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ON SMALL STOCHASTIC PERTURBATIONS OF MAPPINGS OF THE UNIT INTERVAL

BY

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INTRODUCTION

We consider small stochastic perturbations of mappings from the unit interval into itself (for more detailed information see [3]). The general setting of the problem is as follows. Let $I = [0, 1]$ and let $\tau\colon I \to I$ be a piecewise monotonic mapping of class C^1, i.e. there exist $0 = b_0 < b_1 < \ldots < b_{q-1} < b_q = 1$ such that, for $i = 1, 2, \ldots, q$, $\tau|_{(b_{i-1}, b_i)}$ is a monotonic function of class C^1 and, moreover, it can be extended to $[b_{i-1}, b_i]$ as a C^1-function, which will be denoted by τ_i.

Let m be the Lebesgue measure on I and let $L^1 = L^1(I, m)$ be the space of all m-integrable real functions on I. We denote by L^1_+ the subset of L^1 containing all nonnegative functions f satisfying the condition $\int f(x)\,dx = 1$.

With the mapping τ one can associate the Perron-Frobenius operator $P_\tau\colon L^1 \to L^1$ so that

$$(P_\tau f)(y) = \sum_{i=1}^{q} \frac{f(\tau_i^{-1} y)}{|\tau_i'(\tau_i^{-1} y)|}, \qquad y \in I,$$

where $f(\tau_i^{-1} y) = 0$ for $y \notin \tau_i([b_{i-1}, b_i])$. It is well known that $P_\tau(L^1_+) \subset L^1_+$ and $P_\tau f = f$ if and only if the measure fm is invariant under τ. Let $L(\tau)$ denote the set of functions in L^1_+ invariant under P_τ.

For any positive integer n, we consider a family of probability densities $q^n(x, \cdot)$, $x \in I$, with respect to the measure m. The densities q^n considered below are bounded and measurable as functions of two variables. The family of transition densities $p^n(x, \cdot) = q^n(\tau(x), \cdot)$, $n = 1, 2, \ldots$, with respect to m is called a *stochastic perturbation* of the mapping τ. It is called *small* if for any $r > 0$ we have

$$\inf_{x \in I} \int_{x-r}^{x+r} q^n(x, y)\,dy \to 1 \quad \text{as } n \to \infty.$$

Perturbations considered in the sequel are small as they are local, i.e. for

$n = 1, 2, \ldots$ there exists $r_n > 0$ such that $q^n(x, y) = 0$ for $|y-x| > r_n$, and $r_n \to 0$ as $n \to \infty$.

We define operators Q_n and P_n, $n = 1, 2, \ldots$, from L^1 into itself as follows:

$$(Q_n f)(y) = \int q^n(x, y) f(x) dx, \quad (P_n f)(y) = \int p^n(x, y) f(x) dx, \quad y \in I.$$

Here and throughout the paper we neglect the indication of the range of integration if that range is the interval I. It is easy to see that $P_n = Q_n \circ P_\tau$, $n = 1, 2, \ldots$

Under our assumptions the transition density p^n $(n = 1, 2, \ldots)$ has at least one invariant probability measure μ_n, i.e.,

$$\mu_n(A) = \int \int_A p^n(x, y) dy\, d\mu_n(x)$$

for any Borel subset A of I. The measure μ_n is of the form $\mu_n = f_n m$, where $f_n \in L^1_+$ and $P_n f_n = f_n$ (see [2]).

Our aim is to find the limit points of the set $\{\mu_n\colon n = 1, 2, \ldots\}$ in the weak topology of measures ($\mu_n \to \mu \Leftrightarrow \mu_n(g) \to \mu(g)$ for any continuous function $g\colon I \to R$). Any such limit point will be called the *limit measure for the perturbation* p^n $(n = 1, 2, \ldots)$.

In the paper, we discuss mappings τ from 3 different classes. In part I, τ is a piecewise monotonic mapping of class C^2 as in [4] and [5]. Our results can be easily generalized to mappings τ that are piecewise monotonic and of class C^1 with $|1/\tau'|$ of bounded variation. Such mappings have been considered by Wong [11]. In part II, τ is a piecewise monotonic mapping of class $C^{1+\varepsilon}$ as in [9].

The perturbations we consider are of two classes. Their definitions are given in Sections I.A and I.B, respectively. Perturbations of Section I.A are connected with one of Ulam's problems [10] (see Example I.A.1).

The main result of the paper is the proof of the theorem that under our assumptions the limit measures for small stochastic perturbations are of the form fm, where $f \in L(\tau)$. This result may be understood as a stability of absolutely continuous τ-invariant measures under some classes of small stochastic perturbations.

In the paper we use the methods analogous to those of Li [6].

