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# ON INVERSE QUEUING PROCESSES

**Summary.** If in a queuing process we interchange the interarrival times and the service times, we obtain the inverse process of the original one. In this paper we shall show that if for a single-server queuing process we know the distribution of the maximal queue size during a busy period and the distribution of the maximal waiting time during a busy period, then we can obtain immediately the same distributions for the inverse queuing process.

**1. Introduction.** Suppose that in the time interval $(0, \infty)$ customers arrive at a counter at times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ where $0 < \tau_1 < \tau_2 < \ldots$ $\ldots < \tau_n < \ldots$ are random variables. The customers are served by a single server who starts working at time $t = 0$. The successive service times, $\chi_1, \chi_2, \ldots, \chi_n, \ldots$, are positive random variables. The order of serving is not specified. We suppose only that the server is busy if there is at least one customer in the system.

We shall denote by $\xi(t)$ the *queue size* at time $t$, that is, the total number of customers in the system at time $t$. $\xi(0)$ is the initial queue size, that is, the number of customers already waiting for service at time $t = 0$.

We shall denote by $\eta(t)$ the *waiting time* at time $t$, that is, the time that a customer would have to wait if he arrived at time $t$ and if the customers are served in the order of their arrivals. $\eta(0)$ is the initial waiting time, that is, the occupation time of the server at time $t = 0$.

Denote by $\varrho_0$ the number of customers served during the initial busy period, and by $\theta_0$ the length of the initial busy period. If $\xi(0) = 0$, then $\varrho_0 = 0$, and if $\eta(0) = 0$, then $\theta_0 = 0$.

For $0 \leqslant i \leqslant k$ define

$$P(k\,|\,i) = \boldsymbol{P}\{\sup_{0 \leqslant t \leqslant \theta_0} \xi(t) \leqslant k \,|\, \xi(0) = i\} \tag{1}$$

as the probability that the maximal queue size during the initial busy period is $\leqslant k$ given that the initial queue size is $i$.

For $0 \leqslant c \leqslant x$ define

$$G(x \mid c) = \boldsymbol{P}\{\sup_{0 \leqslant t \leqslant \theta_0} \eta(t) \leqslant x \mid \eta(0) = c\} \tag{2}$$

as the probability that the maximal waiting time during the initial busy period is $\leqslant x$ given that the initial waiting time is $c$.

Now we shall define the *inverse process* of the queuing process defined before. Suppose that in the time interval $[0, \infty)$ customers arrive at a counter at times $\tau_0^*, \tau_1^*, \ldots, \tau_n^*, \ldots$ where $\tau_0^* = 0$ and $\tau_n^* = \chi_1 + \ldots + \chi_n$ $(n = 1, 2, \ldots)$. The customers are served by a single server who starts working at time $t = 0$. The successive service times are $\chi_1^*, \chi_2^*, \ldots, \chi_n^*, \ldots$ where $\chi_n^* = \tau_n - \tau_{n-1} (n = 1, 2, \ldots; \tau_0 = 0)$. We suppose that the server is busy if there is at least one customer in the system. This queuing process is called the inverse process of the previous one. That is, we obtain the inverse process of a queuing process if we interchange the interarrival times and service times.

For the inverse queuing process denote by $\xi^*(t)$ the queue size at time $t$. $\xi^*(0)$ is the initial queue size, that is, the queue size immediately before $t = 0$. The customer arriving at time $t = 0$ is not counted in the initial queue size. Further denote by $\eta^*(t)$ the waiting time at time $t$ in the inverse queuing process. $\eta^*(0)$ is the initial waiting time (immediately before $t = 0$). The service time of the customer arriving at time $t = 0$ is not included in the initial waiting time.

Denote by $\varrho_0^*$ the number of customers served in the initial busy period, and by $\theta_0^*$ the length of the initial busy period.

