 0$ and $a_0(t) \leqslant 0$), then (4.1) *becomes*

$$(4.3) \qquad x > 0, \quad z > 0, \quad x'/x \leqslant a < z'/z.$$

(ii) *If n is odd and $P_n(t, D)w \geqslant 0$ [or $P_n(t, D)w \leqslant 0$], then* (1.1) *has a solution $x(t)$ [or $z(t)$] satisfying, for t near ω,*

$$(4.4) \qquad x > 0, \quad x'/x \leqslant w'/w \quad [or \quad z > 0, \quad w'/w < z'/z].$$

In particular, if $w(t) = e^{at}$ and $P_n(t, a) \geqslant 0$ [or $P_n(t, a) \leqslant 0$] (e.g., $a = 0$ and $a_0(t) \geqslant 0$ [or $a_0(t) \leqslant 0$]), then (4.4) *becomes*

$$(4.5) \qquad x > 0, \quad x'/x \leqslant a \quad [or \quad z > 0, \quad a < z'/z].$$

(iii) *In* (i) *and* (ii), *the solution $x(t)$ [or $z(t)$] can be chosen to be a first principal solution [or an n-th principal solution] of* (1.1). *Furthermore, strict inequalities hold in* (4.1) *and* (4.4) *if $w(t)$ is not a solution of* (1.1) *for t near ω.*

This result generalizes a part of Theorem 14.1_n, [4], p. 440, but its proof depends on that theorem. The next theorem is also related to Theorem 14.1_n.

THEOREM 4.2. *Assume that* (1.1) *is disconjugate on* $0 \leqslant t < \omega$. *Let* w_1, w_2 *be functions of class* $C^n[0, \omega)$ *satisfying*

$$(4.6) \quad w_1 > 0, \quad w_2 > 0, \quad w_1'/w_1 < w_2'/w_2, \quad w_1/w_2 \to 0 \quad as \quad t \to \omega,$$

$$(4.7) \quad P_n(t, D)w_1 \not\equiv 0, \quad P_n(t, D)w_2 \not\equiv 0 \quad for \quad t \quad near \quad \omega,$$

and, for a choice of $\varepsilon = \pm 1$,

$$(4.8) \quad \varepsilon P_n(t, D)w_1 \geqslant 0 \quad and \quad \varepsilon P_n(t, D)w_2 \leqslant 0.$$

Then (1.1) *has a solution satisfying*

$$(4.9) \quad w_1'/w_1 < y'/y < w_2'/w_2 \quad for \quad t \quad near \quad \omega.$$

For example, if $a < \beta, P_n(t, a) \not\equiv 0$ *and* $P_n(t, \beta) \not\equiv 0$ *for t near* ω, $\varepsilon P_n(t, a) \geqslant 0$ *and* $\varepsilon P_n(t, \beta) \leqslant 0$, *then* (4.9) *becomes*

$$(4.10) \quad a < y'/y < \beta.$$

Theorem 4.1 will be proved in Section 8 and Theorem 4.2 in Section 9.

5. Proof of Theorem 3.1. We first verify the following:

LEMMA 5.1. *Let* (1.1) *be disconjugate on* $0 \leqslant t < \omega$. *Let* $1 \leqslant k < n$ *and* $u_1, \ldots, u_k$ *linearly independent solutions of* (1.1). *Then* $W(u_1, \ldots, u_k) \neq 0$ *for t near* ω.

Proof. The lemma is correct if $n = 2$. Assume its validity for equations of order $n-1$ ($\geqslant 2$). Let $y = \eta_1(t)$ be a first principal solution for (1.1). Then there is a non-singular disconjugate equation of order $n-1$ on $0 < t < \omega$ such that $z(t)$ is a solution if and only if $z(t) = W(\eta_1, y)$ for some solution $y(t)$ of (1.1); Theorem 7.2_n(iv), [3], p. 332. If $\eta_1, u_1, \ldots, u_k$ are linearly independent solutions of (1.1), then $z_j = W(\eta_1, u_j)$, $1 \leqslant j \leqslant k$, are linearly independent and, by the induction hypothesis, $W(z_1, \ldots, z_j) \neq 0$ for t near ω, that is,

$$(5.1) \quad W(z_1, \ldots, z_j) = \eta_1^{j-1} W(\eta_1, u_1, \ldots, u_j) \neq 0 \quad for \quad t \quad near \quad \omega;$$

cf. [3], p. 310. By a standard formula (cf. [3], p. 315),

$$(5.2) \quad [W(u_1, \ldots, u_j)/W(\eta_1, u_1, \ldots, u_{j-1})]'$$
$$= W(u_1, \ldots, u_{j-1}) W(\eta_1, u_1, \ldots, u_j)/W^2(\eta_1, u_1, \ldots, u_{j-1}).$$

An induction on $j = 2, \ldots, k$ shows that the right-hand side is not 0 for t near ω, and so $W(u_1, \ldots, u_j) \neq 0$ for t near ω.