The author is much indebted to K. Krzyżewski for suggesting the subject and many inspiring talks.

I. PIECEWISE C^2-MAPPINGS

In this part of the paper, τ is a piecewise monotonic mapping of class C^2, i.e. for $i = 1, 2, \ldots, q$ the function τ_i is monotonic and of class C^2.

We shall use the following lemma from [5]:

LEMMA I. *For any* $f \in L^1_+$ *and*

$$K = (\sup|\tau''|)(\inf|\tau'|)^{-2} + 2(\inf|\tau'|)^{-1}\big(\min_{1 \leqslant i \leqslant q} (b_i - b_{i-1})\big)^{-1}$$

we have

$$V_0^1 (P_\tau f) \leqslant 2(\inf|\tau'|)^{-1} V_0^1 (f) + K,$$

where $V_a^b(g)$ *is the variation of the function* g *on the interval* $[a, b]$.

I.A. "Average-like" perturbations. In this section we claim that $|\tau_i'| > 2$ for $i = 1, 2, \ldots, q$.

Let $\pi_n = \{I_{n,1}, \ldots, I_{n,m(n)}\}$ be a partition of I into closed intervals such that $I_{n,i-1} \cap I_{n,i}$ is a single point ($n = 1, 2, \ldots$ and $i = 1, 2, \ldots, m(n)$). We claim that

$$\max\{m(I_{n,i}):\ i = 1, 2, \ldots, m(n)\} \to 0 \quad \text{as } n \to \infty.$$

Let us define

$$q(\pi_n)(x, y) = \begin{cases} (m(I_{n,i}))^{-1} & \text{for } x, y \in I_{n,i}, \\ 0 & \text{otherwise.} \end{cases}$$

The definitions of $p(\pi_n)$, $Q(\pi_n)$, and $P(\pi_n)$ are analogous to those of p_n, Q_n, and P_n in the Introduction ($n = 1, 2, \ldots$).

Since $Q(\pi_n)$ is an operator of conditional expectation, we call the perturbations generated by $q(\pi_n)$'s *average-like perturbations.*

The main technical result of this section is the following

PROPOSITION I.A. *Let* f_n *belong to* L^1_+ *and let* $P(\pi_n) f_n = f_n$ *for* $n = 1, 2, \ldots$ *Then the set* $\{f_n:\ n = 1, 2, \ldots\}$ *is relatively compact in* L^1 *and its limit points belong to* $L(\tau)$.

The proof of Proposition I.A is based on two lemmas.

LEMMA I.A.1. *For any positive integer* n *and for any* $f \in L^1$ *we have* $V_0^1 (Q(\pi_n) f) \leqslant V_0^1 (f)$.

LEMMA I.A.2. *For any* $f \in L^1$ *we have* $Q(\pi_n) f \to f$ *as* $n \to \infty$ *in the* L^1*-norm. The convergence is uniform on relatively compact subsets of* L^1.

The proofs of Lemmas I.A.1 and I.A.2 are analogous to the proofs of the appropriate lemmas in [6].

Proof of Proposition I.A. Let $f_n \in L^1_+$ be invariant for $P(\pi_n)$, where $n = 1, 2, \ldots$ Since $P(\pi_n) = Q(\pi_n) \circ P_\tau$, by Lemmas I and I.A.1 we have

$$V_0^1 (f_n) = V_0^1 (P(\pi_n) f_n) = V_0^1 \big((Q(\pi_n)(P_\tau f_n))\big)$$
$$\leqslant V_0^1 (P_\tau f_n) \leqslant 2(\inf|\tau'|)^{-1} V_0^1 (f_n) + K.$$

Hence $V_0^1 (f_n) \leqslant K\big(1 - 2(\inf|\tau'|)^{-1}\big)^{-1}$ for $n = 1, 2, \ldots$ Since $\|f_n\|_{L^1} = 1$ for any positive integer n, we infer that $\|f_n\|_{L^\infty}$ ($n = 1, 2, \ldots$) are uniformly bounded.

Applying Helly's theorem [8] we see that the set $\{f_n\colon n = 1, 2, \ldots\}$ is relatively compact in L^1.

Let f_{n_i} $(i = 1, 2, \ldots)$ be a subsequence converging to a function f in L^1 as $i \to \infty$. We prove that $f \in L(\tau)$. We have

$$\|P_\tau f - f\|_{L^1} \leqslant \|P_\tau f - P_\tau f_{n_i}\|_{L^1} + \|P_\tau f_{n_i} - Q(\pi_{n_i}) P_\tau f_{n_i}\|_{L^1} + \\ + \|Q(\pi_{n_i}) P_\tau f_{n_i} - f_{n_i}\|_{L^1} + \|f_{n_i} - f\|_{L^1}$$

with all summands on the right-hand side being arbitrarily small. Thus $P_\tau f = f$, which completes the proof.