For the inverse process we use the same notation as for the original process, except that an asterisk is added. Thus we use the notation

$$P^*(k \mid i) = \boldsymbol{P}\{\sup_{0 \leqslant t \leqslant \theta_0^*} \xi^*(t) \leqslant k \mid \xi^*(0) = i\} \tag{3}$$

$(0 \leqslant i \leqslant k)$ for the probability that the maximal queue size during the initial busy period is $\leqslant k$ if the initial queue size is $i$, and

$$G^*(x \mid c) = \boldsymbol{P}\{\sup_{0 \leqslant t \leqslant \theta_0^*} \eta^*(t) \leqslant x \mid \eta^*(0) = c\} \tag{4}$$

$(0 \leqslant c \leqslant x)$ for the probability that the maximal waiting time during the initial busy period is $\leqslant x$ if the initial waiting time is $c$.

**2. Dual theorems.** In this section we shall show that there are simple relations between the distributions (1) and (3) as well as between (2) and (4). We shall always suppose that

$$\boldsymbol{P}\{\sup_{1 \leqslant n < \infty} |\chi_1 + \ldots + \chi_n - \tau_n| = \infty\} = 1. \tag{5}$$

THEOREM 1. *If* $0 < i \leqslant k$, *then*

$$P^*(k \mid k-i) = 1 - P(k \mid i). \tag{6}$$

Proof. Define a stochastic process $\{\delta(t), 0 \leqslant t < \infty\}$ in the following way: $\delta(0) = 0$ and $\delta(t)$ changes only in jumps. A jump of magnitude $+1$ occurs at times $t = \tau_1, \tau_2, \ldots, \tau_n, \ldots$ and a jump of magnitude $-1$ occurs at times $t = \tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots$. Then evidently $P(k\,|\,i)$ is the probability that $\delta(t), 0 \leqslant t \leqslant \infty$, reaches the line $z = -i$ first without touching the line $z = k+1-i$ in the meantime. $P^*(k\,|\,k-i)$ is the probability that $\delta(t), 0 \leqslant t < \infty$, reaches the line $z = k+1-i$ first without touching the line $z = -i$ in the meantime. If (5) is satisfied, then $\delta(t)$, $0 \leqslant t < \infty$, will sooner or later reach either $z = -i$ or $z = k+1-i$ with probability 1. Hence $P(k\,|\,i)+P^*(k\,|\,k-i) = 1$ which was to be proved.

THEOREM 2. *If $0 < c \leqslant x$, then we have*

$$G^*(x\,|\,x-c) = 1-G(x\,|\,c). \tag{7}$$

Proof. Define a stochastic process $\{\chi(t), 0 \leqslant t < \infty\}$ in the following way: $\chi(0) = 0$ and

$$\chi(t) = \sum_{0<\tau_n\leqslant t} \chi_n \tag{8}$$

for $0 \leqslant t < \infty$. Then $G(x\,|\,c)$ can be interpreted as the probability that $\chi(t), 0 \leqslant t < \infty$, intersects the line $z = t-c$ first without intersecting the line $z = t+x-c$ in the meantime. $G^*(x\,|\,x-c)$ is the probability that $\chi(t), 0 \leqslant t < \infty$, intersects the line $z = t+x-c$ first without intersecting the line $z = t-c$ in the meantime. If (5) is satisfied, then $\chi(t), 0 \leqslant t < \infty$, will sooner or later intersect either $z = t-c$ or $z = t+x-c$ with probability 1. Hence $G(x\,|\,c)+G^*(x\,|\,x-c) = 1$ which was to be proved.

In the next two sections we shall give some examples for the applications of the above theorems. We shall suppose that $\tau_n-\tau_{n-1}$ ($n = 1, 2, \ldots$; $\tau_0 = 0$) and $\chi_n$ ($n = 1, 2, \ldots$) are independent sequences of mutually independent and identically distributed positive random variables.

If

$$\mathbf{P}\{\chi_n = \tau_n-\tau_{n-1}\} < 1, \tag{9}$$

then (5) is satisfied and Theorem 1 and Theorem 2 are applicable.

We shall use the following notation:

$$\mathbf{P}\{\tau_n-\tau_{n-1} \leqslant x\} = F(x), \tag{10}$$

$$\mathbf{P}\{\chi_n \leqslant x\} = H(x), \tag{11}$$

$$\psi(s) = \int_0^\infty e^{-sx}dH(x), \tag{12}$$

$$a = \int_0^\infty x\,dH(x). \tag{13}$$

**3. Queues with Poisson input.** In this section we shall give direct proofs for some theorems which have been found by the author [10]. (See also [11] and [12].)

