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On a certain transformation of complex series

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1. Notations. Throughout this text we will apply the following notations:

H_0 — the complex plane, i.e., the set of all complex numbers $z = x+iy$.

H_1 — the set of all pairs (∞, φ), $0 \leqslant \varphi < 2\pi$, for which we assume $|(\infty, \varphi)| = \infty$ and $\arg(\infty, \varphi) = \varphi$.

$K(z_0; \varepsilon) = \{z: |z-z_0| < \varepsilon\}$ if $z_0 \in H_0$; for $z_0 = (\infty, \varphi_0)$, $K(z_0; \varepsilon)$ is the open angular region whose vertex, bisector and angle are, respectively, $\varepsilon^{-1}e^{i\varphi_0}$, $\varphi = \varphi_0$ and 2ε; the arc of the infinite circle H_1 containing z_0 and adjacent to the described region is included in $K(z_0; \varepsilon)$.

H is the set $H_0 \cup H_1$ with topology generated by the system of neighbourhoods $K(z_0; \varepsilon)$.

Σ is the family of all complex sequences $S = \{z_n\}_{n=1}^{\infty}$ such that $z_n = x_n + iy_n \to 0$ as $n \to \infty$, and $z_n \neq 0$ for infinitely many n.

$S = X+iY$ means that $X = \{x_n\}_{n=1}^{\infty}$ and $Y = \{y_n\}_{n=1}^{\infty}$.

$|S| = |z_1|+|z_2|+\dots$

$L(S)$ is the set of all limit points of the sequence $S(+) = \{z_1+z_2+\dots+z_n\}_{n=1}^{\infty}$ in the space H.

θ — the family of all sequences $T = \{t_n = \pm 1\}_{n=1}^{\infty}$.

$TS = \{t_n z_n\}_{n=1}^{\infty}$.

$\Lambda(S)$ is the family of all sets $L(TS)$ for a fixed S from Σ and T running over θ.

$\Lambda_0(S)$ is the family of all those one-point sets in H_0 which belong to $\Lambda(S)$; the same symbol denotes the union of all such sets.

Γ is the family of all continua in H; one-point sets are regarded as continua.

G_1 is the point (∞, π) and the one-point set $\{(\infty, \pi)\}$.

G_2 is the point $(\infty, 0)$ and the one-point set $\{(\infty, 0)\}$.

2. Subject of the paper. It is well known that if $S \in \Sigma$ and $S = X$, then $L(S)$ coincides with the closed interval $[a, b]$, where $a = \varliminf X(+)$

and $b = \overline{\lim} X(+)$. The interval can reduce to a single point, and the latter may be $-\infty$ or $+\infty$. Moreover, if $|X| = \infty$, then $\Lambda(X)$ coincides with the family of all closed intervals $[a, b] \subset [-\infty, +\infty]$, i.e. with the family of all continua in the extended x-axis. In the case $|X| < \infty$, $\Lambda(X)$ reduces to $\Lambda_0(X)$, and the latter is a perfect set on the x-axis.

The simplicity of the structure of the sets $L(S)$ is extended to the complex case. The fact that $L(S)$ is a non-empty closed subset of H is an evident consequence of compactness of the space H. The assumption $z_n \to 0$ implies connectivity of $L(S)$. (For a simple proof of the latter property, see [1].)

What makes the complex case be more complicated than the real one are the distribution of sets from $\Lambda(S)$ in the space H and the fact that two sequences $S_1 = X_1 + iY_1$ and $S_2 = X_2 + iY_2$ with $|X_1| = |Y_1| = |X_2| = |Y_2| = \infty$ can determine two families $\Lambda(S_1)$ and $\Lambda(S_2)$ one of which, say $\Lambda(S_1)$, coincides with Γ whereas $\Lambda(S_2)$ consists of certain simple continua.

Several theorems about the family $\Lambda(S)$ as a function of S have been proved in paper [3] ([4] and [5] are its two-part summary). However, in order to be able to quote these theorems and to formulate the problem we are going to deal with, some simple notations need to be introduced.

A line P passing through the origin is called *principal axis of S* if for every open angular region R with vertex at the origin and with bisector P we have $\sum |z_n| = \infty$, where the summation runs exactly over all z_n which belong to R. Let $A(S)$ be the set of all principal axes of S. It is evident that $A(S) = \emptyset$ if $S \in \Sigma_1^0 = \{S \in \Sigma\colon |S| < \infty\}$. Now we define the following three disjoint subfamilies of Σ: Σ_2^0 consists of all S such that $|X| = \infty$, and $|Y| < \infty$; Σ_3^0 — the family of all S such that $|X| = |Y| = \infty$ and the x-axis is the only principal axis of S; Σ_4^0 — the family of all S such that both of the coordinate axes belong to $A(S)$. We put $\Sigma^0 = \bigcup_{k=1}^{4} \Sigma_k$.

Of course, Σ is essentially larger than Σ^0. Nevertheless, we may confine ourselves to the case $S \in \Sigma^0$. In fact, let $S \in \Sigma \setminus \Sigma^0$. Hence, $S \notin \Sigma_1^0$ and $A(S) \neq \emptyset$. If $\bar{\bar{A}}(S) = 1$ and a is the angle between the x-axis and the principal axis of S, let f be the rotation of H_0 by the angle $-a$. If $\bar{\bar{A}}(S) \geqslant 2$, let v_1 and v_2 be any two linearly independent unit vectors lying on principal axes of S and let f denote the affine transformation generated by the conditions $f(v_1) = 1$ and $f(v_2) = i$. Thus, with every S from $\Sigma \setminus \Sigma^0$ there are associated a non-generated affine mapping $f = f_S$ and a sequence $S' = \{f(z_k)\}_{k=1}^{\infty}$. It is easily seen that S' belongs always to Σ^0. We extend f to H_1 putting $z' = (\infty, \varphi') = f(z)$ for $z = (\infty, \varphi)$ in such a way that $\varphi' = \arg f(e^{i\varphi})$. This extended f is one-to-one and continuous in H. Every continuum $L(TS)$ from $\Lambda(S)$ is the image of $L(TS')$ from $\Lambda(S')$, under the map f^{-1}, and it does not change any of these essential properties of $L(TS')$ we are interested in.