This completes the proof if $\eta_1, u_1, \ldots, u_k$ are linearly independent. But if they are linearly dependent, then $W(u_1, \ldots, u_k)$ is, up to a constant $W(\eta_1, v_1, \ldots, v_{k-1})$, where $\eta_1, v_1, \ldots, v_{k-1}$ are linearly independent, and the desired results follows at once from the induction hypothesis; cf. (5.1).

Proof of Theorem 3.1. *On* (3.1) *in* (i). Choose $u_{j+1}, \ldots, u_n$ such that $u_1, \ldots, u_n$ are linearly independent solutions of (1.1). Then, by Lemma 5.1, there exists a T, $0 \leqslant T < \omega$, such that $(u_1, \ldots, u_{n-1})$ satisfies condition W on $T \leqslant t < \omega$.

From the formula for the derivative of the ratio of two Wronskians,

$$(5.3) \quad [W(u_1, \ldots, u_j, w)/W(u_1, \ldots, u_j, u_{j+1})]'$$

$$= W(u_1, \ldots, u_j) \, W(u_1, \ldots, u_{j+1}, w)/W^2(u_1, \ldots, u_{j+1}).$$

Since $P_n(t, D)w = W(u_1, \ldots, u_n, w)/W(u_1, \ldots, u_n)$, it follows that if $j = n-1$, then the right-hand side of (5.3) does not change signs for $T \leqslant t < \omega$. This proves (3.1) for $j = n-1$ and (3.1) follows for $0 \leqslant j < n$ by an induction on decreasing j.

On (3.2) *in* (i). If $u_1, \ldots, u_j$ and y are linearly dependent, then (3.2) is trivial. If $u_1, \ldots, u_j$ and y are linearly independent, then it can be supposed that $y = u_{j+1}$ in the proof above. In this case, (3.2) follows from (5.3).

On (ii). By Lemma 5.1, there exists a T, $0 \leqslant T < \omega$, such that $(u_1, \ldots, u_j, w)$ satisfies condition W on $T \leqslant t < \omega$.

On (iii). By a similar argument, there exists a T', $0 \leqslant T' < \omega$, such that $W(u_1, \ldots, u_k, w) \neq 0$ for $T' \leqslant t < \omega$, $0 \leqslant k \leqslant j$. Hence $((u_1/w)', \ldots, \ldots, (u_j/w)')$ satisfies condition W on $T' \leqslant t < \omega$; cf. the identity in (3.6).

6. Proof of Theorem 3.2. Let $(u_1, \ldots, u_n)$ be solutions of (1.1) satisfying condition W on $0 < t < \omega$. Then (3.3) implies that $(u_1, \ldots, u_n, w)$ satisfies condition W on $0 < t < \omega$. If the functions $u_1, \ldots, u_n$, are of class $C^{n+1}[0, \omega)$, then they satisfy a non-singular, disconjugate differential equation of order $n+1$ on $0 \leqslant t < \omega$, and the result follows. Actually, Pólya's proof [7] does not require this extra smoothness; cf., e.g., Corollary A2, [3], p. 354. (One can avoid this technical difficulty by a proof by induction on n.)

Proof of Corollary 3.2. After the change of variables $y \to y/w$, we can suppose that $w(t) \equiv 1$. Also, we suppose that $u_j > 0$ for t near ω. We shall use the notion of principal solution (at $t = \omega$) of the linear family of functions spanned by $u_1, \ldots, u_n, 1$. If $d = 0$, then $(1, u_1, \ldots, u_n)$ is an ordered set of principal solutions. If $d \geqslant 1$, then $u_j \to 0$ as $t \to \omega$ for $1 \leqslant j < d$, $u_d \to c$ with $0 \leqslant c < \infty$, and, if $n \geqslant d+1$, $u_{d+1} \to \infty$. Thus, in the case $d = 1$, $(u_1 - c, 1, u_2, \ldots, u_{n-1})$ is an ordered set of principal solutions. For $d \geqslant 2$, we have a set of principal solutions $(\xi_1, \ldots, \xi_d, 1, u_{d+1}, \ldots, u_{n-1})$, where ξ_j is a linear combination of $u_1, \ldots, u_{d-1}$ and $u_d - c$.