The main result of this section is the following theorem, which is an immediate corollary to Proposition I.A.

THEOREM I.A. *If, for any positive integer n, μ_n is a probability Borel measure invariant for $p(\pi_n)$, then the limit points of the set $\{\mu_n\colon n = 1, 2, \ldots\}$, in the weak topology, are of the form fm, $f \in L(\tau)$. Moreover, the convergence is in the total variation norm.*

Remark I.A. For mappings τ considered in this part of the paper Kosjakin and Sandler [4] and Li and Yorke [7] have proved that the set of ergodic probability measures is finite and the support of any such measure is a union of a finite number of intervals. Hence any absolutely continuous τ-invariant measure can be obtained as a limit measure for a perturbation $p(\pi_n)$, $n = 1, 2, \ldots$ It is enough to make a suitable choice of the sequence ot partitions π_n.

Example I.A.1. Choose $\pi_n^0 = \{I_{n,1}, \ldots, I_{n,n}\}$ with $I_{n,i} = [i-1/n, i/n]$, where $i = 1, 2, \ldots, n$, and $n = 1, 2, \ldots$ Ulam [10] has defined an operator $P_n(\tau)\colon D_n \to D_n$ with $D_n = \mathrm{Span}\{\chi_{n,i}\colon i = 1, 2, \ldots, n\}$ in L^1 and $\chi_{n,i}$ the characteristic function of the interval $I_{n,i}$ as follows:

$$P_n(\tau)(\chi_{n,i}) = \sum_{j=1}^{n} P_{ij}\chi_{n,j},$$

where

$$P_{ij} = \frac{m(I_{n,i} \cap \tau^{-1}(I_{n,j}))}{m(I_{n,i})}, \quad 1 \leqslant i, j \leqslant n.$$

Ulam has conjectured that if, for any positive integer n, $f_n \in D_n$ is invariant for $P_n(\tau)$, then the L^1 limit points of the set $\{f_n\colon n = 1, 2, \ldots\}$ belong to $L(\tau)$. Li has answered the conjecture positively for τ discussed in this section (see [6]). It is easy to check that for any positive integer n the operator $P(\pi_n^0)$ is an extension of $P_n(\tau)$ to the whole L^1. Moreover, $P(\pi_n^0)(L^1) \subset D_n$, and so $P(\pi_n^0) f = f$ implies $P_n(\tau) f = f$. Hence Proposition I.A is a generalization of Li's result.

Example I.A.2 (see [1]). The example shows that for mappings τ in a very special class one can obtain an absolutely continuous invariant measure for τ directly as an invariant measure for a stochastic perturbation.

Let τ be a mapping of the unit interval I into itself for which there exist $0 = b_0 < b_1 < \ldots < b_{q-1} < b_q = 1$ such that $\tau|_{(b_{i-1}, b_i)}$ is a linear function, $\tau(\{b_0, b_1, \ldots, b_q\}) \subset \{b_0, b_1, \ldots, b_q\}$, and $\tau' \neq 0$. Let $\pi = \{I_i: i = 1, 2, \ldots, q\}$ denote the partition of I into intervals $I_i = [b_{i-1}, b_i]$ and let $q(\pi)$, $p(\pi)$, $P(\pi)$ be defined as above. The set

$$D = \{\sum_{i=1}^{q} \alpha_i \chi_i : \sum_{i=1}^{q} \alpha_i m(I_i) = 1\}$$

is compact and convex in L^1 (χ_i is the characteristic function of I_i). For $y \in I$ we have

$$\begin{aligned}(P(\pi)\chi_i)(y) &= \int \chi_i(x) p(\pi)(x, y) dx = \sum_{j: \tau I_i \supset I_j} \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_j)} \chi_j(y) \\ &= \frac{1}{|\tau_i'|} \chi_{\tau(I_i)}(y) = P_\tau(\chi_i)(y), \quad i = 1, 2, \ldots, q,\end{aligned}$$

so $P(\pi)|_D = P_\tau|_D$ and $P(\pi)(D) \subset D$. Hence there exists a piecewise constant function $f \in D$ such that $P_\tau f = P(\pi) f = f$.

I.B. "Convolution-like" perturbations. In this section we claim that $|\tau_i'| > 4$ for $i = 1, 2, \ldots, q$.

Fix the sequence of positive numbers r_n ($n = 1, 2, \ldots$), monotonically tending to zero, $r_1 < 1/4$. Let s^n: $R \to R^+$ ($n = 1, 2, \ldots$) be an m-measurable bounded function satisfying the following conditions:

(i) $s^n(t) = 0$ for $|t| > r_n$,

(ii) $s^n(-t) = s^n(t)$,

(iii) $\int_{-r_n}^{r_n} s^n(t) dt = 1$.

