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On the behaviour of the curvature lines in the neighbourhood of an isolated ombilic point

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In the present paper we show that the curvature lines of a two-dimensional surface form a phase space in the neighbourhood of an isolated ombilic point; more exactly there appear two conjugate phase spaces which admit ovals, spirals or one, two, three or four sectors which are parabolic or hyperbolic. We give this solution only in the most common case where in the equation describing the curvature lines the first degree terms dominate, i.e. the linear terms in the expansion of the first and second metric tensors are not all proportional. This problem has been tacled by G. Darboux [3]. The results which he has obtained, however, are not quite correct because he has left out a certain member from his differential equation. Although this member is multiplied by a small parameter, it is unbounded.

1. The equations of curvature lines. We assume that the surface in question is represented in the neighbourhood of the ombilic point under investigation by the first and second fundamental tensors $g_{ik} = g_{ik}(u, v)$ and $h_{ik} = h_{ik}(u, v)$ $(i, k = 1, 2)$ and that this ombilic point corresponds to $u = 0$, $v = 0$. We assume further that in a certain neighbourhood of $(0, 0)$ we have

$$(1)\qquad \begin{aligned} g_{ik}(u, v) &= \overset{\circ}{g}_{ik} + a_{ik}u + \beta_{ik}v + r_{ik}(u, v) \\ h_{ik}(u, v) &= \varrho\overset{\circ}{g}_{ik} + \gamma_{ik}u + \varkappa_{ik}v + s_{ik}(u, v) \end{aligned} \qquad (i, k = 1, 2),$$

where $r_{ik}(u, v)$ and $s_{ik}(u, v)$ are small and of a lower order than $\sqrt{u^2+v^2}$. We assume also that $\varrho \neq 0$, which means that the ombilic point under investigation is not a flat point. Moreover, we assume that the first degree membres in the expansions of g_{ik} and h_{ik} are not proportional, i.e. $\varrho g_{ik} - h_{ik} \neq o(\sqrt{u^2+v^2})$.

When we substitute (1) in the left-hand member of the equation of the curvature lines

$$\begin{vmatrix} \dot{u}^2 & \dot{u}\dot{v} & \dot{v}^2 \\ g_{11} & g_{12} & g_{22} \\ h_{11} & h_{12} & h_{22} \end{vmatrix} = 0 \qquad \left(\cdot = \frac{d}{dt}\right),$$

then after simple calculations we obtain the equation

$$(2)\qquad \big(a_1u+b_1v+r_1(u,v)\big)\dot{u}^2+\big(a_2u+b_2v+r_2(u,v)\big)\dot{u}\dot{v}+ \\ +\big(a_3u+b_3v+r_3(u,v)\big)\dot{v}^2=0\,,$$

where $r_i(u,v)$ $(i=1,2,3)$ are small in comparision with $\sqrt{u^2+v^2}$, and a_i and b_i are bilinear functions of $\mathring{g}_{ik}$, ϱ, α_{ik}, β_{ik}, γ_{ik}, $\varkappa_{ik}$. By our assumption we have

$$\sum_{i=1}^{3}(a_i^2+b_i^2)>0\,.$$

Equation (2) will be satisfied along the trajectories of the two following dynamic systems:

$$(3)\qquad \begin{cases} \dot{u}=a_2u+b_2v+ \\ \qquad +\sqrt{(a_2^2-a_1a_3)u^2+(2a_2b_2-a_1b_3-a_3b_1)uv+(b_2^2-b_1b_3)v^2}+o(u,v)\,, \\ \dot{v}=a_1u+b_1v+o(u,v); \end{cases}$$

$$(3^*)\qquad \begin{cases} \dot{u}^*=a_2u^*+b_2v^*- \\ \qquad -\sqrt{(a_2^2-a_1a_3)u^{*2}+(2a_2b_2-a_1b_3-a_3b_1)u^*v^*+(b_2^2-b_1b_3)v^{*2}}+o(u,v)\,, \\ \dot{v}^*=a_1u^*+b_1v^*+o(u,v)\,. \end{cases}$$

The quadratic form under the root sign is positive. This fact follows from the existence of two orthogonal families of the curvature lines, or it may be proved by a calculus. Then a proper affine transformation $(u,v)\to(x,y)$ yields the following form of equations (3) and (3*)

$$(4)\qquad \begin{aligned} \dot{x}&=Ax+By\pm P\sqrt{x^2+y^2}+o(x,y)\,, \\ \dot{y}&=Cx+Dy\pm Q\sqrt{x^2+y^2}+o(x,y)\,. \end{aligned}$$