On (i). According as $d = 0$ or $d = 1$, the sets of functions $(1, u_1, \ldots, u_d)$ or $(u_1 - c, 1, u_2, \ldots, u_{n-1})$ satisfy condition W on $0 < t < \omega$. Hence, $(u_1', \ldots, u_{n-1}')$ satisfy condition W on $0 < t < \omega$.

On (ii). When $d = 2$, we have the ordered set of principal solutions $(\xi_1, \xi_2, 1, u_3, \ldots, u_{n-1})$. Hence $W(\xi_1, \xi_2, 1) \neq 0$ on $0 < t < \omega$. It follows that $W(u_1, u_2, 1) \neq 0$ and hence $W(u_1', u_2') \neq 0$ on $0 < t < \omega$.

But $u_1 > 0$, $u_2 > 0$, $W(u_1, u_2) > 0$, $W(u_1', u_2') \neq 0$ imply that u_1', u_2' each have at most one (necessarily simple) zero on $\gamma < t < \omega$. For example, if $u_2'(t_0) = 0$, then $u_1'(t_0) < 0$, and $u_2''(t_0) \neq 0$ has the opposite sign to $W(u_1', u_2')$. Thus u_2' cannot have two zeros on $\gamma < t < \omega$.

The remainder of assertion (ii) is clear, for either $u_1' \neq 0$ or $u_2' \neq 0$, $W(u_1', u_2') \neq 0$, while $W(\xi_1, \xi_2, 1, u_3, \ldots, u_j) \neq 0$ implies $\pm W(u_1', \ldots, u_j')$ $= W(u_1, \ldots, u_j, 1) \neq 0$ on $T' < t < \omega$.

7. Proof of Theorem 3.3. After the change of variables $y \to y/w$, we can suppose $w \equiv 1$. Thus (3.4) means that $(-1)^{n-1} a_0(t) \geq 0$. If $Z(t)$ is defined to be

(7.1) $Z(t) = W\big(u_1, \ldots, u_{n-1}, (-1)^{n-1}\big) = W(u_1', \ldots, u_{n-1}')$,

then $Z'(t) + a_{n-1}(t)Z = (-1)^{n-1} a_0(t) \, W(u_1, \ldots, u_{n-1})$ by (1.1). Thus $Z' + a_{n-1}Z \geq 0$, and so $Z(\gamma) > 0$ implies that $Z(t) > 0$ for $\gamma \leq t < \omega$. Hence, $u_1, \ldots, u_{n-1}, 1$ are solutions of a non-singular n-th order equation on $\gamma \leq t < \omega$,

(7.2) $D^n y - (Z'/Z) D^{n-1} y + \ldots = 0$,

which is disconjugate since the solutions $u_1, \ldots, u_{n-1}, 1$ satisfy condition W on $\gamma < t < \omega$. This gives the theorem.

Proof of Corollary 3.3. This is similar to the proof of Corollary 3.2 and will be omitted.

8. Proof of Theorem 4.1. We begin with a collection of simple facts which follow from Theorems 3.1, 3.3 and their corollaries.

PROPOSITION 8.1. *Assume hypothesis* (H) *of Section 3, and* $w(t) \equiv 1$, *so that* $a_0(t)$ *does not change signs. Let d be the dimension of the linear manifold of bounded solutions of* (1.1). *Let* $0 < \gamma < \omega$ *and* $\eta_1, \ldots, \eta_n$ *be special principal solutions of* (1.1), *so that the Wronskian inequalities of* (2.4) *hold.*

(i) *Then, for t near ω,*

$$0 < \eta_2'/\eta_2 < \ldots < \eta_n'/\eta_n \quad \text{if} \quad d = 0,$$

$\eta_1'/\eta_1 < \ldots < \eta_{n-1}'/\eta_{n-1} < 0$ *and either* $\pm \eta_n'/\eta_n > 0$ *if* $d = n$.