We define a family of probability densities q^n ($n = 1, 2, \ldots$) with respect to the Lebesgue measure m as follows:

$$q^n(x, y) = \begin{cases} s^n(y-x) & \text{for } x \in [r_n, 1-r_n], \\ s^n(y-x) + s^n(\tilde{y}-x) & \text{for the remaining } x \in I, \end{cases}$$

where $\tilde{y} = -y$ for $y \in [0, 1/4]$ and $\tilde{y} = 1 + (1-y)$ for $y \in [3/4, 1]$. Let p^n, Q_n, and P_n be defined as in the Introduction.

The perturbations generated by the probability densities q^n are similar at all points of I (except for the points near the ends of I). We call them *convolution-like perturbations* (see Lemma I.B.1).

We shall prove the following proposition analogous to Proposition I.A.

PROPOSITION I.B. *Let f_n belong to L^1_+ and let $P_n f_n = f_n$ for $n = 1, 2, \ldots$ Then the set $\{f_n\colon n = 1, 2, \ldots\}$ is relatively compact in L^1 and its limit points belong to $L(\tau)$.*

Before proving Proposition I.B, we give the following definition and some lemmas.

For any $f\colon I \to R$ we define its extension $\tilde{f}\colon R \to R$ as follows:

$$\tilde{f}(x) = \begin{cases} f(-x) & \text{for } x \in [-1/4, 0), \\ f(x) & \text{for } x \in I, \\ f(1-(x-1)) & \text{for } x \in (1, 5/4], \\ 0 & \text{for the remaining } x \in R. \end{cases}$$

LEMMA I.B.1. *For any positive integer n and for any function $f \in L^1$ we have*

$$(Q_n f)(y) = \int_{-r_n}^{1+r_n} \tilde{f}(x) s^n(y-x)\,dx = (\tilde{f} * s^n)(y), \qquad y \in I.$$

Proof. Let $y \in [0, r_n]$. We have

$$\begin{aligned}(Q_n f)(y) &= \int f(x) q^n(x, y)\,dx \\ &= \int_0^{y+r_n} f(x) s^n(y-x)\,dx + \int_0^{\tilde{y}+r_n} f(x) s^n(\tilde{y}-x)\,dx \\ &= \int_0^{y+r_n} f(x) s^n(y-x)\,dx + \int_{y-r_n}^{0} \tilde{f}(\tilde{x}) s^n(y-\tilde{x})\,d\tilde{x} = \int_{y-r_n}^{y+r_n} \tilde{f}(x) s^n(y-x)\,dx.\end{aligned}$$

The proof for $y \in [1-r_n, 1]$ is analogous and for $y \in (r_n, 1-r_n)$ it is trivial.

LEMMA I.B.2. *For any $f \in L^1_+$ and for any positive integer n we have $V_0^1(Q_n f) \leqslant 2V_0^1(f)$.*

Proof. By Lemma I.B.1, $Q_n f = \tilde{f} * s^n$. Fix a positive integer N and a sequence $0 = t_0 < t_1 < \ldots < t_N = 1$; we then have

$$\begin{aligned}\sum_{i=1}^{N} |(Q_n f)(t_i) - (Q_n f)(t_{i-1})| &= \sum_{i=1}^{N} |(\tilde{f} * s^n)(t_i) - (\tilde{f} * s^n)(t_{i-1})| \\ &= \sum_{i=1}^{N} |(s^n * \tilde{f})(t_i) - (s^n * \tilde{f})(t_{i-1})| \\ &= \sum_{i=1}^{N} \Big| \int_{-r_n}^{r_n} s^n(t)\tilde{f}(t_i - t)\,dt - \int_{-r_n}^{r_n} s^n(t)\tilde{f}(t_{i-1} - t)\,dt \Big| \\ &\leqslant \int_{-r_n}^{r_n} \Big(\sum_{i=1}^{N} |\tilde{f}(t_i - t) - \tilde{f}(t_{i-1} - t)|\Big) s^n(t)\,dt \\ &\leqslant \int_{-r_n}^{r_n} V_{-r_n}^{1+r_n}(\tilde{f}) s^n(t)\,dt = V_{-r_n}^{1+r_n}(\tilde{f}) \leqslant 2V_0^1(f).\end{aligned}$$

LEMMA I.B.3. *For any $f \in L^1$ we have $Q_n f \to f$ as $n \to \infty$ in the L^1-norm. The convergence is uniform on relatively compact subsets of L^1.*

Proof. Since for any positive integer n the operator norm of Q_n is equal to 1 and since continuous functions are dense in L^1, it is enough to prove that $Q_n g \to g$ in L^1 as $n \to \infty$ for any continuous function g.