In view of our assumptions we have

$$(5)\qquad A^2+B^2+C^2+D^2>0\,,\qquad P^2+Q^2>0\,.$$

Since the perturbations $o(u,v)$ have not any essential influence on the qualitative shape of the phase plane (excluding the case of spirals and ovals), we shall restrict ourselves to the consideration of the system

$$(6)\qquad \begin{aligned} \dot{x}&=Ax+By+P\sqrt{x^2+y^2}\,, \\ \dot{y}&=Cx+Dy+Q\sqrt{x^2+y^2}\,. \end{aligned}$$

2. The general form of the solution. We introduce the polar coordinates (r, ϑ), which transform system (6) into

$$(7)\qquad \begin{aligned} r^{-1}\dot{r} &= A\cos^2\vartheta + (B+C)\cos\vartheta\sin\vartheta + D\sin^2\vartheta + P\cos\vartheta + Q\sin\vartheta\,,\\ \dot{\vartheta} &= C\cos^2\vartheta + (D-A)\cos\vartheta\sin\vartheta - B\sin^2\vartheta + Q\cos\vartheta - P\sin\vartheta\,. \end{aligned}$$

We denote the right-hand members of equations (7) for $r^{-1}\dot{r}$ and $\dot{\vartheta}$ by $R(\vartheta)$ and $T(\vartheta)$ respectively. The solution in the polar coordinates has the form

$$(8)\qquad r(\vartheta) = C\exp\int\frac{R(\vartheta)}{T(\vartheta)}\,d\vartheta\,,$$

where C is a positive integration constant. We shall transform system (7) into a simpler form with less parameters. We have to consider several cases, which we divide into two groups (A) and (B) according to

$$(\mathrm{A})\qquad B+C = D-A = 0$$

or

$$(\mathrm{B})\qquad (B+C)^2 + (D-A)^2 > 0\,.$$

(A) In this case we have

$$\begin{aligned} r^{-1}\dot{r} &= A + P\cos\vartheta + Q\sin\vartheta\,,\\ \dot{\vartheta} &= C + Q\cos\vartheta - P\sin\vartheta\,. \end{aligned}$$

The substitution

$$\varphi = \vartheta - \operatorname{arctg}(Q/P)$$

leads to a simpler form,

$$\begin{aligned} r^{-1}\dot{r} &= a + s\cos\varphi\,,\\ \dot{\varphi} &= b + s\sin\varphi\,, \end{aligned}$$

where $s = \sqrt{P^2+Q^2}$. Then we have

$$r(\varphi) = C\exp\int\frac{\varrho + \cos\varphi}{\sigma + \sin\varphi}\,d\varphi\,,$$

where we have substituted $\varrho = s^{-1}a$, $\sigma = s^{-1}b$. We have to investigate the following three cases:

(A_1) $|\sigma| > 1$. We calculate the integral in (7) by substituting $t = \operatorname{tg}\frac{\varphi}{2}$. We obtain

$$r(\varphi) = C\,|\sigma + \cos\varphi|^{\sigma-1}\exp\left(\frac{-\varrho}{\sigma-1}\,\varphi\right)\exp\left(\operatorname{arctg}\sqrt{\frac{\sigma-1}{\sigma+1}}\operatorname{tg}\frac{\varphi}{2}\right).$$

.The trajectories appear as spiral lines if $\varrho \neq 0$ and as oval lines if $\varrho = 0$.

(A_2) $|\sigma| = 1$. Then we have

$$r(\varphi) = C\cos^2\frac{\varphi}{2}\exp\left(\varrho\operatorname{tg}\frac{\varphi}{2}\right).$$

The phase space appears as one parabolic sector ([1]). The case $\varrho = 0$ is impossible because then both $\dot{x}$ and $\dot{y}$ would be equal to 0 along a straight line and H the ombilic point in question would not be isolated. In the following we shall shown that neither can a case of four elliptic sectors occur in our geometrical problem.

(A$_3$) $|\sigma| < 1$. In this case we have

$$r(\varphi) = C\left|\sin\frac{\varphi - a}{2}\right|^{\sigma-1}\left|\sin\frac{\varphi + a}{2}\right|^{\sigma-1}\exp\left(\frac{-a}{\sigma-1}\varphi\right)\exp\left(\operatorname{ctg}\frac{a}{2}\operatorname{tg}\frac{\varphi}{2}\right),$$

where $a = \arcsin|\sigma|$. The phase plane consists of two hyperbolic sectors.