(ii) *If $d < n$, then $\eta_j' \to 0$, $\eta_k' \to \infty$ as $t \to \omega$ for $1 \leq j < d < k \leq n$ and, for t near ω,* $0 < \eta_{d+1}'/\eta_{d+1} < \ldots < \eta_n'/\eta_n$. *Also, if $d > 2$,* $\eta_n'/\eta_1 < \ldots < \eta_{d-1}'/\eta_{d-1}$ < 0 *and either* $\eta_{d-1}'/\eta_{d-1} < \eta_d'/\eta_d < 0$ *or* $\eta_{d-1}'/\eta_{d-1} < 0 < \eta_d'/\eta_d$ *for t near ω; while if $d = 2$, either $\eta_1'/\eta_1 < \eta_2'/\eta_2 < 0$ or $\eta_2' > 0$.*

Proof of Theorem 4.1(i). *The solution $x(t)$.* It is sufficient to suppose that $w(t) \equiv 1$ and that $(-1)^{n-1} a_0(t) = -a_0(t) \geq 0$ for $0 \leq t < \omega$. We

first prove the existence of a solution $x(t)$ satisfying (4.1). Let $\eta_1, \dots, \eta_n$ be special principal solutions belonging to some γ, $0 < \gamma < \omega$. By the last proposition, it suffices to consider the cases $d \leqslant 1$ and $d = 2$.

If $d \leqslant 1$, then $(u_1, \dots, u_{n-1}) = (1, \eta_2, \dots, \eta_{n-1})$ are the first $n-1$ principal solutions of (7.2) and, by (2.4) and Corollary 3.3(i), satisfy $W(u_i, \dots, u_j) > 0$ on $\gamma < t < \omega$ for $1 \leqslant i \leqslant j < n$; cf. Proposition 2.2, [3], p. 311. If $\varepsilon > 0$, the existence of the solution $x(t)$ satisfying $x > 0$ and $x'/x \leqslant u_1'/u_1 = 0$ for $\gamma + \varepsilon \leqslant t < \omega$ follows from Theorem 14.1$_n$, [4], p. 440, since $(-1)^{n-1}P_n(t, D)u_1 = (-1)^{n-1}a_0 \geqslant 0$ and $P_n(t, D)u_j = 0$ for $2 \leqslant j < n$.

Consider the case $d = 2$. Since $(\eta_2', \dots, \eta_n')$ satisfies condition W on $\gamma < t < \omega$, by Corollary 3.3, it follows that $\eta_2' \neq 0$ on $\gamma < t < \omega$. But $\eta_2'(\gamma) > 0$ by (2.3), so we have $\eta_2' > 0$ for $\gamma \leqslant t < \omega$. We can assume that $\eta_2(\infty) = 1$, for otherwise η_2 can be multiplied by a suitable positive constant. Then $(1 - \eta_2, \eta_2, \dots, \eta_n)$ are principal solutions of (7.2) and hence satisfy condition W on $\gamma < t < \omega$. By (2.4) and Corollary 3.3, $(u_1, \dots, u_{n-1}) = (1 - \eta_2, \eta_2, \dots, \eta_{n-1})$ satisfy $W(u_i, \dots, u_j) > 0$ on $\gamma < t < \omega$ for $1 \leqslant i \leqslant j < n$, while $(-1)^{n-1}P_n(t, D)u_1 = (-1)^{n-1}a_0 \geqslant 0$ and $P_n(t, D)u_j = 0$ for $2 \leqslant j < n$. Hence, Theorem 14.1$_n$, [4], implies that if $\varepsilon > 0$, then (1.1) has a solution $x(t)$ satisfying $x > 0$, $x'/x \leqslant u_1'/u_1 = -\eta_2'/(1 - \eta_2) < 0$ for $\gamma + \varepsilon \leqslant t < \omega$.

On (i). *The solution* $z(t)$. We now verify the existence of a solution $z(t)$ satisfying (4.1). In view of Proposition 8.1, it suffices to consider the case $d = n$ (so that $\eta_j \to 0$ as $t \to \omega$ for $1 \leqslant j < n$) and $\eta_n' < 0$ for t near ω. Since $\eta_2', \dots, \eta_n'$ are solutions of a disconjugate differential equation (3.7) for t near ω, $\lim \eta_j'/\eta_{j+1}'$ exists (possibly $\pm\infty$) as $t \to \omega$ for $2 \leqslant j < n$. From $\eta_j \to 0$ and $\eta_j/\eta_{j+1} \to 0$ for $1 \leqslant j < n$, it follows that $\eta_j'/\eta_{j+1}' \to 0$ for $2 \leqslant j \leqslant n - 2$.