For every $S \in \Sigma^0$, the following subfamilies of $\Lambda(S)$ will be distinguished: $\Phi_0(S)$ — the family of all sets from $\Lambda(S)$ which reduce to single points in H_0 or to finite segments parallel to the x-axis; $\Phi_1(S)$ — the family consisting of two one-point sets G_1 and G_2 and of all half-lines and lines from $\Lambda(S)$ which are parallel to the x-axis; $\Phi(S) = \Phi_0(S) \cup \Phi_1(S)$; $\Omega(S)$ — the family of all sets from $\Lambda(S)$ which are unions of half-lines parallel to the x-axis and of arcs in the infinite circle H_1 which contain at least one of the points G_1 and G_2. (A more strict definition of $\Omega(S)$ is given in [3], p. 34.) Of course, we have $\Lambda_0(S) \subset \Phi_0(S) \subset \Phi(S)$ and $\Phi_1(S) \subset \Omega(S) \subset \Lambda(S)$.

The case $S \in \Sigma_1^0$ is well-known: $\Lambda(S)$ reduces to $\Lambda_0(S)$ which is a perfect set.

The case $S \in \Sigma_4^0$ is completely described by Theorem 2, [3], p. 20, saying that

$$\Lambda(S) = \Gamma \tag{1}$$

for every $S \in \Sigma_4^0$.

For $S \in \Sigma_2^0$ we have $\Lambda(S) = \Phi(S)$, which is an evident consequence of the definitions. In this case the family $\Omega(S)$ reduces to $\Phi_1(S)$.

The case $S \in \Sigma_3^0$ is the most difficult of all of the four. The details concerning the family $\Phi_0(S)$ are given in [3], Chapter IV. It is worth mentioning that the distribution of the segments from $\Phi_0(S)$ in the plane H_0 may be very complicated. As for the family $\Omega(S)$ it has been fully described in [3], Chapter V. One of the examples given in [3], Chapter VI, shows that there are $S \in \Sigma_3^0$ for which we have

$$\Lambda(S) = \Phi_0(S) \cup \Omega(S) . \tag{1'}$$

On the other hand, another example from the same chapter shows that there are $S \in \Sigma_3^0$ realising (1).

The two examples gave rise to pose the question whether cases (1) and (1′) are the only ones which can occur within Σ_3^0. The positive answer to this question is the main subject of this paper.

3. Scheme of the paper. Let $W = \{w_j\}_{j=1}^k$ be any sequence of complex numbers or vectors $w_j = u_j + iv_j$ and let $c_0 = a_0 + ib_0$ be any complex number or vector. Placing the origin of the vector w_1 at the point c_0 we obtain a point $c_1 = a_1 + ib_1 = c_0 + w_1$. In the same way we define points $c_j = a_j + ib_j = c_0 + w_1 + w_2 + \ldots + w_j$ for $j = 2, 3, \ldots, k$. We denote by $B[c_0, W]$ the orientated polynomial line with successive vertices $c_0, c_1, \ldots, c_k$. $B_x[c_0, W]$ and $B_y[c_0, W]$ are lengths of the projections of $B[c_0, W]$ on the coordinate axes.

We shall express properties (1) and (1′) of $S = X+iY$ from Σ^0 by means of the following simple conditions:

(2) *There is a $T \in \theta$ such that $TX(+)$ converges whereas $TY(+)$ diverges.*

(2′) *There is a $T \in \theta$ such that $TX(+)$ converges whereas* $\overline{\underline{\lim}}\, TY(+) = \begin{cases} +\infty \\ -\infty \end{cases}$.

(2″) *There is a $T \in \theta$ and an infinite sequence of positive integers $n_1 < p_1 < n_2 < p_2 < \ldots$ such that $B_x[0, TS(n_k, p_k)] \to 0$, and $B_y[0, TS(n_k, p_k)] \to \delta$, $\delta > 0$, as $k \to \infty$, where $TS(n_k, p_k)$ is the sequence with successive terms $t_{n_k+1}z_{n_k+1}, t_{n_k+2}z_{n_k+2}, \ldots, t_{p_k}z_{p_k}$.*

(3) *There is a $T \in \theta$ giving $TX(+)$ bounded an $TY(+)$ divergent.*

(3′) *There is a $T \in \theta$ for which $TX(+)$ is bounded whereas* $\overline{\underline{\lim}}\, TY(+) = \begin{cases} +\infty \\ -\infty \end{cases}$.

(3″) *There is a $T \in \theta$, two positive numbers a and b and a sequence of positive integers $n_1 < p_1 < n_2 < p_2 < \ldots$ such that $B_y[0, TS(n_k, p_k)] \geqslant b$ and $B_x[0, TS(n_k, p_k)] \leqslant a$ for all k.*

(4) *There is a $T \in \theta$ such that $TY(+)$ diverges and $L(TS)$ contains none of the points G_1 and G_2.*

(5) *There is a sequence $I = \{i_j = 0, \pm 1\}_{j=1}^{\infty}$, a positive number r and an infinite sequence of positive integers $n_1 < p_1 < n_2 < p_2 < \ldots$ such that $B_y[0, IS(n_k, p_k)] \geqslant r\big(B_x[0, IS(n_k, p_k)]+1\big)$ for all k.*

It is evident that none of the ten conditions hold for $S \in \Sigma_1^0 \cup \Sigma_2^0$, if the trivial case $\Omega(S) = \Phi_1(S)$ is excluded, and that (1) implies all the others with the exception of (1′). Hence, in view of Theorem 2 we have mentioned, every S from Σ_4^0 satisfies all the ten conditions with the exception of (1′).