(B) $(B+C)^2 + (A-D)^2 > 0$. A rotation through and angle $\gamma = 2^{-1}\operatorname{arctg}\dfrac{B+C}{A-D}$ and a homothety with a coefficient $\sqrt{(B+C)^2+(A-D)^2}$ will transform our system into

$$(9)\qquad \begin{aligned} 'r^{-1}\,'\dot{r} &= a + \sin 2\,'\vartheta + p\cos'\vartheta + q\sin'\vartheta \equiv 'R('\vartheta)\,, \\ '\dot{\vartheta} &= b + \cos 2\,'\vartheta + q\cos'\vartheta - p\sin'\vartheta \equiv 'T('\vartheta)\,. \end{aligned}$$

It will not lead to confusion if we denote the right-hand member again by r and ϑ respectively and leave out the sign$'$. We shall compute the integral in (8) using classical substitution and then decomposition into simple fractions. The following cases are possible:

(B$_1$) The denominator $T(\vartheta)$ has no real zeros. Then the solution has the shape

$$r(\vartheta) = C\exp(\lambda\vartheta)\prod_{i=1}^{2}\left(a_i\cos^2\frac{\vartheta}{2} + \beta_i\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} + \gamma_i\sin^2\frac{\vartheta}{2}\right)^{\sigma_i}\times \\ \times\exp\left(\eta_i \operatorname{arc\,tg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right| + \nu_i\right)\right)$$

where $a_i, \beta_i, \gamma_i, \sigma_i, \eta_i, \mu_i, \nu_i, \sigma_i$ are constants and the quadratic forms

$$a_i\cos^2\frac{\vartheta}{2} + \beta_i\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} + \gamma_i\sin^2\frac{\vartheta}{2}$$

are both positive. The trajectories are spirals or ovals.

(B$_2$) The denominator has two distinct real zeros, a_1 and a_2. Then we have

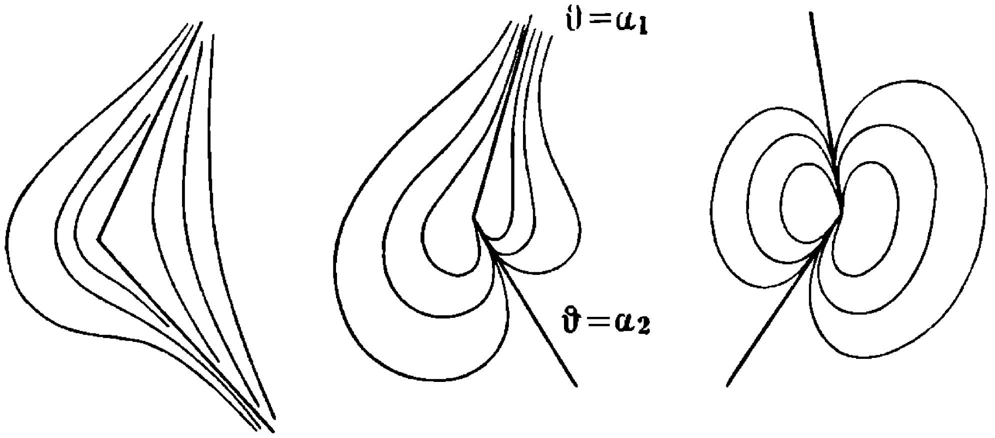
$$r(\vartheta) = C\prod_{i=1}^{2}\left|\sin\frac{\vartheta - a_i}{2}\right|^{A_i}\exp\left(\eta_i\operatorname{arctg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right| + \nu_i\right)\right)\times \\ \times\left(a\cos^2\frac{\vartheta}{2} + \beta\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} + \gamma\sin^2\frac{\vartheta}{2}\right)^{\sigma}\exp\left(\eta\operatorname{arctg}\left(\mu\left|\operatorname{tg}\frac{\vartheta}{2}\right| + \nu\right)\right),$$

where

$$(10)\qquad A_i = R(a_i)/T'(a_i)\,.$$

([1]) For the definition of a sector cf. [4].

No A_i may be equal to zero because in that case the exit system (7) would have a singular line. According to the signs of the parameters A_i we might obtain the following reliefs



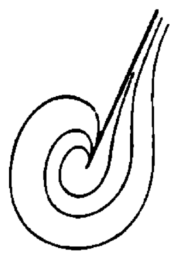
$A_1 < 0,\ A_2 < 0$ $A_1 < 0,\ A_2 > 0$ $A_1 > 0,\ A_2 > 0$

In the last part of the paper we shall exclude the case of two elliptic sectors.