Case 1. $\eta_{n-1}'/\eta_n' \to 0$ *as* $t \to \omega$. (This holds, for example, if $\eta_n \to 0$ as $t \to \omega$.) In this case, $-\eta_2', \dots, -\eta_n'$ are (positive) principal solutions of (3.7) for t near ω and so, $W(-\eta_2', \dots, -\eta_j') > 0$ for $2 \leqslant j < n$; Theorem 7.2$_n$(iii), [3], p. 332. Hence $W(-\eta_i', \dots, -\eta_j') > 0$ for t near ω, $2 \leqslant i \leqslant j < n$; [6], p. 63–64; cf. Lemma 5.1 and its proof. It follows from (2.4) that if $(u_1, \dots, u_{n-1}) = (\eta_2, \dots, \eta_{n-1}, 1)$, then $W(u_i, \dots, u_j) > 0$ for t near ω, $1 \leqslant i \leqslant j < n$, while $P_n(t, D)u_j = 0$ for $1 \leqslant j < n - 1$ and $(-1)^{n+n-1}P_n(t, D)u_{n-1} = -a_0(t) \geqslant 0$. Thus, Theorem 14.1$_n$, [4], implies the existence of a solution $z(t)$ of (1.1) satisfying $z > 0$, $0 = u_{n-1}'/u_{n-1} < z'/z$ for t near ω.

Case 2. $\lim \eta_n = c > 0$ and $\lim \eta_{n-1}'/\eta_n' = \sigma$, $0 < \sigma < \infty$. Let $0 < \tau < \sigma$ and put $z(t) = \tau\eta_n(t) - \eta_{n-1}(t)$. Then $z \to \tau c > 0$, $t \to \omega$, so that $z > 0$ for t near ω. Also $z' = \tau\eta_n' - \eta_{n-1}' = \sigma\eta_n' - \eta_{n-1}' + (\sigma - \tau)(-\eta_n') \sim (\sigma - \tau)(-\eta_n')$ as $t \to \omega$, so that $z' > 0$ for t near ω.

Case 3. $\lim \eta_n = c > 0$ and $\lim \eta'_{n-1}/\eta'_n = \infty$. Put $z(t) = \eta_n(t) -$
$- \eta_{n-1}(t)$. Then $z \to c$ and $z' = \eta'_n - \eta'_{n-1} \sim - \eta'_{n-1}$ as $t \to \omega$, so that $z > 0$,
$z' > 0$ for t near ω.

Proof of Theorem 4.1(ii). The existence of $x(t)$ in the case $a_0(t) \geqslant 0$
and of $z(t)$ in the case of $a_0(t) \leqslant 0$ is identical with the corresponding
proofs in case (i), for $(-1)^{n-1} P_n(t, D)1 = a_0(t)$ and $(-1)^{n+n-1} P_n(t, D)1$
$= -a_0(t)$ when n is odd.

Proof of Theorem 4.1(iii). First, it is clear that $x(t)$ can be taken
to be $\eta_1(t)$ if $d > 2$, or $d = 1$, or $d = 2$ and $\eta'_2 < 0$ for t near ω. Consider
the remaining case, $d = 2$ and $\eta'_2 > 0$ for t near ω. Since $x(t)$ is bounded,
it is of the form $x = c_1 \eta_1 + c_2 \eta_2$, where $c_1 \neq 0$ and $c_2 \geqslant 0$ since $x > 0$
for t near ω. Thus $0 \geqslant x' = c_1 \eta'_1 + c_2 \eta'_2 \geqslant c_1 \eta'_1$. Thus η'_1 does not change
signs for t near ω. But since $0 < \eta_1 \to 0$ as $t \to \omega$, it follows that $\eta'_1 < 0$ and
so, x can be taken to be η_1.

The proof that $z(t)$ can be taken to be an n-th principal solution is
simple in view of the proofs of parts (i) and (ii).

If $P_n(t, D)1 = a_0(t) \neq 0$ for t near ω, then $W(x, 1) \neq 0$ for t near ω
by Corollary 3.1. This completes the proof of (iii).

9. Proof of Theorem 4.2. Let $0 < \gamma < \omega$ and $\eta_1, \ldots, \eta_{n-1}$ the first
$n-1$ special principal solutions of (1.1). By Theorem 4.1, there is a posi-
tive n-th principal solution η_n satisfying $\eta'_n/\eta_n > w'_2/w_2 > w'_1/w_1$ or η'_n/η_n
$> w'_1/w_1$ for t near ω according as $\varepsilon = 1$ or $\varepsilon = -1$, while $\eta'_1/\eta_1 < w'_1/w_1$
$< w'_2/w_2$ or $\eta'_1/\eta_1 < w'_2/w_2$ for t near ω according as $\varepsilon = (-1)^{n-1}$ or $\varepsilon = (-1)^n$.
Also, by Corollary 3.1, $W(\eta_j, w_i) \neq 0$ for t near ω, $1 \leqslant j \leqslant n$ and $i = 1, 2$.