We first prove that for any $y \in I$

$$\int q^n(x, y)\,dx = 1, \qquad n = 1, 2, \ldots$$

Let $y \in [0, r_n]$; we have

$$\int q^n(x, y)\,dx = \int_0^{r_n} \left(s^n(y-x) + s^n(\tilde{y}-x)\right)dx + \int_{r_n}^{y+r_n} s^n(y-x)\,dx.$$

Since

$$\int_0^{r_n} s^n(\tilde{y}-x)\,dx = \int_{-r_n}^{0} s^n(y-x)\,dx = \int_{y-r_n}^{0} s^n(y-x)\,dx,$$

we have

$$\int q^n(x, y)\,dx = \int_{y-r_n}^{y+r_n} s^n(y-x)\,dx = 1,$$

as desired. For $y \in [1-r_n, 1]$ the proof is analogous and for $y \in (r_n, 1-r_n)$ it is trivial.

Hence

$$\int |g(y) - (Q_n g)(y)|\,dy = \int \left|g(y) - \int g(x) q^n(x, y)\,dx\right| dy$$
$$\leqslant \int\int |g(y) - g(x)|\, q^n(x, y)\,dx\,dy \leqslant \int\int \omega(r_n) q^n(x, y)\,dx\,dy = \omega(r_n),$$

where ω is the modulus of continuity of the function g and $\omega(r_n) \to 0$ as $n \to \infty$.

We are now in a position to prove Proposition I.B. It is implied by Lemmas I.B.2, I.B.3, and Lemma I in the same way as Proposition I.A follows from Lemmas I.A.1, I.A.2, and Lemma I.

The main result of this section is the following

THEOREM I.B. *If, for any positive integer n, μ_n is a probability Borel measure invariant for p^n, then the limit points of the set $\{\mu_n: n = 1, 2, \ldots\}$, in the weak topology, are of the form fm, $f \in L(\tau)$. Moreover, the convergence is in the total variation norm.*

This theorem is a direct consequence of Proposition I.B.

Remark I.B.1. The assumption (i) can be replaced by a weaker one: (i^+) $s^n(t) = 0$ for $|t| > 1/4$, $n = 1, 2, \ldots$, and for any $r > 0$

$$\int_{-r}^{r} s^n(t)\,dt \to 1 \qquad \text{as } n \to \infty.$$

The proof remains almost unchanged.

Remark I.B.2. Contrary to the situation discussed in Section I.A, there are absolutely continuous τ-invariant measures, which cannot be obtained as limit measures for perturbations considered here. One can formulate the following sufficient conditions for obtaining a measure $m_1 \ll m$ as the limit measure for a perturbation p^n ($n = 1, 2, \dots$):

(a) m_1 is τ-ergodic;

(b) there exists an open set $U \supset \sup m_1$ such that every absolutely continuous τ-ergodic measure m_2 different from m_1 vanishes on U;

(c) $\mathrm{cl}(\tau U) \subset U$, i.e. $\mathrm{supp}\, m_1$ is an "attractor".

For the proof note that, by (c), for n large enough there exists a measure μ_n invariant for p^n and concentrated in U. We know that any limit point of the set $\{\mu_n: n = 1, 2, \dots\}$ (in the weak topology) is an absolutely continuous invariant measure for τ. From (a) and (b) it follows that any such limit point equals m_1.

We believe that condition (c) is necessary for obtaining m_1 as a limit measure, but we have no proof of that.

To illustrate the above remarks, we give two examples (see Figs. 1 and 2).