The non-primed conditions have been arranged in such a way that the implications $(k) \Rightarrow (k+1)$, $k = 1, 2, 3, 4$, are evident from the definitions. Thus we see that the problem we have posed will be solved completely if we prove the following theorems:

THEOREM A. *If $S \in \Sigma_3^0$, then* (5) *implies* (1).

THEOREM B. *If $S \in \Sigma_3^0$ and* (5) *does not hold, then* (1′) *holds.*

In our proof of Theorem A we confine ourselves to implication (5) $\Rightarrow$ (2″), for that (2″) $\Rightarrow$ (1) was proved in [3], Theorem 1, p. 18. The proof of (5) $\Rightarrow$ (2″) will be divided into three steps: (5) $\Rightarrow$ (3″), (3″) $\Rightarrow$ (3′) and (3′) $\Rightarrow$ (2″).

In the proof of Theorem B we shall refer to a partial result from paper [3], Theorem 13, p. 43.

4. Proof of (5) $\Rightarrow$ (3″).

LEMMA I. *Every sequence* $W = \{w_j\}_{j=1}^{k}$ *of complex numbers contains a subsequence* W' *such that*

$$B_y[0, W'] \geqslant 2^{-1} B_y[0, W] \tag{6}$$

and

$$B_x[0, W'] \leqslant 2^{-1} B_x[0, W] + m(W), \tag{7}$$

where $m(W) = \max_{1 \leqslant j \leqslant k} |w_j|$.

Proof. In each of the following cases $a = B_x[0, W] = 0$, $m = m(W) = 0$ and $m \geqslant 2^{-1}a$, $W' = W$ satisfies (6) and (7). In the case $\beta = B_y[0, W] = 0$, $\{w_1\}$ may be taken for W'.

Thus we may assume additionally that

$$a > 0, \quad \beta > 0 \quad \text{and} \quad 2m < a. \tag{8}$$

Let A be the smallest of all the rectangles with sides parallel to the coordinate axes which contain $B[0, W]$ and let $z = x + iy$ be the centre of A. By (8), A reduces neither to a point nor to a segment and the rectangle $e_5 e_7 e_8 e_{10}$ in Fig. 1 is contained in $A = e_1 e_2 e_3 e_4$.

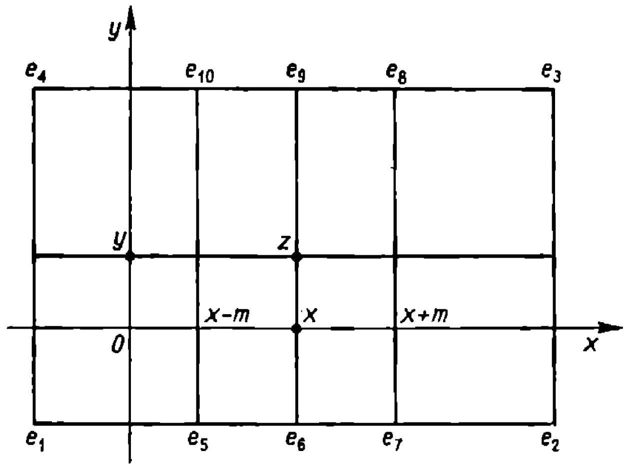

Fig. 1

By the definition of A, the origin 0 belongs to A. We consider the case when the origin 0 belongs to the closed rectangle $e_1 e_6 e_9 e_4$, the other case being analogous. In this case the one-side polygonal line $B[0, W(0, 1)]$ is contained in the rectangle $e_1 e_7 e_8 e_4$, in view of (8).

Let k_1 be the greatest of all the numbers 1 such that $B[0, W(0, 1)] \subset e_1 e_7 e_8 e_4$. If $B_y[0, W(0, k_1)] \geqslant 2^{-1} \cdot \beta$, the proof is finished, for $W' = W(0, k_1)$ satisfies (6) and (7). Thus, let $B_y[0, W(0, k_1)] < 2^{-1} \cdot \beta$. Of course, we have $1 \leqslant k_1 < k$, $c_{k_1} = w_1 + w_2 + \ldots + w_{k_1} \in e_1 e_7 e_8 e_4$ and c_{k_1+1} lies on the right-hand side of the line $e_7 e_8$. Let k_2 be the greatest of all the numbers 1 such that $B[c_{k_1}, W(k_1, 1)] \subset e_5 e_2 e_3 e_{10}$. If it were $B_y[c_{k_1},$

$W(k_1, k_2)] \geqslant 2^{-1}\cdot\beta$, the proof would be completed, for $W' = W(k_1, k_2)$ would satisfy (6) and (7). So, let $B_y[c_{k_1}, W(k_1, k_2)] < 2^{-1}\cdot\beta$. In this case it must be $k_2 < k$, for otherwise we would have $\beta \leqslant B_y[0, W(0, k_1)] + B_y[c_{k_1}, W(k_1, k_2)] < 2^{-1}\cdot\beta + 2^{-1}\cdot\beta = \beta$, which is impossible.