(B_3) The denominator $T(\vartheta)$ has one double root a. In this case the solution is

$$r(\vartheta) = \varphi\left|\sin\frac{\vartheta - a}{2}\right|^{A}\exp\left(\frac{B}{\operatorname{tg}\frac{\vartheta}{2} - \operatorname{tg}\frac{a}{2}}\right)\times$$
$$\times\left(\alpha\cos^2\frac{\vartheta}{2} + \beta\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2} + \gamma\sin^2\frac{\vartheta}{2}\right)^{\sigma}\exp\left(\lambda\vartheta + \operatorname{arc\,tg}\left(\mu\left|\operatorname{tg}\frac{\vartheta}{2}\right| + \nu\right)\right).$$

The corresponding phase plane is as follows:



that is one parabolic sector.

(B_4) $T(\vartheta)$ has four distinct zeros $a_1, \ldots, a_4$. Then we have

$$r(\vartheta) = Ce^{\lambda\vartheta}\prod_{i=1}^{4}\left|\sin\frac{\vartheta - a_i}{2}\right|^{A_i}\exp\left(\eta_i \operatorname{arc\,tg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right| + \nu_i\right)\right).$$

The phase plane is determinated by the signs of the numbers A_i. We shall investigate this case in detail in the next part of the paper. There are six combinations of the parabolic, hyperbolic and elliptic sectors possible a priori but we shall exclude the elliptic.

(B_5) The case of two distinct zeros a_1, a_2 and one double a_3. Then we have

$$r(\vartheta) = Ce^{\lambda\vartheta}\left|\sin\frac{\vartheta-a_3}{2}\right|^{G}\exp\left(\frac{B}{\operatorname{tg}\frac{\vartheta}{2}-\operatorname{tg}\frac{a_3}{2}}\right)\prod_{i=1}^{2}\left|\sin\frac{\vartheta-a_i}{2}\right|^{A_i}\times$$
$$\times\exp\left(\eta_i\operatorname{arctg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right|+\nu_i\right)\right).$$

(B_6) The case of two double roots $a_1 = a_3$, $a_2 = a_4$. Then we have

$$r(\vartheta) = Ce^{\lambda\vartheta}\prod_{i=1}^{2}\left|\sin\frac{\vartheta-a_i}{2}\right|^{G_i}\exp\left(\frac{B_i}{\operatorname{tg}\frac{\vartheta}{2}-\operatorname{tg}\frac{a_i}{2}}\right)\times$$
$$\times\exp\left(\eta_i\operatorname{arctg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right|+\nu_i\right)\right).$$

Neither G_i nor A_i is equal to zero. The cases (B_5) and (B_6) will be considered later. It will be shown that they admit two hyperbolic and one parabolic or two hyperbolic or three parabolic sectors.

It may easily be proved that the equation $T(\vartheta) = 0$ cannot have a fourfold or a threefold root. In order to show this, we substitute $t = \operatorname{tg}\frac{\vartheta}{2}$ and our equation appears as equivalent to the algebraic equation

$$(b+q)t^4-2pt^3+2bt^2-2pt+b-q = 0,$$

which cannot be written in the form $(t-a)^4 = 0$ or $(t-a)^3(t-\beta) = 0$.

3. The index. For a more exact investigation we shall compute the Poincaré index of the singular point $O(0, 0)$. We base our calculations on the formula ([2], Chap. XVI, § 5)

$$\operatorname{ind} O = \frac{1}{2\pi}\oint d\operatorname{arctg}\frac{\dot{y}}{\dot{x}}. \tag{11}$$

(B^{bis}) First we take into consideration a system of the type described as (B) in part 2. We assume that $a^2+b^2 > 0$. Then our system has the following form in the Cartesian coordinates

$$\begin{aligned}\dot{x} &= ax+(1-b)y+pr,\\ \dot{y} &= (1+b)x+ay+qr,\end{aligned} \tag{12}$$

where $r = \sqrt{x^2+y^2}$.

We substitute $a = \varrho^2\cos\alpha$, $b = \varrho^2\sin\alpha$, $p = \sigma\varrho\cos\beta$, $q = \sigma\varrho\sin\beta$. Then system (12) assumes the form

$$(13)\qquad \begin{aligned} \dot{x} &= r\big(\varrho^2\cos(\vartheta+\alpha)+\sin\vartheta+\sigma\varrho\cos\beta\big)\,, \\ \dot{y} &= r\big(\varrho^2\sin(\vartheta+\alpha)+\cos\vartheta+\sigma\varrho\sin\beta\big)\,. \end{aligned}$$

We introduce the complex function $\dot{x}+i\dot{y}$. We have

$$\arg(\dot{x}+i\dot{y}) = \operatorname{arctg}\frac{\dot{y}}{\dot{x}}+K\pi \qquad (K=0 \text{ or } K=1)\,.$$

We put

$$\omega = r^{-1}e^{i\vartheta}(\dot{x}+i\dot{y})\,, \qquad z = \varrho e^{i\vartheta}\,.$$