Thus there exist integers J and K, $0 \leqslant J$, $K \leqslant n$, such that, for t
near ω, $(\eta_j/w_1)' < 0$ for $j \leqslant J$ and $(\eta_j/w_1)' > 0$ for $j > J$, while $(\eta_j/w_2)' < 0$
for $j \leqslant K$ and $(\eta_j/w_1)' > 0$ for $j > K$. Hence $J \leqslant K$ by (4.6).

If $J < K$ and $J \leqslant j \leqslant K$, then $y(t) = \eta_j(t)$ satisfies the conclusion
of the theorem. Suppose therefore that $J = K$, so that $0 < J = K < n$
by the remarks above.

We can suppose that $w_1 \equiv 1$ and write w for w_2. Hence

$$(9.1) \qquad \varepsilon a_0(t) = \varepsilon P_n(t, D)1 \geqslant 0 \qquad \text{and} \qquad \varepsilon P_n(t, D)w \leqslant 0.$$

Thus, for t near ω,

$$(9.2) \qquad \eta'_1/\eta_1 < \ldots < \eta'_J/\eta_J < 0 < w'/w < \eta'_{J+1}/\eta_{J+1} < \ldots < \eta'_n/\eta_n.$$

In what follows, we shall use the following consequence of Lemma 2.6,
[6], p. 63–64.

LEMMA 9.1 (Levin). *Assume hypothesis* (H) *and* $w > 0$. *Suppose that*
(1.1) *has principal solutions* $x_1(t), \ldots, x_n(t)$ *such that* $x_J = o(w)$, $w = o(x_{J+1})$
as $t \to \omega$ *for some integer* J, $0 \leqslant J \leqslant n$ *(where we omit* $x_J = o(w)$ *or* $w = o(x_{J+1})$
if $J = 0$ *or* $J = n$*). Then* $(-1)^{n-J} P_n(t, D)w \geqslant 0$.

Since $w \to \infty$ as $t \to \omega$, it follows from (4.7), (9.1) and (9.2) that not both of the relations $\eta_J = o(1)$ and $w = o(\eta_{J+1})$ can hold as $t \to \omega$.

C a s e 1. $\eta_J = o(1)$, $w \neq o(\eta_{J+1})$. Then $\varepsilon = (-1)^{n-J}$. Also $\lim \eta_{J+1}/w$ exists and is a positive (finite) number which we can suppose is 1. If we consider $(\eta_1, \ldots, \eta_J, w - \eta_{J+1}, \eta_{J+1}, \ldots, \eta_n)$, it follows that $\eta_J \neq o(w - \eta_{J+1})$ as $t \to \omega$, for otherwise $-\varepsilon = (-1)^{n-J}$. Let $c = \lim (w - \eta_{J+1})/\eta_J$, so that $0 \leqslant c < \infty$. Since the functions η_J/w, $(w - \eta_{J+1})/w$ tend to 0 and the ratio of their derivatives have limits (by Theorem 3.1), it follows that $(-\eta_{J+1}/w)'/(\eta_J/w)' \to c$ as $t \to \omega$.

Since $(\eta_J/w)' < 0$, $(-\eta_{J+1}/w)' < 0$ for t near ω, we have that $(1 + c)$ $(\eta_J/w)' < (-\eta_{J+1}/w)' < 0$ for t near ω. Thus $[y(t)/w(t)]' < 0$ for t near ω if $y = \eta_{J+1} + (1 + c)\eta_J$.

In view of $\int^\omega (-\eta_J')\,dt < \infty$ and $\int^\omega \eta_{J+1}'\,dt = \infty$ and the fact that η_J'/η_{J+1}' has a limit (possibly $-\infty$) as $t \to \omega$, it follows that $\eta_J'/\eta_{J+1}' \to 0$ as $t \to \omega$. Thus $y' = \eta_{J+1}' + (1 + c)\eta_J' > 0$ for t near ω. Consequently, $y > 0$ and $0 < y'/y < w'/w$ for t near ω. This proves the case under consideration.