Consider the single-server queuing process defined in the introduction in the case when the interarrival times $\tau_n - \tau_{n-1}$ $(n = 1, 2, \ldots;\ \tau_0 = 0)$ are mutually independent random variables having the common distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if} \quad x \geqslant 0, \\ 0 & \text{if} \quad x < 0, \end{cases} \tag{14}$$

and the service times $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ are also mutually independent random variables having the common distribution function $H(x)$. Further, the two sequences $\{\tau_n - \tau_{n-1}\}$ and $\{\chi_n\}$ are also independent.

In this process the arrivals form a Poisson process of density $\lambda$, and the probability that exactly $j$ $(j = 0, 1, 2, \ldots)$ customers arrive during a service time is given by

$$\pi_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} \, dH(x). \tag{15}$$

The generating function of $\pi_j$ $(j = 0, 1, 2, \ldots)$ is given by

$$\pi(z) = \sum_{j=0}^\infty \pi_j z^j = \int_0^\infty e^{-\lambda(1-z)x} \, dH(x) = \psi\big(\lambda(1-z)\big) \tag{16}$$

and (16) is necessarily convergent for $|z| \leqslant 1$.

THEOREM 3. *For $0 \leqslant i \leqslant k$ we have*

$$P(k \mid i) = \frac{Q_{k-i}}{Q_k} \tag{17}$$

*where*

$$Q(z) = \sum_{k=0}^\infty Q_k z^k = \frac{Q_0 \pi(z)}{\pi(z) - z} \tag{18}$$

*for $|z| < \delta$ and $\delta$ is the smallest nonnegative real root of*

$$\pi(z) = z. \tag{19}$$

*If $\lambda a \leqslant 1$, then $\delta = 1$ and if $\lambda a > 1$, then $\delta < 1$. $Q_0$ is an arbitrary nonnull constant.*

Proof. Define the process $\{\delta(t), 0 \leqslant t < \infty\}$ in the same way as in the proof of Theorem 1. If we measure time from a transition $i+1 \to i$, then independently of the past, the increments of the process $\{\delta(t), 0 \leqslant t < \infty\}$ have the same stochastic behavior as the original process $\{\delta(t),$

$0 \leqslant t < \infty\}$. This implies that

$$P(k\,|\,k-i) = P(j\,|\,j-i)P(k\,|\,k-j) \tag{20}$$

for $0 \leqslant i \leqslant j \leqslant k$. Since $0 < P(k\,|\,i) \leqslant 1$ if $0 \leqslant i \leqslant k$, it follows that $P(k\,|\,i)$ can be represented in the following form

$$P(k\,|\,i) = \frac{Q_{k-i}}{Q_k} \tag{21}$$

for $0 \leqslant i \leqslant k$ where $Q_0 \neq 0$ and $Q_k/Q_0$ $(k = 0, 1, 2, \ldots)$ is a nondecreasing sequence.

If we take into consideration that during the first service time in the initial busy period the number of arrivals may be $j = 0, 1, 2, \ldots,$ then we can write that

$$P(k+i\,|\,i) = \sum_{j=0}^{k} \pi_j P(k+i\,|\,i+j-1) \tag{22}$$

for $i \geqslant 0$, $k \geqslant 0$. If we multiply (22) by $Q_{k+i}$ and use (21), then we get the following recurrence formula for the determination of $Q_k$ $(k = 0, 1, 2, \ldots)$:

$$Q_k = \sum_{j=0}^{k} \pi_j Q_{k+1-j} \quad (k = 0, 1, 2, \ldots). \tag{23}$$

If we introduce generating functions, we get (18).