A simple computation yields us

$$\omega = \omega(z) = e^{i\alpha}z^2+\sigma e^{i\beta}z+i\,.$$

We denote the increase of any complex function w on a circle with centre O and radius ϱ by $\Delta_\varrho w$. In view of (11) and (13) we have

$$\begin{aligned} \frac{1}{2\pi}\oint d\operatorname{arctg}\frac{\dot{y}}{\dot{x}} &= \Delta(\dot{x}+i\dot{y}) = \Delta_\varrho(e^{-i\vartheta}\omega) = \Delta_\varrho e^{-i\vartheta}+\Delta_\varrho\omega \\ &= -1+\Delta_\varrho(e^{i\alpha}z+\sigma e^{i\beta}z+i)\,. \end{aligned}$$

$\Delta_\varrho\omega$ is equal to the number of those zeros of the function ω which are enclosed in the circle $|z|<\varrho$. We denote those zeros by ζ_1 and ζ_2. It follows from the identity

$$\dot{x}^2+\dot{y}^2 = r^2|\omega|^2$$

that on the circle $\varrho = \zeta_i$ both $\dot{x}$ and $\dot{y}$ are equal to 0; this case takes no part in our considerations. Then we see that the index of O is equal to one of the numbers $+1$, 0 and -1, depending on the parameters a, b, p, q.

Now we return to the other cases. If $a = b = 0$, then we have

$$\begin{aligned} \dot{x} &= y+pr = r(\sin\vartheta+p)\,, \\ \dot{y} &= x+qr = r(\cos\vartheta+q) \end{aligned}$$

and a similar method to the above leads us to the conclusion that $\operatorname{ind}O$ is equal to -1 or to 0 in conformity with $\sqrt{p^2+q^2}>1$ or $\sqrt{p^2+q^2}<1$. The case $p^2+q^2=1$ corresponds to a system which has a singular line (because it is easy to compute that in this case $\dot{x}$ and $\dot{y}$ are equal to zero along a line $\vartheta = \arcsin p = \arccos q$).

(A^{bis}) If our system has the form described in part 2 then we substitute, as in part (B^{bis}), $a = \varrho^2\cos\alpha$, $b = \varrho^2\sin\alpha$, $p = \sigma\varrho\cos\beta$, $q = \sigma\varrho\sin\beta$. Hence we find

$$\dot{x} = ax - by + pr = r\big(\varrho^2\cos(\vartheta+\alpha) + \sigma\varrho\cos\beta\big)\,,$$
$$\dot{y} = bx + ay + qr = r\big(\varrho^2\sin(\vartheta+\alpha) + \sigma\varrho\sin\beta\big)$$

and

$$\dot{x} + i\dot{y} = r(e^{i\alpha}z + \sigma e^{i\beta})\,.$$

Finding Δ_ϱ of this function we see that ind O is equal to 0 or to 1 according to whether $\varrho < \sigma$ or $\varrho > \sigma$.

Summing up the above results we obtain

THEOREM 1. *The index of the isolated singular point O for the dynamical system*

$$\dot{x} = Ax + By + P\sqrt{x^2+y^2}\,,$$
$$\dot{y} = Cx + Dy + Q\sqrt{x^2+y^2}$$

is equal to one of the three numbers $-1, 0, 1$.

4. Investigation of the phase planes. Now we shall show, using theorem 1, that elliptic sectors do not appear in the phase spaces of the curvature lines. The starting point is Bendixon's theorem [1], which states that

(14) $\quad \operatorname{ind} O = 1 +$
$\qquad + (\text{number of elliptic sectors} - \text{number of hyperbolic sectors})\,.$

We easily deduce from this formula and theorem 1, that, if the plane has two sectors, then no elliptic sector can appear, and if we have four sectors then only one elliptic sector can appear associated with one hyperbolic and two parabolic sectors. We shall show that even this case does not hold.