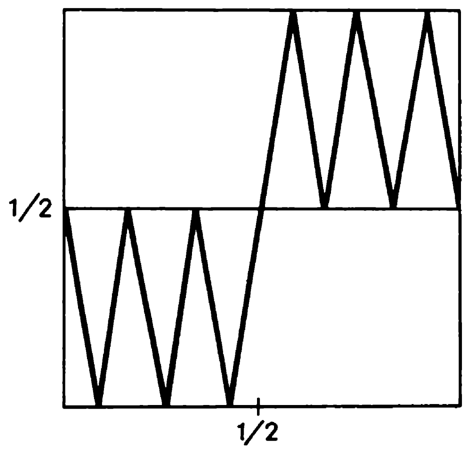

Fig. 1

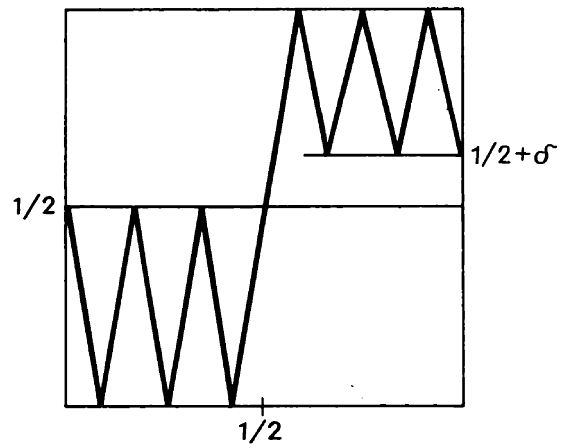

Fig. 2

For τ of Fig. 1 there are two ergodic absolutely continuous probability measures: $m_1 = 2m|_{[0,1/2]}$ and $m_2 = 2m|_{[1/2,1]}$. For any positive integer n the only invariant probability measure for p^n is $m = \frac{1}{2}m_1 + \frac{1}{2}m_2$.

For τ of Fig. 2 there are two ergodic absolutely continuous probability measures: $m_1 = 2m|_{[0,1/2]}$ and m_2 with support in $[1/2+\delta, 1]$. For any n such that $r_n < \delta/2$ there exists only one invariant probability measure μ_n for the transition density p^n. Since $\mu_n([0, 1/2]) = 0$, we have $\mu_n \to m_2$ ($n \to \infty$) in the weak topology.

Remark I.B.3. The results of Sections I.A and I.B remain true for τ considered by Wong [11], i.e. τ piecewise monotonic and of class C^1 with

$|1/\tau'|$ of bounded variation, if we claim in addition that $\inf|\tau'| > 3$ in Section I.A or $\inf|\tau'| > 6$ in Section I.B. The only change in proofs is replacing Lemma I by an analogous one from [11].

II. PIECEWISE $C^{1+\varepsilon}$-MAPPINGS

In this part we discuss small stochastic perturbations of piecewise monotonic expanding mappings τ of class $C^{1+\varepsilon}$, i.e. for any $i = 1, 2, \ldots, q$ the mapping τ_i' satisfies the Hölder condition with a constant α and an exponent ε. We claim that $\inf|\tau_i'| \geqslant \lambda > 1$, $i = 1, 2, \ldots, q$. The existence of absolutely continuous invariant measures for such mappings has been proved by Wong [12] under some additional very restrictive conditions and, recently, without any supplementary assumptions by Rychlik [9].

For functions from L^1 Rychlik has introduced a quantity C_ε which for τ piecewise of class $C^{1+\varepsilon}$ plays an analogous role to that of the variation V_0^1 for τ piecewise of class C^2.

It is worth noting that Rychlik's method applies to expanding mappings τ for which the modulus of continuity ω of τ_i' $(i = 1, 2, \ldots, q)$ satisfies the condition

$$\sup_{\delta>0} \omega(\delta/\lambda)/\omega(\delta) < 1.$$

The results of this part can be easily generalized to small stochastic perturbations of such mappings.

Now we claim additionally that $(\lambda)^{-\varepsilon}+4(\lambda)^{-1} < 1$. Let us denote by a_j $(j = 1, 2, \ldots, \tilde{q})$ all different points $\tau_i(b_{i-1})$, $\tau_i(b_i)$ $(i = 1, 2, \ldots, q)$. Let $\delta_0 > 0$ satisfy the following conditions:

(i) $(\lambda)^{-\varepsilon}+4(\lambda)^{-1}+\alpha(\delta_0)^{\varepsilon}(\lambda)^{-1-2\varepsilon} < 1$;

(ii) intervals $[a_j-\delta_0, a_j+\delta_0]$ for $j = 1, 2, \ldots, \tilde{q}$ are disjoint;

(iii) $\delta_0 < \frac{1}{3} \min\limits_{1\leqslant i\leqslant q} (b_i-b_{i-1})$.

For $f\in L^1$ we write

$$\Lambda(f, \delta, x) = \sup_{|x-y|<\delta} |f(x)-f(y)|$$

and

$$C_\varepsilon(f) = \sup_{0<\delta\leqslant\delta_0} (\delta)^{-\varepsilon} \int \Lambda(f, \delta, x)\,dx.$$

Then C_ε is a seminorm on the subspace of L^1 composed of all functions f such that $C_\varepsilon(f) < \infty$. Any subset of L^1_+ bounded in the seminorm C_ε is relatively compact in L^1. We shall prove this for a countable set $\{f_n\in L^1_+ : n = 1, 2, \ldots\}$.