C a s e 2. $\eta_J \neq o(1)$, $w = o(\eta_{J+1})$. Then $\varepsilon = (-1)^{n-J-1}$. Also $\lim \eta_J$ exists and is a positive number which we can suppose is 1. If we consider $(\eta_1, \ldots, \eta_{J-1}, \eta_J - 1, \eta_J, \ldots, \eta_n)$, it follows that $J > 1$ and $\eta_{J-1} \neq o(\eta_J - 1)$, for otherwise $\varepsilon P_n(t, D)[\eta_J - 1] \leqslant 0$ implies that $-\varepsilon = (-1)^{n-J-1}$. Let $c = \lim (\eta_J - 1)/\eta_{J-1}$, so that $0 \leqslant c < \infty$. Arguing as above with the use of Theorem 3.1, we see that $\eta_J'/\eta_{J-1}' \to c$ and that $y = \eta_J - (1 + c)\eta_{J-1}$ satisfies $y > 0, y' > 0$ for t near ω. Since $0 < y/w \to 0$ as $t \to \omega$ and $(y/w)' \neq 0$ for t near ω, we have $(y/w)' < 0$ for t near ω. Thus $0 < y'/y < w'/w$ for t near ω. This proves the case under consideration.

C a s e 3. $\eta_J \neq o(1)$, $w \neq o(\eta_{J+1})$. We can suppose that $\eta_J \to 1$ and $\eta_{J+1}/w \to 1$ as $t \to \omega$. On the one hand, if $\eta_J \neq o(w - \eta_{J+1})$, then the proof can be completed as in Case 1 or if $J > 1$ and $\eta_{J-1} \neq o(\eta_J - 1)$, then the proof can be completed as in Case 2. On the other hand, one of these two alternatives must hold, for otherwise $-\varepsilon = (-1)^{n-J}$ and $-\varepsilon = (-1)^{n-J-1}$. This completes the proof of Theorem 4.2.

Appendix. There are a number of results of the following type dealing with third order equations:

(*) In the differential equation

$$(1) \qquad P_3(t, D) \equiv y''' + p(t)y'' + q(t)y' + r(t)y = 0,$$

let $p, q, r \in C^0[0, \infty)$ and let

$$(2) \qquad r(t) \geqslant 0.$$

In addition, assume certain other conditions, say (C). Then (1) has a solution $y = y(t)$ satisfying

$$(3) \qquad y > 0, \quad y' \leqslant 0 \quad \text{for large } t.$$

Examples of suitable conditions (C) are as follows: (a) (1) is disconjugate on $[0, \infty)$; cf. Theorem 4.1 above. (b) $q(t) \leqslant 0$ or, more generally, $y'' + p(t)y' + q(t)y = 0$ is disconjugate on $[0, \infty)$; cf. [5]. (c) A generalization of (b) occurs in [3], p. 358–361. (d) See also [A2] which deals with a special case of (1), namely, the Appell equation,

$$(4) \qquad\qquad w''' + 4q(t)w' + 2q'(t)w = 0,$$

in which $q \in C^1[0, \infty)$ and for which the general solution is $w = c_1 u^2(t) + c_2 v^2(t) + c_3 u(t)v(t)$, where $u(t)$, $v(t)$ are linearly independent solutions of

$$(5) \qquad\qquad u'' + q(t)u = 0.$$

Here (4) has a solution $w > 0$, $w' \leqslant 0$ for $t \geqslant 0$ if $q \in C^2[0, \infty)$, $q > 0$, $q' \geqslant 0$ and $q'' \leqslant 0$; [A2], p. 182–183.

These results suggest the question of the validity of $(*)$ if the extra assumptions (C) are omitted. As we show by counter-examples, the answer is in the negative.

PROPOSITION. *Let* $p, q, r \in C^0[0, \infty)$ *and* $r \geqslant 0$. *Then it is possible* (i) *that, for large* t, (1) *has positive solutions, but no monotone solution* $(\neq 0)$ *and* (ii) *that every solution* $y(t)$ *of* (1) *has infinitely many zeros.*

The result (ii) gives another counter-example to a question raised by Mammana [A3] as to whether a real operator $P_3(t, D)$ always has a real factorization $P_3 = P_2 P_1$; cf. Ascoli [A1] and Sansone [A5].

Proof (i). Let $q(t) > 0$ be a continuous non-decreasing function for $t \geqslant 0$ satisfying $q \to \infty$ as $t \to \infty$. Then if $u(t)$ is a solution of (5), $u^2 + u'^2/q$ is non-increasing and there is at least one solution $u(t) \not\equiv 0$ such that $\lim(u^2 + u'^2/q) = 0$ as $t \to \infty$; Milloux [A4], cf. [2], p. 510. For a suitable choice of such a q, (5) has a solution $u = v(t)$ such that $\lim(v^2 + v'^2/q) > 0$; Milloux [A4]. In Milloux's example, q is a step-function but it can be supposed that q is smooth, for otherwise it can be replaced by a smooth q_0 with

$$\int\limits_t^\infty |q(s) - q_0(s)|\, ds$$

small at ∞; cf., e.g., [A6] or [2], p. 370. Thus, the corresponding Appell equation (4), where $q' \geqslant 0$, has a positive solution $w = u^2 + v^2$, but no solution $w = c_1 u^2 + c_2 v^2 + c_3 uv \not\equiv 0$ is positive and monotone.