We consider the formula

$$(15)\qquad r(\vartheta) = Ce^{\lambda\vartheta}\prod_{i=1}^{4}\left|\sin\frac{\vartheta-a_i}{2}\right|^{A_i}\exp\left(\eta_i\operatorname{arctg}\left(\mu_i\left|\operatorname{tg}\frac{\vartheta}{2}\right|+\nu_i\right)\right).$$

We arrange a_i in such a way that $0 \leqslant a_1 < a_2 < a_3 < a_4 < 2\pi$. In view of (9) and (10) we have

$$(16)\qquad A_i = \big(T'(a_i)\big)^{-1}\big(a + \sin 2a_i + p\cos a_i + q\sin a_i\big)\,.$$

The signs of $T'(a_i)$ are alternately $+$ and $-$ because the roots of $T(\vartheta) = 0$ are now all single. Then the formula (16) implies that if

$$|a| > 1 + \sqrt{p^2+q^2}$$

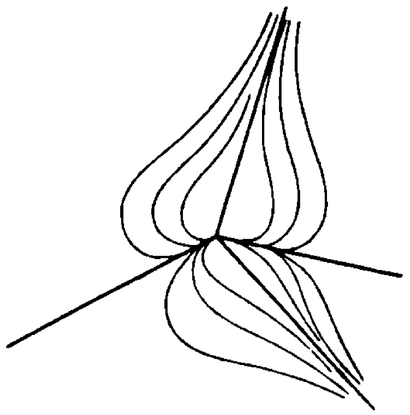
then

$$\operatorname{sign} A_i = \operatorname{sign} a \operatorname{sign} T'(a_i)$$

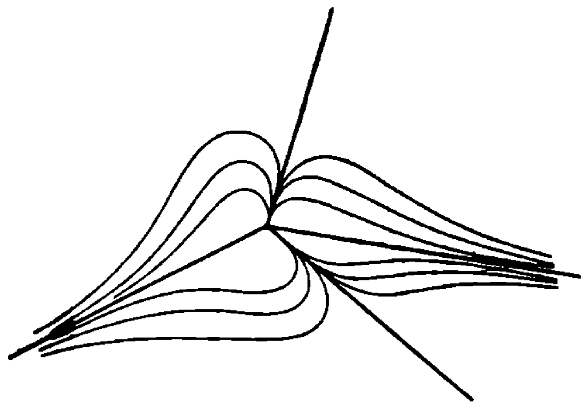
and the signs of $T'(a_i)$ are alternately $+$ and $-$. Hence follows

LEMMA 1. *If* $|a| > 1 + \sqrt{p^2 + q^2}$ *then we have four parabolic sectors.*

Now we fix b, p and q (still such that the equation $T(\vartheta) = 0$ has four real distinct roots) and we let a vary over the interval $(-\infty, \infty)$. The variable $\varrho = \sqrt{a^4 + b^4}$ then runs from $+\infty$ to b^2 and returns to $+\infty$. By theorem 1 the index jumps over the values $+1, 0, -1, 0, +1$. Every jump correponds to a change of the sign of a certain A_j. It follows from (14) that the value 0 of the index corresponds to four hyperbolic sectors. If we choose a value $a = a_1 > 1 + \sqrt{p^2 + q^2}$ and $a_2 = -a_1$, then the corresponding parabolic sectors of the phase planes are opposite; they are for instance the following:



$a = a_1$

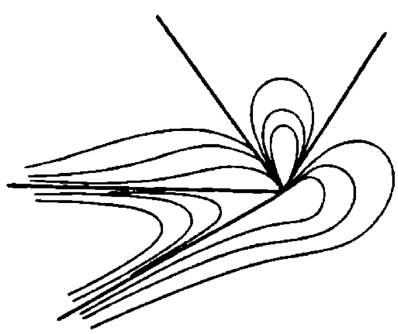


$a = a_2$

since we have the following relations between the corresponding values of A_i, denoted by A_i^1 and A_i^2 respectively:

$$\operatorname{sign} A_i^1 = \operatorname{sign} A_i^2 .$$

The index is equal to zero if and only if we have two parabolic with two hyperbolic sectors. The index can be equal to 1 also in the case of one elliptic with one hyperbolic and two parabolic sectors, as for instance



Then an elliptic sector may appear only if a is large, so that in the circle $|z| < \sqrt{a^4+b^4}$ (in the z-plane) both roots of the equation

$$e^{i\alpha}z^2+\sigma e^{i\beta}z+i = 0$$

are contained. But we have seen that, if a is large enough, then we have four parabolic sectors, and every change of the phase plane must be caused by a change of the sign of some A_j. The four successive changes of the signs of A_j correspond to the jumps of $|z|$ over the regions

$$|z| < |\zeta_1|\,, \quad |\zeta_1| < |z| < |\zeta_2|\,, \quad |\zeta_2| < |z|$$

and in consequence to the jumps of the index over the values $-1, 0, +1$. The only possible corresponding alternation of the phase planes is the following: 4 parabolic, 2 hyperbolic with 2 parabolic, 4 hyperbolic, 2 parabolic with 2 hyperbolic and 4 parabolic sectors and there is no room for any elliptic sector. Summing up our previous considerations we can formulate

LEMMA 2. *If the phase plane of a dynamical system* (5) *consists of one, two or three sectors, then no elliptic sectors appear.*

Now it remains to solve the case of three sectors, i.e. the case of one double and two distinct roots of the equation $T(\vartheta) = 0$. We shall exclude here the possibility of an elliptic sector by "splitting" the double root and making use of lemma 2.