We have explicitly $Q_1 = Q_0/\pi_0$ and for $k = 1, 2, \ldots$

$$Q_{k+1} = Q_0 \sum_{\nu=1}^{k} \frac{(-1)^\nu \nu!}{\pi_0^{\nu+1}} \sum_{\substack{i_1+i_2+\ldots+i_k=\nu \\ i_1+2i_2+\ldots+ki_k=k}} \frac{(\pi_1-1)^{i_1}\pi_2^{i_2}\ldots\pi_k^{i_k}}{i_1!\,i_2!\ldots i_k!}. \tag{24}$$

We remark that if $\lambda a < 1$, then

$$\lim_{k\to\infty} Q_k = \frac{Q_0}{1-\lambda a}, \tag{25}$$

that is, by choosing $Q_0 = 1-\lambda a$ we have $\lim_{k\to\infty} Q_k = 1$. If $\lambda a \geqslant 1$, then $\lim_{k\to\infty} Q_k/Q_0 = \infty$.

If we consider any busy period other than the initial one, then

$$P(k\,|\,1) = \frac{Q_{k-1}}{Q_k} \quad (k = 1, 2, \ldots) \tag{26}$$

is the probability that the maximal queue size during the busy periodis $\leqslant k$.

EXAMPLE. If, in particular, $H(x) = 1-e^{-\mu x}$ for $x \geqslant 0$, that is, the service times have an exponential distribution, then

$$Q_k = \frac{Q_0}{1-\frac{\lambda}{\mu}}\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right] \tag{27}$$

for $\lambda \neq \mu$ and $Q_k = Q_0(k+1)$ for $\lambda = \mu$.

In this case

$$P(k|1) = \frac{1-\left(\frac{\lambda}{\mu}\right)^k}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \tag{28}$$

if $\lambda \neq \mu$ and $P(k|1) = k/(k+1)$ if $\lambda = \mu$.

Formula (28) has been found by S. Karlin and J. McGregor [6].

NOTE. First, we note that if $\lambda a < 1$, then

$$\lim_{t\to\infty} \boldsymbol{P}\{\xi(t) \leqslant k\} = Q_k \quad (k = 0, 1, 2, \ldots) \tag{29}$$

where $Q_0 = 1-\lambda a$. The limit (29) is independent of the initial state. If $\lambda a > 1$, then $\lim_{t\to\infty} \boldsymbol{P}\{\xi(t) \leqslant k\} = 0$ for all $k$.

Second, consider the queuing process studied in this section with the modification that there is a waiting room of size $m$, that is, the number of customers in the system is at most $m+1$. If an arriving customer finds $m+1$ customers in the system, then he departs without being served. In this modified process denote by $\zeta_n$ the queue size immediately after the $n$-th departure. Then $\{\zeta_n\}$ is an irreducible and aperiodic Markov chain with state space $I = \{0, 1, \ldots, m\}$. Consequently the limiting distribution $\lim_{n\to\infty} \boldsymbol{P}\{\zeta_n \leqslant k\} = P_k$ $(k = 0, 1, \ldots, m)$ exists, is independent of the initial state and can be obtained as the unique solution of the following system of linear equations

$$P_k = \sum_{j=0}^{k} \pi_j P_{k+1-j} \quad (k = 0, 1, \ldots, m-1) \tag{30}$$

and $P_m = 1$. A comparison of (23) and (30) shows that

$$P_k = \frac{Q_k}{Q_m} \quad (k = 0, 1, \ldots, m) \tag{31}$$

where $Q_k$ $(k = 0, 1, \ldots)$ is defined in Theorem 3.

If $\xi(t)$ denotes the queue size at time $t$ also for the modified process, then it can easily be proved that $\lim_{t\to\infty} \boldsymbol{P}\{\xi(t) \leqslant k\} = P_k^*$ $(k = 0, 1, \ldots, m+1)$ exists, is independent of the initial state, and

$$P_k^* = \frac{Q_k}{Q_0 + \lambda a Q_m} \tag{32}$$

for $k = 0, 1, \ldots, m$. Obviously $P_{m+1}^* = 1$. If we take into consideration that the number of transitions $i \to i+1$ and $i+1 \to i$ $(i = 0, 1, \ldots, m)$ in any interval $(0, t)$ may differ by at most 1, then we obtain that $P_k^* = P_k P_m^*$ for $k = 0, 1, \ldots, m$ and evidently $P_0^* = P_0/(\lambda a + P_0)$. Thus we get (32). (Cf. also P. D. Finch [5], J. Keilson [7] and J. W. Cohen [2].)