By the definition of k_2, the vertex c_{k_2+1} lies on the left-hand side of the line e_5e_{10}. This means that the polygonal line $B[0, W']$, $W' = \{w_1, w_2, \ldots, w_{k_1}, w_{k_2+1}\}$, is contained in the strip between the lines e_5e_{10} and e_7e_8. Let k_3 be the greatest of all the numbers 1 such that the polygonal line $B[0, W(0, k_1) \cup W(k_2, 1)]$ is contained in the strip between these lines. If we had $B_y[0, W(0, k_1) \cup W(k_2, k_3)] \geqslant 2^{-1}\cdot\beta$, the proof would be completed, for $W' = W(0, k_1) \cup W(k_2, k_3)$ satisfies (6) and (7). This certainly occurs when $k_3 = k$. Indeed, in view of $B_y[c_{k_1}, W(k_1, k_2)] < 2^{-1}\beta$, the complementary subsequence $W(0, k_1) \cup W(k_2, k)$ must satisfy (6). Thus $k_3 < k$.

By the definition of k_3, the point $c' = c_{k_1} + w_{k_2+1} + w_{k_2+2} + \ldots + w_{k_3}$ belongs to the strip between e_6e_9 and e_7e_8 whereas $c' + w_{k_3+1}$ lies on the right-hand side of e_7e_8. Hence, in view of the fact that c_{k_2} belongs to the strip between e_5e_{10} and e_6e_9, the point $c_{k_2} + w_{k_3+1}$ is placed between e_5e_{10} and e_7e_8. We denote by k_4 the greatest of all the numbers 1 such that the polygonal line $B[c_{k_1}, W(k_1, k_2) \cup W(k_3, 1)]$ is contained between e_5e_{10} and e_7e_8. If we had $B_y[c_{k_1}, W(k_1, k_2) \cup W(k_3, k_4)] \geqslant 2^{-1}\cdot\beta$, the proof would be finished. This case certainly occurs when $k_4 = k$. In fact, in view of $B_y[0, W(0, k_1) \cup W(k_2, k_3)] < 2^{-1}\cdot\beta$, the complementary subsequence $W(k_1, k_2) \cup W(k_3, k_4)$ must satisfy (6) and (7).

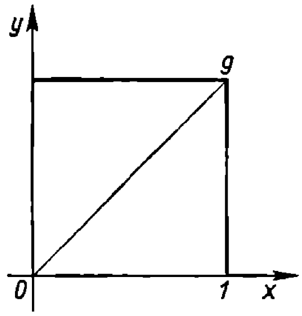

Fig. 2

From the above construction it is seen how to split W into two disjoint subsequences $W' = W(0, k_1) \cup W(k_2, k_3) \cup \ldots$ and $W'' = W(k_1, k_2) \cup W(k_3, k_4) \cup \ldots$ each of which satisfies (7). The fact that for every splitting, $W = W' \circ W''$, of W into two disjoint subsequences W' and W'' the sum $B_y[0, W'] + B_y[0, W'']$ is not less than $B_y[0, W]$, implies evidently that at least one of the two subsequences W and W' must satisfy (6).

Remark. In the general case the coefficient 2^{-1} in (6) cannot be replaced by any number $d = 2^{-1} + e$, where $e > 0$. In fact, if $e > 2^{-1}$, then every sequence W with terms of the type $w_n = iv_n$, $v_n > 0$, can be taken as a counterexample. So, let $e \leqslant 2^{-1}$ and let h be any positive number less than $2^{-1}e$. We denote by W_h any finite sequence such that $B[0, W_h]$ coincides with the diagonal $0g$ in Fig. 2 and $m(W_h) \leqslant h$. It is evident that for every subsequence W'_h of W_h satisfying (7) we have $B_y[0, W'_h] \leqslant 2^{-1}+2^{-1}h = 2^{-1}(1+h)$, whereas (6) with d instead of 2^{-1} gives $B_y[0, W'_h] \geqslant 2^{-1}+e = 2^{-1}(1+2e)$, a contradiction.

LEMMA II. *To every non-negative integer q in every sequence $W = \{w_j\}_{j=1}^{k}$ of complex numbers a subsequence W' can be chosen in such a way that*

$$B_y[0, W'] \geqslant 2^{-q}B_y(0, W) \tag{9}$$

and

$$B_x[0, W'] \leqslant 2^{-q}B_x[0, W]+2m(W) . \tag{10}$$

Proof. For $q = 0$ we may put $W' = W$. For $q = 1$ the lemma is true, in view of Lemma I. If $q \geqslant 2$, then applying q times Lemma I one obtains (9) and (10).

LEMMA III. *To every sequence $W = \{w_j\}_{j=1}^{k}$, a sequence $T = \{t_j = \pm 1\}_{j=1}^{k}$ can be defined in such a way that*

$$|t_1w_1+t_2w_2+\ldots+t_jw_j| < m(W)\sqrt{3} \quad \textit{for } j = 1, 2, \ldots, k . \tag{11}$$

This lemma was proved in [2].

LEMMA IV. *Suppose that a sequence $W = \{w_j\}_{j=1}^{k}$, two positive numbers, a and β, and a sequence $I = \{i_j = 0, \pm 1\}_{j=1}^{k}$ has been chosen in such a way that $B_y[0, IW] \geqslant \beta$ and $B_x[0, IW] \leqslant a$. Let, moreover, q be any non-negative integer satisfying*

$$2^{-q}\cdot\beta \geqslant 2\sqrt{3}\,m(W) . \tag{12}$$

Then there is a sequence $T = \{t_j = \pm 1\}_{j=1}^{k}$ such that

$$B_y[0, TW] \geqslant 2^{-q}\cdot\beta - 2\sqrt{3}\,m(W) \tag{13}$$

and

$$B_x[0, TW] \leqslant 2^{-q}\cdot a + 2(1+\sqrt{3})m(W) . \tag{14}$$

Proof. Let $I = I' \circ I''$, where I'' encloses all zeros of I and only zeros. By Lemma II, $I'W'$ has a subsequence I^*W^* such that