LEMMA 3. *If we are given three distinct numbers $\bar{a}_1, \bar{a}_2, \bar{a}_3$ enclosed in the interval $[0, 2\pi)$ then there exist numbers $\bar{b}, \bar{p}, \bar{q}$ and a_4 such that $\bar{a}_i$ $(i = 1, 2, 3, 4)$ are roots of the equation*

$$\bar{b}+\sin 2\vartheta+\bar{q}\cos\vartheta-\bar{p}\sin\vartheta = 0$$

$\bar{b}, \bar{p}, \bar{q}$ and $\bar{a}_4$ are continuous functions of $\bar{a}_1, \bar{a}_2, \bar{a}_3$.

Proof. We determine $\bar{b}, \bar{p}$ and $\bar{q}$ from the system of equations

$$\bar{b}+\bar{q}\cos\bar{a}_i-\bar{p}\sin\bar{a}_i = -\cos 2\bar{a}_i\,. \tag{17}$$

We put

$$\Delta = \begin{bmatrix} 1 & \cos\bar{a}_1 & -\sin\bar{a}_1 \\ 1 & \cos\bar{a}_2 & -\sin\bar{a}_2 \\ 1 & \cos\bar{a}_3 & -\sin\bar{a}_3 \end{bmatrix}$$

or, using a brief notation,

$$\Delta = [1 \ \ \cos\bar{a}_i \ \ -\sin\bar{a}_i]\,;$$

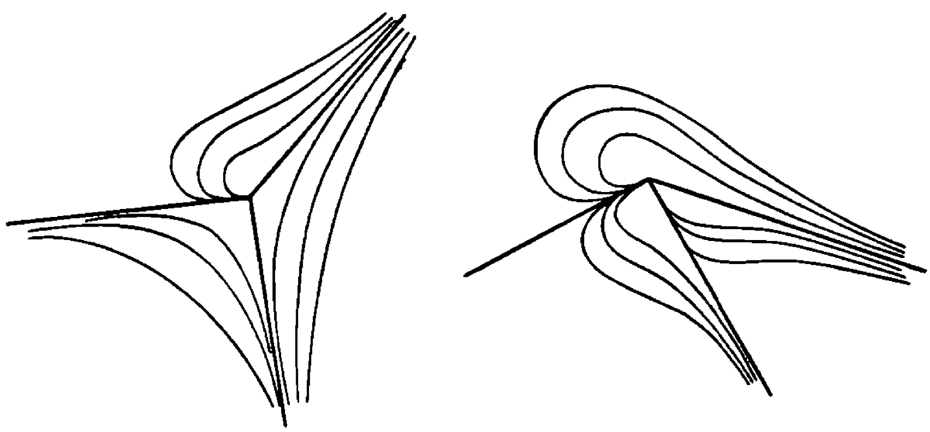
$2^{-1}|\Delta|$ is equal to the area of a triangle whose vertices have the Cartesian coordinates $(\cos\bar{a}_i, \sin\bar{a}_i)$ $(i = 1, 2, 3)$. Then we see that $\det\Delta \neq 0$ and after simple computation we obtain the formulas

$$\bar{b} = \det[\cos 2\bar{a}_i \quad \cos\bar{a}_i \quad \sin\bar{a}_i](\det\Delta)^{-1},$$
$$\bar{p} = 2\det[1 \quad \cos\bar{a}_i \quad \cos^2\bar{a}_i](\det\Delta)^{-1},$$
$$\bar{q} = 2\det[1 \quad \sin\bar{a}_i \quad \sin^2\bar{a}_i](\det\Delta)^{-1}.$$

In order to determine a we write equation (17), where we substitute the values just found for $\bar{b}, \bar{p}, \bar{q}$. We transform equation (17) into an algebraic equation of the fourth degree using the common substitution $t = \mathrm{tg}\frac{\vartheta}{2}$. The three roots of this equation are $\tau_i = \mathrm{tg}\frac{\bar{a}_i}{2}$ $(i = 1, 2, 3)$. The fourth, $\tau_4 = \mathrm{tg}\frac{\bar{a}_4}{2}$, will easily be computed from the Vieta identities. Evidently $\bar{a}_4$ depends continuously on $\bar{a}_1, \bar{a}_2, \bar{a}_3$. Thus we have proved our lemma.