THEOREM 4. *For* $0 \leqslant c \leqslant x$ *we have*

$$G(x\,|\,c) = \frac{W(x-c)}{W(x)} \tag{33}$$

*where*

$$\Omega(s) = \int_0^\infty e^{-sx}\,dW(x) = \frac{W(0)s}{s - \lambda[1-\psi(s)]} \tag{34}$$

*for* $Re(s) > \omega$ *and* $\omega$ *is the largest nonnegative real root of*

$$\lambda[1-\psi(s)] = s. \tag{35}$$

*If* $\lambda a \leqslant 1$, *then* $\omega = 0$, *whereas if* $\lambda a > 1$, *then* $\omega > 0$. $W(0)$ *is an arbitrary nonnull constant.*

Proof. In this case the process $\{\chi(t), 0 \leqslant t < \infty\}$ defined by (8) has nonnegative stationary independent increments. This implies that for $0 \leqslant c \leqslant y \leqslant x$ we have

$$G(x\,|\,x-c) = G(y\,|\,y-c)G(x\,|\,x-y). \tag{36}$$

Since $0 < G(x\,|\,y) \leqslant 1$ if $0 \leqslant y \leqslant x$, it follows that $G(x\,|\,c)$ can be represented in the following form

$$G(x\,|\,c) = \frac{W(x-c)}{W(x)} \tag{37}$$

for $0 \leqslant c \leqslant x$ where $W(0) \neq 0$ and $W(x)/W(0)$ $(0 \leqslant x < \infty)$ is a nondecreasing function of $x$. If we take into consideration that in the time interval $(0, u)$ one customer arrives with probability $\lambda u + o(u)$, and more than one customer arrives with probability $o(u)$, then we can write that for $x \geqslant 0$ and $y \geqslant 0$

$$\begin{aligned} &G(x+y\,|\,y) \\ &\quad = (1-\lambda u)G(x+y\,|\,y-u) + \lambda u \int_0^x G(x+y\,|\,y+z)\,dH(z) + o(u). \end{aligned} \tag{38}$$

If we multiply this equation by $W(x+y)$, then we obtain

$$W(x) = (1-\lambda u)W(x+u)+\lambda u \int_0^x W(x-z)\,dH(z)+o(u) \tag{39}$$

for $x \geqslant 0$. Hence

$$W'(x) = \lambda W(x)-\lambda \int_0^x W(x-z)\,dH(z) \tag{40}$$

for $x > 0$. Forming the Laplace-Stieltjes transform of (40) we get

$$s[\Omega(s)-W(0)] = \lambda\Omega(s)-\lambda\Omega(s)\psi(s), \tag{41}$$

whence

$$\Omega(s) = \frac{W(0)s}{s-\lambda[1-\psi(s)]} \tag{42}$$

for $\mathrm{Re}(s) > \omega$. This proves Theorem 4.

If we suppose that $a$ is a finite positive number and introduce a distribution function $H^*(x)$ for which

$$\frac{dH^*(x)}{dx} = \frac{1-H(x)}{a} \tag{43}$$

if $x > 0$ and $H^*(x) = 0$ if $x \leqslant 0$, then by inverting (42) we obtain

$$W(x) = W(0)\sum_{n=0}^{\infty}(\lambda a)^n H_n^*(x) \tag{44}$$

where $H_n^*(x)$ is the $n$-th iterated convolution of $H^*(x)$ with itself; $H_0^*(x) = 1$ if $x \geqslant 0$ and $H_0^*(x) = 0$ if $x < 0$.

We remark that if $\lambda a < 1$, then

$$\lim_{x\to\infty} W(x) = \frac{W(0)}{1-\lambda a}, \tag{45}$$

that is, by choosing $W(0) = 1-\lambda a$ we have $\lim_{x\to\infty} W(x) = 1$. If $\lambda a \geqslant 1$, then $\lim_{x\to\infty} W(x)/W(0) = \infty$.

If we consider any busy period other than the initial one, then the probability that the maximal waiting time during the busy period is $\leqslant x$ is given by

$$G(x) = \frac{1}{W(x)}\int_0^x W(x-z)\,dH(z) = 1-\frac{W'(x)}{\lambda W(x)} \tag{46}$$

for $x > 0$ which follows from (40).