$$B_y[0, I^*W^*] \geqslant 2^{-q}\cdot\beta , \tag{15}$$

and

$$B_x[0, I^*W^*] \leqslant 2^{-q}\cdot a + 2m(W) . \tag{16}$$

Now let $W^{**} = \{w_{p_1}, w_{p_2}, \ldots, w_{p_s}\}$ be defined by $W = W^* \circ W^{**}$. Lemma III assures the existence of a $T^{**} = \{t_{p_j} = \pm 1\}_{j=1}^{s}$ such that

$$|t_{p_1}w_{p_1}+t_{p_2}w_{p_2}+\ldots+t_{p_j}w_{p_j}| < \sqrt{3}\,m(W) , \quad i = 1, 2, \ldots, s . \tag{17}$$

Putting $T = I^* \circ T^{**}$ we see that (13) and (14) are immediate consequences of (15)-(17).

LEMMA V. (5) $\Rightarrow$ (3″).

Proof. Let (5) hold. Without any restriction of the generality we may confine ourselves to two cases: (a) there is an l such that $l_k = B_x[0, IS(n_k, p_k)] \leqslant l$ for all k and (b) $2^k < l_k \leqslant 2^{k+1}$ for all k.

Case (a). We have $B_x[0, IS(n_k, p_k)] \leqslant l$ and $B_y[0, IS(n_k, p_k)] \geqslant r$ for all k, and $r \geqslant 2\sqrt{3}\,m\big(IS(n_k, p_k)\big)$ for all k sufficiently large, in view of the fact that $z_j \to 0$ as $j \to \infty$. By Lemma IV, if $q = 0$, there is a block of signes $T(n_k, p_k)$ such that $B_y[0, TS(n_k, p_k)] \geqslant r - 2\sqrt{3}\,m\big(TS(n_k, p_k)\big)$ and $B_x[0, TS(n_k, p_k)] \leqslant l + 2(1+\sqrt{3})m\big(TS(n_k, p_k)\big) \leqslant l + 2(1+\sqrt{3})$ for all k sufficiently large. Hence, (3″) holds for $b = 2^{-1}r$ and $a = l + 2(1+\sqrt{3})$.

Case (b). By (5), we have $B_y[0, IS(n_k, p_k)] \geqslant r(2^k+1)$ for all k. Putting $\beta = r(2^k+1)$, $a = 2^{k+1}$, $q = k$, and applying Lemma IV we see that there is a sequence $T(n_k, p_k)$ such that $B_y[0, TS(n_k, p_k)] \geqslant 2^{-k} r(2^k+1) - 2\sqrt{3}\,m\big(TS(n_k, p_k)\big)$ and $B_x[0, TS(n_k, p_k)] \leqslant 2^{-k}\cdot 2^{k+1} + 2(1+\sqrt{3})m\big(TS(n_k, p_k)\big)$. In other words, (3″) holds for $a = 3$ and $b = 2^{-1}r$.

5. Proof of (3″) $\Rightarrow$ (3′). In view of property (3″), we shall not restrict the generality if we assume that there are two positive numbers a and b, and an infinite sequence of segments $S(n_k, p_k)$, $n_1 < p_1 < n_2 < p_2 < \ldots$, such that the following conditions hold for all k:

(18) $B_y[0, S(n_k, p_k)] \geqslant b$ *and* $B[0, S(n_k, p_k)]$ *is completely above the* x*-axis*;

(19) *the projection of* $B[0, S(n_k, p_k)]$ *on the* x*-axis is contained in the interval* $[-a, a]$;

$$|z_j| < 2^{-6}\cdot 2^{-k}\min(a, b) \quad \textit{for all } j > n_k. \tag{20}$$

It is clear that (3″) implies $S \notin \Sigma_1^0$. Since, moreover, $S \in \Sigma^0$, we may refer to Lemma 15, [3], p. 15. By this lemma, we may assume our sequence S have the property expressed by the following

LEMMA VI. *To every positive integer k and to every sequence of signes $T(n_1, p_k)$ there is a number l_k, $n_{l_k} > p_k$, and a sequence of signes $T(p_k, n_{l_k})$ such that putting $\gamma_{p_k} = a_{p_k} + i\beta_{p_k} = t_{n_1+1}z_{n_1+1} + t_{n_1+2}z_{n_1+2} + \ldots + t_{p_k}z_{p_k}$ we obtain a polygonal line $B_k = B[\gamma_{p_k}, TS(p_k, n_{l_k})]$ for which the following conditions hold*:

(21) B_k *is contained in the rectangle* $a_1a_2a_3a_4$ *in Fig.* 3, *where* $|a_1a_4| = |a_1a_5| = |a_8a_2| = 2^{-3}\cdot 2^{-k}\min(a, b)$;

(22) *the last vertex of* B_k *belongs to the square* $a_1a_5a_8a_4$.

Proof. We have to prove the existence of a $T \epsilon \theta$ for which $\overline{\lim}\, TY(+) = \begin{cases} +\infty \\ -\infty \end{cases}$, and $TX(+)$ is bounded. Such a T will be constructed in blocks of the types $T(n_k, p_k)$ and $T(p_k, n_{l_k})$.