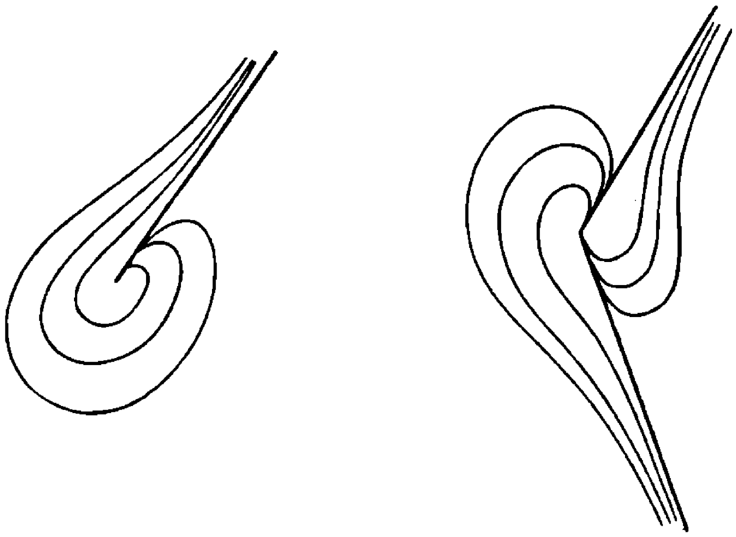
Now we consider a phase plane of system (10) which consists of three sectors, i.e. the case of one double root a_1 and two distinct a_3, a_4. Suppose that we have an elliptic sector. We split the boundary line $\vartheta = a_1$ putting $\bar{a}_1 = a_1$ and choosing $\bar{a}_2$ near a_1 but not inside the hypothetic elliptic sector. We put $\bar{a}_3 = a_3$ and we find $\bar{a}_4$ by means of lemma 3. We have constructed in this way a four-sector field which does contain any elliptic sector. Now we let a_2 convergence to a_1. Then the fields in those sectors which are not contained between $\bar{a}_1$ and a_2 vary continuously converging towards those which correspond to the exit situation. A hyperbolic or a parabolic sector cannot be transformed into an elliptic one by such a continuous transformation. Then our hypothesis on the existence of an elliptic sector is false in this case also. Thus follows

LEMMA 4. *If the phase plane of* (10) *consists of three sectors, then they are parabolic or hyperbolic, for instance*



In the same way we obtain

LEMMA 5. *If the phase plane consists of one sector* (*a case of one double root*), *then this sector is parabolic*; *and if the equation* $T(\vartheta) = 0$ *has two double roots, then the phase plane consists of two parabolic sectors*



As we have said, a threefold or a fourfold root of the equation $T(\vartheta) = 0$ cannot appear.

Summing up all these results, we can formulate

THEOREM 2. *The possible phase spaces of the curvature lines determined by* (3) *and* (3*) *are the following*:

1° *spiral lines or oval lines*; *the conjugate family forms parabolic sectors*,

2° *one, two, three or four sectors*: *some of them, even in number, are hyperbolic, the remainder are parabolic.*

The boundary lines between the sectors are the curvature lines.

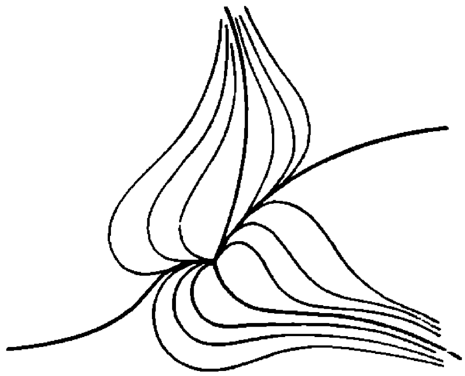
COROLLARY. *A curvature line which goes to an ombilic point does not return to it* (*i.e. it does not form a loop which determines an elliptic sector*) *but goes to another ombilic or to a flat point.*

If we have a phase which consist of two or four hyperbolic sectors, then the conjugate family consists also of two or four hyperbolic sectors respectively. Then we conclude by means of theorem 2 and the corollary that the following theorem holds:

THEOREM 3. *At least four curvature lines with their tangents go to an ombilic point.*

We remember that these results have been obtained on the assumption that in the equations determining the curvature lines the first degree members dominate. In more general cases more sectors can appear but we may only suspect that also here elliptic sectors cannot appear.

Remark. More accurate considerations of the phase planes, which correspond to the cases of double roots where perturbations play a part, also depend on investigation of the system in which the higher terms dominate. We know that there may appear degenerated sectors, i.e. sectors which have the vertex angle equal to 0, as for instance



In the case of single roots the perturbations can only curve the boundary lines of the sectors or change ovals into spirals.

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