If ω_n is the integral modulus of continuity of the function f_n, then for $\delta \leqslant \delta_0/2$ we have

$$\omega_n(\delta) = \int |f_n(\delta+x)-f_n(x)|\,dx \leqslant \int \Lambda(f_n, 2\delta, x)\,dx \leqslant \tilde{C}_\varepsilon \cdot 2^\varepsilon \delta^\varepsilon,$$

where $\tilde{C}_\varepsilon = \sup\{C_\varepsilon(f_n)\colon n = 1, 2, \ldots\}$. Thus the functions f_n $(n = 1, 2, \ldots)$, uniformly bounded in the L^1-norm, have also a common integral modulus of continuity. Hence they form a relatively compact set in L^1.

Now, we recall two lemmas from [9].

LEMMA II.1. *For any $f \in L^1$ we have*

$$C_\varepsilon(P_\tau f) \leqslant (\lambda^{-\varepsilon}+4\lambda^{-1}+\alpha\delta_0^\varepsilon \lambda^{-1-2\varepsilon})\,C_\varepsilon(f)+ \\ +(\alpha\lambda^{-1-\varepsilon}+2\lambda^{-1}\delta_0^{-\varepsilon})\,\|f\|_{L^1}.$$

LEMMA II.2. *If $f \in L^1$ and x, y belong to an interval J such that $m(J) = k\delta$ for some positive integer k, then*

$$|f(x)-f(y)| \leqslant (2/\delta) \int_J \Lambda(f, \delta, x)\,dx.$$

II.A. "Average-like" perturbations. In this section, for τ as above we consider small stochastic perturbations of the type discussed in Section I.A. Let π_n, $q(\pi_n)$, $p(\pi_n)$, $Q(\pi_n)$, and $P(\pi_n)$ be as in Section I.A.

LEMMA II.A.1. *For any $f \in L^1_+$ and for any positive integer n we have $C_\varepsilon(Q(\pi_n)f) \leqslant 18\,C_\varepsilon(f)$.*

Proof. Fix a function $f \in L^1_+$ and a positive integer n. We put $g = Q(\pi_n)f$. The function g is constant on elements of the partition π_n. On any such element the value of g is equal to the mean value of the function f on this interval.

Fix $\delta \leqslant \delta_0$ and consider intervals $I_i = [(i-1)\delta, i\delta]$ for $i = 1, 2, \ldots, w$ and $I_{w+1} = [w\delta, 1]$, where $w = \mathrm{E}(\delta^{-1})$. Moreover, assume that $I_i = \emptyset$ for $i \notin \{1, 2, \ldots, w+1\}$.

Let

$$O(h, J) = \sup_J h - \inf_J h$$

for any interval J and any function h from L^1_+. For $x \in I_i$, $i = 1, 2, \ldots, w+1$, we have

$$\Lambda(g, \delta, x) \leqslant O(g, I_{i-1} \cup I_i \cup I_{i+1}).$$

Thus

$$\int \Lambda(g, \delta, x)\,dx \leqslant \delta \sum_{i=1}^{w+1} O(g, I_{i-1} \cup I_i \cup I_{i+1}).$$

The sets $I_{i-1} \cup I_i \cup I_{i+1}$ $(i = 1, 2, \ldots, w+1)$ cover the interval $I = [0, 1]$ at most three times. They form three families, say F_1, F_2, F_3, that cover or almost cover I. We consider the first of them. Let F_1 be the family of intervals

$$J_j = [(2+3(j-1))\delta, (2+3j)\delta] \cap I, \quad j = 0, 1, \ldots, \tilde{w}.$$

We assume the worst possibility, i.e. that F_1 is a covering of I ($\tilde{w} = (w-1)/3$ is a positive integer). Let $j_1 < j_2 < \ldots < j_k$, $j_i \in \{0, 1, \ldots, \tilde{w}\}$, $i = 1, 2, \ldots, k$, be all indices j such that $j = 0$ or $j = \tilde{w}$ or the function g is not continuous on the interval J_j. Of course, we have

$$\sum_{j=0}^{\tilde{w}} O(g, J_j) = \sum_{j=j_1,\ldots,j_k} O(g, J_j).$$

Define sets

$$A_i = \bigcup_{j_{i-1} \leqslant j \leqslant j_{i+1}} J_j \quad \text{for } i = 1, 2, \ldots, k.$$

There exist points $x_i, y_i \in A_i$ ($i = 1, 2, \ldots, k$) such that

$$f(x_i) \leqslant \inf_{J_{j_i}} g \leqslant \sup_{J_{j_i}} g \leqslant f(y_i).$$

Thus $O(g, J_{j_i}) \leqslant O(f, A_i)$, $i = 1, 2, \ldots, k$. Using Lemma II.2 we obtain the estimate

$$O(g, J_{j_i}) \leqslant O(f, A_i) \leqslant (2/\delta) \int_{A_i} \Lambda(f, \delta, x)\, dx$$

for $i = 1, 2, \ldots, k-1$. For $i = k$ there are three possibilities:

