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On internal approximations of parabolic problems

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Dedicated to the memory of Jacek Szarski

Abstract. We consider in this paper the approximate solutions of a class of initial-boundary value problems for the equation

$$Au + u_t = f, \tag{1}$$

where A is a linear elliptic differential operator of order $2m$ in space variables, with time-depending coefficients. We start with a weak formulation of the exact problem, which yields a linear system of equations for finding the approximate solution. Then it is proved that the estimation of the error may be reduced to an approximation problem in suitably defined Sobolev-type space. In particular, we may look for approximate solutions which are piecewise polynomials in both space and time variables. In this manner we are led to a finite element method based on a triangulation of the space-time domain in which equation (1) is considered.

1. Basic definitions and assumptions. We denote

$$u_t = \frac{\partial u}{\partial t}, \quad D_j = \frac{\partial}{\partial x_j}, \quad D_x^a = D_1^{a_1} \ldots D_n^{a_n}$$

with $a = (a_1, \ldots, a_n)$ and $|a| = a_1 + \ldots + a_n$ for arbitrary non-negative integers a_j. For $x = (x_1, \ldots, x_n) \in R^n$, $t \in R$, we write $x' = (x_1, \ldots, x_{n-1})$, and (x, t) for a point of R^{n+1}. All derivations in the sequel will be understood in the distributional sense. For simplicity we suppose all functions considered to be real-valued.

Equation (1) will be considered in a bounded domain D_T of R^{n+1} satisfying the conditions:

(C_1) $\partial D_T = \bar{\Omega}_0 \cup \bar{\Omega}_T \cup \Gamma$, where Ω_0, Ω_T are two domains in the planes $t = 0$, $t = T$, respectively, and Γ is the part of ∂D_T lying in the strip $0 < t < T$.

(C_2) There exists a finite covering $\bar{\Gamma} \subset \bigcup_{j=1}^{p} U_j$ and positive numbers a, β, γ_j, δ_j such that the set $\Gamma \cap U_j$ may be described in a suitably given

x-coordinate system in R^n by the equation

$$x_n = \varphi_j(x', t) \qquad ((x', t) \in \Delta_j),$$

where

(i) Δ_j is the cube: $|x_s| < a$ $(s = 1, \ldots, n-1)$, $\gamma_j < t < \delta_j$;

(ii) φ_j is Lipschitz continuous in $\bar{\Delta}_j$;

(iii) the set $\{(x, t)\colon (x', t) \in \Delta_j,\ \varphi_j(x', t) < x_n < \varphi_j(x', t) + \beta\}$ is contained in D_T;

(iv) the set $\{(x, t)\colon (x', t) \in \Delta_j,\ \varphi_j(x', t) - \beta < x_n < \varphi_j(x', t)\}$ is outside of $\bar{D}_T$.

We denote by $(\ ,\ )_{\Omega_j}$ $(j = 0, T)$ the scalar product in $L^2(\Omega_j)$. For the scalar product in $L^2(D_T)$ we write simply $(\ ,\ )$. In the sequel we shall use the following Hilbert spaces (given a positive integer k):

$$H_{k,0} = \{v \in L^2(D_T)\colon D_x^\alpha v \in L^2(D_T),\ |\alpha| \leqslant k\},$$
$$H_{k,1} = \{v \in H_{k,0}\colon v_t \in L^2(D_T)\}$$

with the corresponding norms $\|\ \|_{k,0}$ and $\|\ \|_{k,1}$ defined by the scalar products

$$(u, v)_{k,0} = \sum_{|\alpha| \leqslant k} (D_x^\alpha u, D_x^\alpha v)$$

and

$$(u, v)_{k,1} = (u, v)_{k,0} + (u_t, v_t),$$

respectively. Assumptions (C_1), (C_2) yield the following well-known lemmas (see [6]):

LEMMA 1. *For $v \in H_{k,0}$ the trace $v|_\Gamma$ is well defined and*

$$\|v|_\Gamma\|_{L^2(\Gamma)} \leqslant c\|v\|_{k,0}.$$

LEMMA 2. *For $v \in H_{k,1}$ the traces $v(\cdot, 0)$, $v(\cdot, T)$, are well defined and*

$$\max\big(\|v(\cdot, 0)\|_{L^2(\Omega_0)}, \|v(\cdot, T)\|_{L^2(\Omega_T)}\big) \leqslant c\|v\|_{k,1}.$$