Let r_1 be the smallest of all positive integers r such that $rb - 2^{-3}b \geqslant 1$. The signes of the first block $T(n_{q_0}, p_{q_0})$, $q_0 = 1$, will be all $+1$, i.e., $TS(n_1, p_1) = S(n_1, p_1)$. The second block $T(p_{q_0}, n_{q_1})$ $q_1 = l_1$, will be defined so that (21) and (22) hold for $k = 1$. Suppose that all the blocks $T(n_{q_0}, p_{q_0})$, $T(p_{q_0}, n_{q_1})$, ..., $T(n_{q_{j-1}}, p_{q_{j-1}})$, $T(p_{q_{j-1}}, n_{q_j})$, $q_j = l_{q_{j-1}}$, $j < r_1$, have already been defined. Then the blocks $T(n_{q_j}, p_{q_j})$ and $T(p_{q_j}, n_{q_{j+1}})$, $q_{j+1} = l_{q_j}$, are defined in the same way as $T(n_{q_0}, p_{q_0})$ and $T(p_{q_0}, n_{q_1})$. Thus we may assume that the blocks $T(n_{q_j}, p_{q_j})$ and $T(p_{q_j}, n_{q_{j+1}})$ are defined for $j = 0, 1, 2, ..., r_1 - 1$.

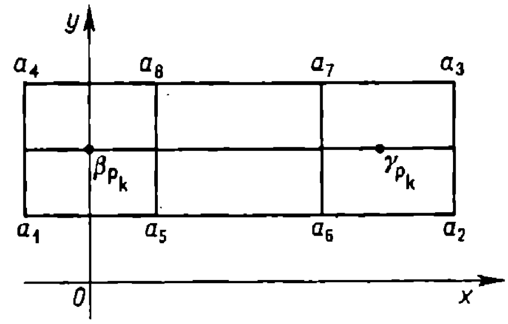

Fig. 3

By (18) and by our definition of the blocks $T(n_{q_j}, p_{q_j})$, the sum of all $t_m y_m$ which belong to the segments $TS(n_{q_j}, p_{q_j})$, $0 \leqslant j \leqslant r_1 - 1$, is non-smaller than $r_1 b$. On the other hand, in view of (21), the absolute value of the sum of all $t_m y_m$ which belong to the blocks $TS(p_{q_j}, n_{q_j})$, $0 \leqslant j \leqslant r_1 - 1$, is non-greater than $2^{-3}b$. Hence and from the definition of r_1, we see that

$$(23) \qquad b_1 = t_{n_1+1} y_{n_1+1} + t_{n_1+2} y_{n_1+2} + \ldots + t_{n_{q_{r_1}}} y_{n_{q_{r_1}}} \geqslant 1 \,.$$

Now, let r_2 be the smallest of all positive integers $r \geqslant r_1$ such that $2^{-1} \cdot 2^{-q_{r_1}} \cdot b - (r_2 - r_1) b \leqslant -2$ and let the signes of the block $T(n_{q_{r_1}}, p_{q_{r_1}})$ be all -1. The signes of the next block $T(p_{q_{r_1}}, n_{q_{r_1+1}})$ are defined so that (21) and (22) hold for $k = q_{r_1}$. The same method is applied to all the further pairs of blocks $T(n_{q_j}, p_{q_j})$ and $T(p_{q_j}, n_{q_{j+1}})$, $j = r_1 + 1, r_1 + 2, ..., r_2 - - r_1 - 1$. By (18), (21) and the definition of r_2, we have

$$(24) \qquad b_2 = t_{n_1+1} y_{n_1+1} + t_{n_1+2} y_{n_1+2} + \ldots + t_{n_{q_{r_2}}} y_{n_{q_{r_2}}} \leqslant -2 \,.$$

We have shown a construction of the block $T(n_{q_0}, n_{q_{r_2}}) = T(n_{q_0}, n_{q_{r_1}}) \cup \cup T(n_{q_{r_1}}, n_{q_{r_2}})$ which gives (23) and (24). From this construction it is seen how we have to define the blocks $T(n_{q_{r_j}}, n_{q_{r_{j+1}}})$, $j = 2, 3, \ldots$, giving

$$(25)\qquad b_j = t_{n_1+1}y_{n_1+1} + t_{n_1+2}y_{n_1+2} + \ldots + t_{n_{q_{r_j}}}y_{n_{q_{r_j}}}\begin{cases}\geqslant j\,, & j = 2m-1\,,\\ \leqslant -j\,, & j = 2m\,.\end{cases}$$

Let $T = \{1\}_{j=1}^{n_1} \cup T(n_{q_0}, n_{q_{r_1}}) \cup T(n_{q_{r_1}}, n_{q_{r_2}}) \cup \ldots$ In view of (25), $\overline{\underline{\lim}}\, TY(+) = \begin{cases}+\infty\\ -\infty\end{cases}$. What is still to be proved is the boundedness of $TX(+)$. But the latter property is an evident consequence of the construction, in view of (19), (21) and (22).

6. Proof of $(3') \Rightarrow (2'')$. Let S have property $(3')$. We shall not restrict the generality if we assume

$$(26)\qquad \overline{\underline{\lim}}\, Y(+) = \begin{cases}+\infty\\ -\infty\end{cases}, \quad |x_1+x_2+\ldots+x_j| < A\,, \quad j = 1, 2, \ldots$$

It suffices to prove that given $\varepsilon > 0$, natural numbers n and p, $n < p$, and a block of signes $T(n, p)$ can be found so that the polygonal line $B[0, TS(n, p)]$ satisfies the following conditions:

$$(27)\qquad B_x[0, TS(n, p)] < \varepsilon\,,$$

$$(28)\qquad 1-\varepsilon \leqslant B_y[0, TS(n, p)] \leqslant 1+\varepsilon\,.$$