(i) $x_k, y_k \in A_k \cap [0, w\delta]$; then the estimate analogous to the above one is true;

(ii) exactly one of the points x_k, y_k (say x_k) belongs to the interval $[w\delta, 1]$; then

$$O(g, J_{j_k}) \leqslant |f(x_k) - f(y_k)| \leqslant |f(x_k) - f(w\delta)| + |f(w\delta) - f(y_k)|$$
$$\leqslant (2/\delta) \int_{1-\delta}^{1} \Lambda(f, \delta, x)\, dx + (2/\delta) \int_{A_k \cap [0, w\delta]} \Lambda(f, \delta, x)\, dx;$$

(iii) both of the points x_k, y_k belong to $[w\delta, 1]$; then

$$O(g, J_{j_k}) \leqslant (2/\delta) \int_{1-\delta}^{1} \Lambda(f, \delta, x)\, dx + (2/\delta) \int_{1-\delta}^{1} \Lambda(f, \delta, x)\, dx.$$

In any of these cases, the sets we integrate over cover the interval I at most three times. Hence

$$\sum_{j=0}^{\tilde{w}} O(g, J_j) \leqslant 3(2/\delta) \int \Lambda(f, \delta, x)\, dx.$$

Since for F_2 and F_3 the analogous estimates are true, we obtain

$$\int \Lambda(g, \delta, x)\,dx \leqslant 3\cdot 3\cdot 2 \int \Lambda(f, \delta, x)\,dx.$$

Hence $C_\varepsilon(g) \leqslant 18\, C_\varepsilon(f)$, which completes the proof.

Lemmas II.A.1, II.1, and I.A.2 imply the following

THEOREM II.A. *Let τ be as described above. If* $18(\lambda^{-\varepsilon}+4\lambda^{-1}) < 1$, *then the analogues of Proposition* I.A *and Theorem* I.A *are true.*

II.B. "Convolution-like" perturbations. In this section we discuss, for τ as above, small stochastic perturbations of the type considered in Section I.B. Let r_n, s^n, q^n, p^n, Q_n, and P_n be as in Section I.B.

LEMMA II.B.1. *For any $f \in L^1_+$ and for any positive integer n we have* $C_\varepsilon(Q_n f) \leqslant 2C_\varepsilon(f)$.

Proof. Since $Q_n f = \tilde{f} * s^n = s^n * \tilde{f}$, for any $\delta \leqslant \delta_0$ we have

$$\int \Lambda(Q_n f, \delta, x)\,dx = \int \sup_{|x-y|<\delta} \Big| \int_{-r_n}^{r_n} (\tilde{f}(x-t) - \tilde{f}(y-t))\, s^n(t)\,dt \Big|\,dx$$
$$\leqslant \int_{-r_n}^{r_n} \int \Lambda(\tilde{f}, \delta, x-t)\,dx\, s^n(t)\,dt \leqslant \int_{-r_n}^{1+r_n} \Lambda(\tilde{f}, \delta, x)\,dx.$$

Condition (iii) on δ_0 implies $\delta_0 < 1/6$, so we obtain

$$\Lambda(\tilde{f}, \delta, x) = \begin{cases} \Lambda(f, \delta, -x) & \text{for } x \in [-1/4, 0), \\ \Lambda(f, \delta, x) & \text{for } x \in I, \\ \Lambda\big(f, \delta, 1-(x-1)\big) & \text{for } x \in (1, 5/4). \end{cases}$$

Thus $C_\varepsilon(Q_n f) \leqslant 2C_\varepsilon(f)$.

Lemmas II.B.1, II.1, and I.B.2 imply the following

THEOREM II.B. *Let τ be as described above. If* $2(\lambda^{-\varepsilon}+4\lambda^{-1}) < 1$, *then the analogous of Proposition* I.B *and Theorem* I.B *are true.*

III. CONJECTURES

At the end of the paper we formulate two conjectures.

CONJECTURE I. **(P 1280)** If there exists a positive integer k such that the mapping τ^k belongs to one of the classes discussed in the paper, then the analogues of all our propositions and theorems are true for the mapping τ.

CONJECTURE II. **(P 1281)** Let for any positive integer n the mapping q^n: $I \times I \to R^+$ be a measurable bounded function and let for all $x, y \in I$

$$\int q^n(x, y)\,dy = \int q^n(x, y)\,dx = 1.$$

Then all our results are valid for small stochastic perturbations generated by the family of densities q^n, $n = 1, 2, \ldots$ (q^n discussed in the paper are examples of such densities).

Added in proof. A more general treatment of the problem can be found in: G. Keller, *Stochastic stability in some chaotic dynamical systems*, Monatshefte für Mathematik 94 (1982), p. 313-334.

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