In both lemmas c denotes a positive constant not depending on v. Now we can introduce for $u, v \in H_{k,1}$ the scalar product

$$[u, v]_k = (u, v)_{k,0} + \big(u(\cdot, 0), v(\cdot, 0)\big)_{\Omega_0} + \big(u(\cdot, T), v(\cdot, T)\big)_{\Omega_T}$$

and the corresponding norm $|||\ |||_k$. We define also

$$H_{k,1}^0 = \{v \in H_{k,1}\colon v(\cdot, T) = 0\}.$$

As regards the operator A, we suppose that it is of the form

$$Au = \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\alpha|} D_x^\alpha\big(a_{\alpha\beta}(x, t) D_x^\beta u\big) \tag{2}$$

with measurable and bounded in D_T coefficients $a_{\alpha\beta}$. In the sequel we shall consider its bilinear Dirichlet form

$$(3) \qquad a(u, v) = \sum_{|\alpha|,|\beta|\leqslant m} (a_{\alpha\beta} D_x^\beta u, D_x^\alpha v).$$

Let $C_{0,x}^\infty(D_T)$ be the set of all infinitely differentiable in $\bar{D}_T$ functions which vanish in some neighbourhood of Γ. To describe formally the boundary condition on Γ we introduce a linear subspace V of $H_{m,0}$ containing the set $C_{0,x}^\infty(D_T)$ and we make the following assumption:

(C_3) The form $a(\cdot, \cdot)$ is coercive over the space V; this means that there are two constants $c > 0$ and $\lambda \geqslant 0$ such that the estimate

$$a(v, v) + \lambda(v, v) \geqslant c\|v\|_{m,0}^2$$

holds for $v \in V$.

Condition (C_2) is obviously satisfied if Γ is a polyhedral surface, so that D_T is a polyhedron. We give now some examples to illustrate condition (C_3).

EXAMPLE 1. Let V be the closure of $C_{0,x}^\infty(D_T)$ in $H_{m,0}$ and suppose that A is elliptic, uniformly in D_T with continuous coefficients $a_{\alpha\beta}$ ($|\alpha| = |\beta| = m$). Then (C_3) follows from the well-known theorem due to L. Gårding [2], [5]. According to Lemma 1 the space V consists of functions $v \in H_{m,0}$ for which the derivatives $D_x^\alpha v$ ($|\alpha| \leqslant m-1$) vanish on Γ almost everywhere with respect to the surface measure. So v satisfies on Γ homogeneous boundary conditions of Dirichlet type in a slightly generalized form.

EXAMPLE 2. Let $m = 1$ and $V = H_{1,0}$. Then (C_3) holds if we suppose that A is elliptic, uniformly in D_T.

2. Formulation of the exact problem. Let us introduce for $u, v \in H_{m,1}$ the bilinear form

$$B(u, v) = a(u, v) - (u, v_t) + \big(u(\cdot, T), v(\cdot, T)\big)_{\Omega_T} + \int_\Gamma uv\cos\nu\, d\sigma,$$

where ν denotes the angle between the positive t-half axis and the outside normal to Γ (according to (C_2) it exists on Γ almost everywhere, see [6]).

We formulate our initial-boundary value problem in the following weak form:

(P) Find an $u \in V$ satisfying for all $v \in V \cap H_{m,1}^0$ the identity

$$(4) \qquad B(u, v) = \big(u_0, v(\cdot, 0)\big)_{\Omega_0} + (f, v)$$

with given $u_0 \in L^2(\Omega_0)$ and $f \in L^2(D_T)$.

In the sequel we denote by $l_{f,u_0}(v)$ the right-hand side of (4).

The above formulation is justified by the following

PROPOSITION 1. *Suppose $u \in H_{m,1}$. Then u is a solution of* (P) *if and only if it satisfies the differential equation*

(5) $$Au + u_t = f$$

in D_T together with the initial condition

(6) $$u(\cdot, 0) = u_0$$

and boundary conditions

(b_1) $$u \in V,$$

(b_2) $$(Au, v) = a(u, v)$$

for all $v \in V \cap H^0_{m,1}$.