Let q be the smallest of all non-negative integers r such that $2^{-r}\cdot 4A < \varepsilon$ and let $b = 2^q$. In view of (26), there exists a segment $S(n, p)$ such that $s_n = z_1+z_2+\ldots+z_n \in a_1a_2a_8a_5$, Fig. 4, $s_p = z_1+z_2+\ldots+z_p \in a_8a_7a_3a_4$, $|a_1a_2| = 2A$, and

$$(29)\qquad B[s_n, S(n, p)] \subset a_1a_2a_3a_4\,.$$

Of course, we may assume that

$$(30)\qquad |z_j| < 2^{-4}\cdot\varepsilon \quad \text{for all } j \geqslant n+1\,.$$

By Lemma II, $S(n, p)$ has a subsequence $S'(n, p)$ for which we have

$$(31)\qquad B_y[0, S'(n, p)] \geqslant 2^{-q}\cdot B_y[0, S(n, p)] \geqslant 2^{-q}\cdot 2^{+q} = 1$$

and

$$(32)\qquad B_x[0, S'(n, p)] \leqslant 2^{-q}\cdot 2A + 2\cdot 2^{-4}\cdot\varepsilon < (2^{-1}+2^{-3})\varepsilon\,.$$

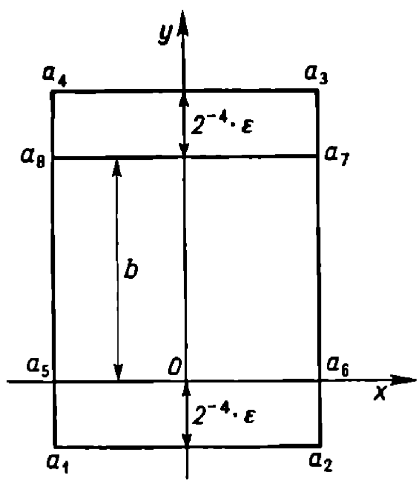

Fig. 4

Considering (30), we may additionally impose

$$(33)\qquad B_y[0, S'(n, p)] \leqslant 1+2^{-4}\cdot\varepsilon\,.$$

We denote by $S''(n, p)$ the complementary subsequence of $S'(n, p)$. In the same way we split sequence $T(n, p)$ we are seeking for all the terms of $T'(n, p)$ will be $+1$. Referring to Lemma III and to (30) we define $T''(n, p)$ in such a way that

$$(34) \qquad B_t[0, T''S''(n, p)] \leqslant 2\cdot 2^{-4}\sqrt{3}\cdot\varepsilon \quad \text{for } t = x \text{ and } t = y\,.$$

It is seen that (31)-(34) imply (27) and (28), which ends the proof.

7. Proof of Theorem B. We have to prove that the negation of (5) implies (1′). Let $S \in \Sigma_3^0$, $T \in \theta$, and let us suppose (5) does not hold. We shall consider all the cases that can occur.

(a) $TY(+)$ converges. In this case only the following eventualities can occur: $\lim TX(+) = -\infty$ and $L(TS) = G_1$, $\lim TX(+) = +\infty$ and $L(TS) = G_2$, $-\infty = \underline{\lim}\, TX(+) < \overline{\lim}\, TX(+) < +\infty$ and $L(TS)$ is a 'left' half-line parallel to the x-axis, $-\infty < \underline{\lim}\, TX(+) < \overline{\lim}\, TX(+) = +\infty$ and $L(TS)$ is a 'right' half-line parallel to the x-axis, $-\infty = \underline{\lim}\, TX(+) < \overline{\lim}\, TX(+) = \infty$ and $L(TS)$ is a line parallel to the x-axis, $-\infty < \underline{\lim}\, TX(+) \leqslant \overline{\lim}\, TX(+) < +\infty$ and $L(TS)$ is a segment parallel to the x-axis. In other words, in each of the above eventualities $L(TS)$ belongs to $\Phi_0(S) \cup \Phi_1(S)$.

(b) $TY(+)$ diverges and $L(TS) \cap H_0 = \emptyset$. Since (4) does not hold either, at least one of the points G_1 and G_2 must belong to $L(TS)$. If exactly one of the points G_1 and G_2 belongs to $L(TS)$, say G_1, then $L(TS)$ as a connected subset of H_1 is an arc containing G_1, and as such, by the definition, belongs to $\Omega(S)$. If $G_1 \cup G_2 \subset L(TS)$, then $L(TS)$ is a union of two arcs containing G_1 and G_2, respectively, and as such, by the definition, belongs to $\Omega(S)$.

(c) $TY(+)$ diverges and $L(TS) \cap H_0 \neq \emptyset$. Let us decompose $L(TS)$ into $L_0(TS) = L(TS) \cap H_0$ and $L_1(TS) = L(TS) \cap H_1$. Of course, $L_0(TS) \neq \emptyset$. Since (4) does not hold either, $L_1(TS) \neq \emptyset$, too.

Referring to the definition of $\Omega(S)$, it suffices to prove that each of the components $L_0(TS)$ and $L_1(TS)$ belongs to Ω, i.e., that the components are unions of sets from the family $G_1 \cup G_2 \cup \Omega_1 \cup \Omega_2$, where $\Omega_1(\Omega_2)$ consists of all left (right) half-lines parallel to the x-axis and of all arcs of H_1 containing $G_1(G_2)$.

A particular case of $L_0(TS)$ has already been examined in [3], p. 43, Theorem 13. This theorem says that if an $S \in \Sigma_3^0$ has a property $(-)$, p. 42, and if

$$(35) \qquad -\infty < \underline{\lim}\, TY(+) < \overline{\lim}\, TY(+) < +\infty,$$

then $L_0(TS) \in \Omega(S)$.

Hence, since $(5) \Leftrightarrow (3'')$ (Lemma V) and the negation of (3″) implies property $(-)$, $L_0(TS) \in \Omega(S)$ for every T satisfying (35).