Proof (ii). Let $p_0(t)$, $q_0(t)$ be step-functions defined for $t \geqslant 0$ by

$$p_0(t) = 2n \quad \text{and} \quad q_0(t) = 1 + n^2 \quad \text{on } [2n\pi, 2(n+1)\pi)$$

$$\text{for } n = 0, 1, \ldots$$

Then the homogeneous equation

$$u'' - p_0(t)u' + q_0(t)u = 0$$

has the linearly independent $C^1[0, \infty)$ solutions

(6) $u_0(t) = e^{-n(n+1)\pi} e^{nt} \cos(t + \gamma_n)/\cos\gamma_n$ on $[2n\pi, 2(n+1)\pi)$,

(7) $v_0(t) = e^{-n(n+1)\pi} e^{nt} \sin t$ on $[2n\pi, 2(n+1)\pi)$,

where

$$0 \leqslant \gamma_n < \pi/2, \quad \tan\gamma_n = n, \quad \text{so that} \quad \gamma_n \to \pi/2,$$

as $n \to \infty$. The solutions (6), (7), satisfy $u_0'(t) = 0$ and $v_0(t) = 0$ for $t = 2n\pi$ and $n = 0, 1, \ldots$

The inhomogeneous equation

$$w'' - p_0(t)w' + q_0(t)w = 1$$

has the $C^1[0, \infty)$ solution

$$w_0(t) = (1 + n^2)^{-1} - \left[(1 + n^2)^{-1} + e^{n(n-1)\pi} \sum_{k=1}^{n-1} \lambda_k \right] e^{-2n^2\pi} e^{nt} \cos(t + \gamma_n)/\cos\gamma_n,$$

on $[2n\pi, 2(n+1)\pi)$ for $n = 0, 1, \ldots$, where

$$\gamma_k = (1 + k^2)^{-1} (e^{2k\pi} - 1) e^{-k(k-1)\pi}.$$

Note that, as $n \to \infty$,

$$w_0(2n\pi) = -e^{n(n+1)\pi} \sum_{k=1}^{n-1} \lambda_k \to -\infty,$$

$$w_0(2n\pi + \pi) = (1 + n^2)^{-1}(1 + e^{n\pi}) + e^{n^2\pi} \sum_{k=1}^{n-1} \lambda_k \to \infty.$$

Also, if we put

$$c_0 = \sum_{k=1}^{\infty} \lambda_k, \quad \sigma_n = \sum_{k=n}^{\infty} \lambda_k,$$

then $\sigma_n \geqslant \lambda_n$ gives.

$$w_0(2n\pi) + c_0 u_0(2n\pi) = e^{n(n+1)\pi} \sigma_n \geqslant (1 + n^2)^{-1}(e^{2n\pi} - 1)e^{2n\pi} \to \infty,$$

$$w_0(2n\pi + \pi) + c_0 u_0(2n\pi + \pi) = (1 + n^2)^{-1}(1 + e^{n\pi}) - e^{n(n+1)\pi} \sigma_n \to -\infty,$$

as $n \to \infty$.

Thus the three functions $u_0(t)$, $v_0(t)$, $w_0(t)$ are linearly independent and every linear combination $c_1 u_0(t) + c_2 v_0(t) + c_3 w_0(t)$ has arbitrarily large zeros.

Therefore, there exist $C^1[0, \infty)$ functions $p(t)$ and $q(t)$, which are "near" to $p_0(t)$ and $q_0(t)$, such that $q(t)$ is increasing and

(8) $$u'' - p(t)u' + q(t)z = 0$$

has a pair of solutions $u(t)$, $v(t)$ and

(9) $$w'' - p(t)w' + q(t)w = 1$$

has a solution $w(t)$ with the properties that $u(t)$, $v(t)$, $w(t)$ are linearly independent and every linear combination has arbitrarily large zeros. Since $u(t)$, $v(t)$, $w(t)$ are solutions of the third order equation

$$y''' - p(t)y'' + [q(t) - p'(t)]y' + q'(t)y = 0,$$

obtained by differentiating (9), and since $q'(t) \geqslant 0$, assertion (ii) follows.

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