Proof. Identity (4) with $v \in C_0^\infty(D_T)$ yields

(7) $$Au + u_t = f.$$

So $Au \in L^2(D_T)$. After integrating by parts the scalar product (u, v_t) we obtain from (4)

(8) $$a(u, v) + \big(u(\cdot, 0), v(\cdot, 0)\big)_{\Omega_0} = (Au, v) + \big(u_0, v(\cdot, 0)\big)_{\Omega_0}$$

for $v \in V \cap H^0_{m,1}$; this yields for $v \in C_0^\infty(D_T)$

(9) $$a(u, v) = (Au, v).$$

The last identity remains valid for $v \in C^\infty_{0,x}(D_T)$. To see this it suffices to replace v in (9) by $v_k = \eta_k(t)v$, where $\eta_k \in C_0^\infty\big((0, T)\big)$, $0 \leqslant \eta_k \leqslant 1$ and $\eta_k = 1$ in $[1/k, T-1/k]$. Then $v_k \to v$ in $\| \ \|_{m,0}$-norm and one may pass to the limit in (9). If, in particular, $v(\cdot, T) = 0$, then subtracting (9) from (8) yields

(10) $$\big(u(\cdot, 0), v(\cdot, 0)\big)_{\Omega_0} = \big(u_0, v(\cdot, 0)\big).$$

As it can be easily shown that $v(\cdot, 0)$ may be an arbitrary function in $C_0^\infty(\Omega_0)$, identity (10) yields the initial condition (6) and so (b_2) follows from (8). Thus we have shown that each solution of (P) solves the initial-boundary value problem formulated in our proposition. The converse statement can be easily proved, by taking the $L^2(D_T)$-scalar product of both sides of (5) with $v \in V \cap H^0_{m,1}$ and integrating by parts with respect to t.

Remark 1. We explain the meaning of condition (b_2) for the spaces V considered in examples given in Section 1. In Example 1 condition (b_2) is automatically verified, it follows simply from the definition of the derivation with respect to x in the sense of distributions. In Example 2 let us suppose that Γ, the solution u and the coefficients $a_{\alpha\beta}$ are smooth enough. Then integration by parts shows that u satisfies the homogeneous Neumann condition $\left.\frac{\partial u}{\partial \tau}\right|_{\Gamma} = 0$, where τ is the vector conormal to Γ with respect to the operator A. So (b_2) may be viewed as a kind of "natural boundary condition" on Γ.

Remark 2. Suppose that for each $a \in C^{\infty}([0, T])$

$$(C_4) \qquad a(t)V \subset V$$

(this is obviously satisfied in the above two examples). Then introducing in (P) the new unknown function $\tilde{u} = e^{-\lambda t}u$ we are led to the bilinear form $\tilde{a}(\ ,\ )$, which is positive-definite on V. In the sequel we shall suppose that (C_4) is true and that (C_3) holds with $\lambda = 0$.

In the sequel of this paper we shall suppose that one of the following assumptions is satisfied:

(C_5) $\cos\nu \geqslant 0$ on Γ (in particular, D_T may be a cylindrical domain, then $\cos\nu$ vanishes identically on Γ);

(C_6) V is the closure in $H_{m,0}$ of $C_{0,x}^{\infty}(D_T)$ (see Example 1).

We have the following

PROPOSITION 2. *$B(\ ,\ )$ is positive-definite on the set $V \cap H_{m,1}$ equipped with the norm $|||\ |||_m$.*

The proof follows immediately if we apply the Green formula to the scalar product (v, v_t).

PROPOSITION 3. *Problem* (P) *is solvable.*

The proof goes in a standard way. For fixed $v \in V \cap H_{m,1}^0$ the linear functional $V \ni u \to B(u, v)$ is bounded in the norm $\|\ \|_{m,0}$, and so, according to the Riesz theorem, it may be represented as

$$(12) \qquad B(u, v) = (u, Sv)_{m,0},$$

where $S\colon V \cap H_{m,1}^0 \to V$. Puting $u = v$ and applying Proposition 2 we get

$$(13) \qquad |||v|||_m \leqslant c_1 \|Sv\|_{m,0}.$$

Then it follows from (13) that

$$|l_{f,u_0}(v)| \leqslant c_2 \|Sv\|_{m,0},$$

so that $Sv \to l_{f,u_0}(v)$ is a linear functional bounded on the range of S. Extending this functional to the whole space V (equipped with the norm of $H_{m,0}$) and applying once more the Riesz theorem we obtain

$$(14) \qquad l_{f,u_0}(v) = (\bar{u}, Sv)_{m,0}$$

with some $\bar{u} \in V$, which is the desired solution, according to (12), (14).

3. The approximate problem. We are going to approximate the sufficiently smooth solutions of (P), namely those belonging to $H_{m,1}$. For such u we can reformulate the exact problem as follows:

PROPOSITION 4. *If* $u \in H_{m,1}$ *is a solution of* (P), *then it is unique and* (4) *holds for all* $v \in V \cap H_{m,1}$.