The conclusion of Theorem 13 is true for every T satisfying a weaker condition

$$-\infty \leqslant \underline{\lim}\, TY(+) < \overline{\lim}\, TY(+) \leqslant +\infty\,, \tag{36}$$

i.e. for every T we deal with in case (c). Since the proof of this new theorem is almost identical with the proof of Theorem 13, we omit the details.

As for $L_1(TS)$ we have to prove that to every point $z = (\infty, \varphi) \in L_1(TS)$ there is an arc A contained in $L_1(TS)$, such that at least one of the points G_1 and G_2 belongs to A.

The case $L_1(TS) = G_1$, $L_1(TS) = G_2$ and $L_1(TS) = G_1 \cup G_2$ are evident. So, let $L_1(TS)$ contain a point $z = (\infty, \varphi)$ different from G_1 and G_2. It suffices to prove that to every $\varepsilon > 0$ there is a segment $TS(n, p)$ such that the polygonal line $B = B[w_n, TS(n, p)]$, $w_n = u_n + iv_n = t_1z_1 + t_2z_2 + \ldots + t_nz_n$, satisfies the following conditions:

$$\begin{gathered} B \cap K(0;\varepsilon^{-1}) = \emptyset\,, \qquad B \cap K(z;\varepsilon) \neq \emptyset\,, \\ B \cap [K(G_1;\varepsilon) \cup K(G_2;\varepsilon)] \neq 0\,. \end{gathered} \tag{37}$$

We confine ourselves to the case $0 < \varphi < 2\pi$, the case $\pi < \varphi < 2\pi$ being analogous.

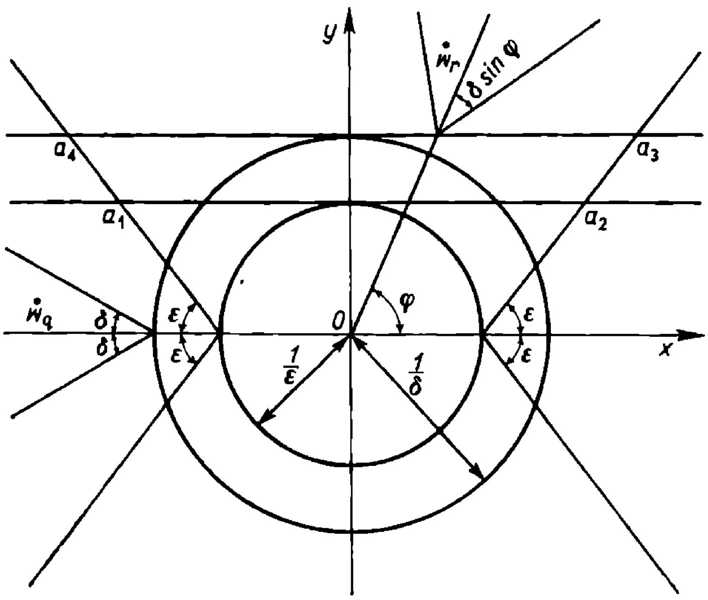

Fig. 5

Let ε be any fixed positive number. For the sake of simplicity we assume $\varepsilon \leqslant \min(\varphi, \pi - \varphi)$. Since $z = (\infty, \varphi) \in L(TS)$ and at least one of the points G_1 and G_2, say G_1, belongs to $L(TS)$, to every δ, $0 < \delta \leqslant \varepsilon$, a pair of positive integers q and r, $q < r$, can be chosen in such a way that

$$w_q \in K(G_1;\delta)\,, \qquad w_r \in K(z;\delta \sin\varphi)\,. \tag{38}$$

Fig. 5 illustrates the position of the partial sums w_q and w_r.

There are only two possibilities: (i) there is a positive $\delta \leqslant \varepsilon$ such that $B[w_q, TS(q, r)] \cap K(0; \varepsilon^{-1}) = \emptyset$ and (ii) $B[w_q, TS(q, r)] \cap K(0; \varepsilon^{-1}) \neq \emptyset$ for every $\delta \leqslant \varepsilon$.

In case (i) $TS(q, r)$ may be taken for $TS(n, p)$, for (37) holds for such a substitution.

In case (ii) we cut $TS(q, r)$ into two pieces $TS(q, n)$ and $TS(n, r)$ in such a way that n is the smallest of all k such that $TS(k, r)$ is wholly above the line a_1a_2 in Fig. 5. Of course, $TS(n, r)$ and $K(0; \varepsilon^{-1})$ are disjoint for every $\delta \leqslant \varepsilon$. Now, again, there are only two possibilities: 1° there is a $\delta \leqslant \varepsilon$ such that $B[w_n, TS(n, r)] \cap [K(G_1; \varepsilon) \cup K(G_2; \varepsilon)] \neq \emptyset$ and 2° for every $\delta \leqslant \varepsilon$, $B[w_n, TS(n, r)]$ is contained in the trapezoidal region between the lines a_1a_4, a_1a_2 and a_2a_3 in Fig. 5.

In case 1°, $TS(n, p) = TS(n, r)$ satisfies (37). We shall show that case 2° does not hold. In fact, we have $B_y = B_y[w_n, TS(n, r)] \geqslant \delta^{-1} - \varepsilon^{-1} - a$, where $a = \max\limits_{j \geqslant n} |z_j|$, and $B_x = B_x[w_n, TS(n, r)] \leqslant 2 \cdot \varepsilon^{-1} + 2\varepsilon^{-1} \operatorname{ctg} \varepsilon + 2(B_y + a) \operatorname{ctg} \varepsilon$. Hence, after a very simple calculation, $B_y \geqslant (1 + 2 \operatorname{ctg} \varepsilon)^{-1}(1 + B_x)$ for all δ sufficiently small, which implies property (5), contrary to our assumption. This ends the proof of Theorem B.

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