EXAMPLE. If, in particular, $H(x) = 1-e^{-\mu x}$ for $x \geqslant 0$, then

$$W(x) = \frac{W(0)}{\mu-\lambda}[\mu-\lambda e^{(\lambda-\mu)x}] \tag{47}$$

for $\lambda \neq \mu$ and $W(x) = W(0)(1+\mu x)$ for $\lambda = \mu$.

In this case

$$G(x) = \frac{\mu-\mu e^{(\lambda-\mu)x}}{\mu-\lambda e^{(\lambda-\mu)x}} \tag{48}$$

if $\lambda \neq \mu$ and $G(x) = \mu x/(1+\mu x)$ if $\lambda = \mu$.

NOTE. First, we note that if $\lambda a < 1$, then

$$\lim_{t\to\infty} \boldsymbol{P}\{\eta(t) \leqslant x\} = W(x) \qquad (0 \leqslant x < \infty) \tag{49}$$

where $W(0) = 1-\lambda a$. The limit (49) is independent of the initial state. If $\lambda a > 1$, then $\lim_{t\to\infty} \boldsymbol{P}\{\eta(t) \leqslant x\} = 0$ for all $x$.

Second, consider the queuing process studied in this section with the modification that the total time spent in the system by a customer cannot increase above $m$ where $m$ is a positive number. That is, if a customer already spent time $m$ in the system, then he departs even if his serving has not yet been completed. (This model can be used also in the theory of dams where there is an overflow if the dam is full.)

If $\eta(t)$ denotes the waiting time at time $t$ also for the modified process, then we have

$$\lim_{t\to\infty} \boldsymbol{P}\{\eta(t) \leqslant x\} = \frac{W(x)}{W(m)} \tag{50}$$

for $0 \leqslant x \leqslant m$, where $W(x)$ is defined in Theorem 4. (Cf. also I. N. Kovalenko [8].) If $\eta_n = \eta(\tau_n-0)$, that is, $\eta_n$ is the waiting time immediately before the arrival of the $n$-th customer, then we have also

$$\lim_{n\to\infty} \boldsymbol{P}\{\eta_n \leqslant x\} = \frac{W(x)}{W(m)} \tag{51}$$

for $0 \leqslant x \leqslant m$. This result is due to D. J. Daley [4].

**4. Queues with exponentially distributed service times.** Now consider the inverse process of the queuing process discussed in the previous section. In this case the service times have the distribution function (14).

THEOREM 5. *If $0 \leqslant i \leqslant k$, then we have*

$$P^*(k\,|\,i) = 1-\frac{Q_i}{Q_k} \tag{52}$$

*where $Q_k$ ($k = 0, 1, 2, \ldots$) is defined in Theorem 3.*

Proof. (52) follows immediately from Theorem 1 and Theorem 3.

The probability that in any busy period other than the initial one the maximal queue size is $\leqslant k$ is given by

$$P^*(k\,|\,0) = 1 - \frac{Q_0}{Q_k} \tag{53}$$

for $k = 0, 1, 2, \ldots$ (See J. W. Cohen [3].)

NOTE. First, let us remark that if $\lambda a < 1$, then

$$\boldsymbol{P}\{\varrho_0^* < \infty \,|\, \xi^*(0) = i\} = 1 - Q_i \tag{54}$$

for $i = 0, 1, 2, \ldots$ where $Q_0 = 1 - \lambda a$.

Second, consider the queuing process investigated in this section with the modification that there is a waiting room of size $m-1$, that is, the number of customers in the system is at most $m$. If an arriving customer finds $m$ customers in the system, then he departs without being served. In this modified process denote by $\xi_n$ the queue size immediately before the arrival of the $n$-th customer, and by $\xi(t)$ the queue size at time $t$. Then independently of the initial state we have

$$\lim_{n\to\infty} \boldsymbol{P}\{m - \xi_n \leqslant k\} = \frac{Q_k}{Q_m} \tag{55}$$

for $k = 0, 1, \ldots, m$ and if $H(x)$ is not a lattice distribution function, then

$$\lim_{t\to\infty} \boldsymbol{P}\{m - \xi(t) \leqslant k\} = \frac{Q_{k+1} - Q_0}{\lambda a Q_m} \tag{56}$$

for $k = 0, 1, \ldots, m-1$. (See reference [9] and p. 28 in the discussion of [13].)