Proof. Let η_k be as in the proof of Proposition 1 and $w \in V \cap H_{m,1}$. Then $\eta_k(t)w \in V \cap H^0_{m,1}$ and tends to w in the norm $\| \ \|_{m,0}$; therefore (b_2) is valid with v replaced by w. So the second part of our statement follows after integrating by parts the scalar product (u, v_t) and applying Proposition 1. The uniqueness of u is now readily proved using Proposition 2.

Let now W_h be a finite-dimensional subspace of $V \cap H_{m,1}$. We formulate the approximate problem as follows:

(P_h) Find a $u_h \in W_h$ such that for all $\varphi \in W_h$ the identity

$$B(u_h, \varphi) = l_{f,u_0}(\varphi) \tag{15}$$

holds.

PROPOSITION 5. *Problem* (P_h) *has a unique solution.*

The proof follows directly from Proposition 2 after applying the lemma of Lax–Milgram (see [1], [6]) to form $B(\ ,\ )$ and functional l_{f,u_0} considered on the space W_h (note that on the finite-dimensional space W_h the two norms $|||\ |||_m$ and $\|\ \|_{m,1}$ are equivalent).

We now pass to the estimation of the error. Subtracting (15) from (4) with $v = \varphi$ yields

$$B(u-u_h, \varphi) = 0; \tag{16}$$

therefore

$$B(u-u_h, u-u_h) = B(u-u_h, u-\varphi) \tag{17}$$

with arbitrary $\varphi \in W_h$. Identity (16) is an analogue of the principle of orthogonal projection known in the case of a selfadjoint elliptic problem.

Applying Proposition 2 to (17) we obtain

$$d_1 |||u-u_h|||_m^2 \leqslant B(u-u_h, u-\varphi); \tag{18}$$

but

$$|B(u-u_h, u-\varphi)| \leqslant d_2 |||u-u_h|||_m \|u-\varphi\|_{m,1} \tag{19}$$

with positive constants d_j depending on the coefficients of A and the domain D_T. Now (18), (19) yield

$$|||u-u_h|||_m \leqslant d_2 d_1^{-1} \inf_{\varphi \in W_h} \|u-\varphi\|_{m,1} \tag{20}$$

and in this manner we are led to the approximation problem in the space $H_{m,1}$. Estimate (20) is an analogue of the Céa's lemma in elliptic problems (see [1]).

We discuss more detailly the case $m = 1$. Assuming that D_T is a polyhedron let us consider a regular family $\{T_h\}$ of its triangulations and let W_h be the corresponding finite element space of Lagrange type [1], [8]. Using the known properties of finite element approximations in Sobolev spaces we obtain from (20) the following

PROPOSITION 6. *Suppose that*

(i) *for each* $K \in T_h$ *the restriction of* W_h *to* K *contains all polynomials of order* $\leqslant r$,

(ii) $u \in H_{r+1}(D_T)$ *with* $r > \frac{1}{2}(n-1)$.

Then

$$|||u-u_h|||_1 = O(h^r)$$

where h *denotes the maximal diameter of the triangulation* T_h.

4. Final remarks. Internal approximations of parabolic problems in a cylindrical domain $\Omega \times (0, T)$ have been investigated by many authors (see [7], [9], [10], [11], where other references are given). The usual way is as follows: to choose a family of triangulations of the space domain Ω and look for an approximate solution $\tilde{u}$ such that for fixed t the function $\tilde{u}(\cdot, t)$ should belong to a corresponding finite element space X. Denoting by $\{e_j\}_{j=1}^N$ the basis in X and putting

$$\tilde{u}(x, t) = \sum_{j=1}^{N} u_j(t) e_j(x) \qquad (x \in \Omega)$$

we are led to a system of ordinary differential equations for the unknown functions u_j. So the second step is to solve approximately this system. This is usually done by difference methods and one obtains finally an approximate solution of the considered problem as a function, which is piecewise constant, and therefore discontinuous, with respect to t [7].

The method proposed in this paper yields approximate solutions in the form of spline functions with respect to all the variables x and t. The assumption of the regularity of the family $\{T_h\}$ means, roughly speaking, that the time step tends to zero with the same order of convergence as the space step.

Taking for φ in (15) the basis functions in the space W_h we are led directly to a system of linear equations, which gives the coordinates of the approximate solution u_h in the considered basis.

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