THEOREM 6. *If* $0 \leqslant c \leqslant x$, *then we have*

$$G^*(x\,|\,c) = 1 - \frac{W(c)}{W(x)} \tag{57}$$

*where* $W(x)$ *is defined in Theorem* 4.

Proof. This result immediately follows from Theorem 2 and Theorem 4.

The probability that in any busy period other than the initial one the maximal waiting time is $\leqslant x$ is given by

$$G^*(x) = 1 - \frac{W(0)}{W(x)} \tag{58}$$

for $x \geqslant 0$. (See J. W. Cohen [3].)

NOTE. First, let us remark that if $\lambda a < 1$, then

$$P\{\theta_0^* < \infty \mid \eta^*(0) = c\} = 1 - W(c) \tag{59}$$

where $W(0) = 1 - \lambda a$.

Second, consider the queuing process investigated in this section with the modification that the total time spent in the system by a customer cannot be greater than $m$ where $m$ is a positive number. In this case if $\eta_n$ denotes the waiting time immediately before the arrival of the $n$-th customer, then

$$\lim_{n\to\infty} P\{\eta_n \leqslant x\} = 1 - \frac{1}{W(m)} \int_0^{m-x} W(m-x-z)\,dH(z) \tag{60}$$

$$= 1 - \frac{W(m-x)}{W(m)} + \frac{W'(m-x)}{\lambda W(m)}$$

for $0 \leqslant x < m$. This result is due to D. J. Daley [4].

**References**


[1] J. W. Cohen, *The distribution of the maximum number of customers present simultaneously during a busy period for the queueing systems M/G/1 and G/M/1*, J. Appl. Prob. 4 (1967), pp. 162-179.

[2] J. W. Cohen, *The M/G/1 queueing system with finite waiting room*, Mathematisch Instituut der Technische Hogeschool Delft, Nederland, February 1967.

[3] J. W. Cohen, *Extreme value distribution for the M/G/1 and the G/M/1 queueing systems*, Mathematisch Instituut der Technische Hogeschool Delft, Nederland, March 1967.

[4] D. J. Daley, *Single-server queueing systems with uniformly limited queueing time*, J. Austral. Math. Soc. 4 (1964), pp. 489-505.

[5] P. D. Finch, *The effect of the size of the waiting room on a simple queue*, J. Roy. Stat. Soc. B 20 (1958), pp. 182-186.

[6] S. Karlin and J. McGregor, *Many server queueing processes with Poisson input and exponential service times*, Pacific J. Math. 8 (1958), pp. 87-118.

[7] J. Keilson, *The ergodic queue length distribution for queueing systems with finite capacity*, J. Roy. Stat. Soc. B 28 (1966), pp. 190-201.

[8] I. N. Kovalenko, *Some queuing problems with restrictions*, Theor. Prob. and Appl. 6 (1961), pp. 204-208.

[9] L. Takács, *On a combined waiting time and loss problem concerning telephone traffic*, Ann. Univ. Sci. Budapest Eötvös. Sect. Math. 1 (1958), pp. 73-82.

[10] L. Takács, *From ballot theorems to the theory of queues*, Columbia University Report CU-41-64-Nonr-266 (59), MS, March, 1964. (Abstract: Post Office Telecommunications Journal, Special Issue, Report of the Proceedings of the Fourth International Teletraffic Congress, London 1964, pp. 25-26.)

[11] L. Takács, *Application of ballot theorems in the theory of queues*, Proccedings of the Symposium on Congestion Theory, Chapel Hill, North Carolina, August 24-26.

1964. Edited by W. L. Smith and W. E. Wilkinson, University of North Carolina Press, Chapel Hill, North Carolina 1965, pp. 337-398.

[12] L. Takács, *Combinatorial methods in the theory of stochastic processes*, John Wiley and Sons, New York 1967.

[13] C. B. Winston, *Geometric distributions in the theory of queues*, J. Roy. Stat. Soc. B 21 (1959), pp. 1-35.

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