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ON APPROXIMATE SYMMETRIC DERIVATIVE

BY

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1. Let f be a real function defined in an open interval I of which $[a, b]$ is a closed subinterval. Let

$$\varphi(x, h) = \frac{f(x+h)-f(x-h)}{2h}, \quad x\in[a, b], x\pm h\in I, h \neq 0.$$

Then the *upper* and the *lower symmetric derivative* of f at x is defined by

$$\bar{f}^{(1)}(x) = \limsup_{h\to 0}\varphi(x, h) \quad \text{and} \quad \underline{f}^{(1)}(x) = \liminf_{h\to 0}\varphi(x, h),$$

respectively [5]. If $\bar{f}^{(1)}(x) = \underline{f}^{(1)}(x)$, then the common value, denoted by $f^{(1)}(x)$, is the *symmetric derivative* [3] of f at x.

The approximate upper limit and the approximate lower limit of $\varphi(x, h)$ as $h\to 0$ are respectively called the *approximate upper* and the *approximate lower symmetric derivative* of f at x and are denoted by $\bar{f}^{(1)}_{\mathrm{ap}}(x)$ and $\underline{f}^{(1)}_{\mathrm{ap}}(x)$ respectively [6]. That is, $\bar{f}^{(1)}_{\mathrm{ap}}(x)$ is the lower bound of all numbers y (including $+\infty$) for which the set $\{h: h > 0;\ \varphi(x, h) > y\}$ has the point zero as the point of dispersion ([10], p. 128).

Similar is the case of $\underline{f}^{(1)}_{\mathrm{ap}}(x)$. If $\bar{f}^{(1)}_{\mathrm{ap}}(x) = \underline{f}^{(1)}_{\mathrm{ap}}(x)$, then the common value is the *approximate symmetric derivative* of f at x and is denoted by $f^{(1)}_{\mathrm{ap}}(x)$.

2. Definitions and notations. A set E is said *to satisfy the property* M_2 iff E is an F_σ and every one-sided neighbourhood of each point in E intersects E in a set of positive measure. A function f is said *to satisfy the property* M_2 iff, for every real a, the sets $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ satisfy the property M_2 [12]. The oscillation of a function f at a point x will be denoted by $\omega(x, f)$. The Lebesgue measure of a measurable set A will be denoted by $m(A)$. Throughout the paper R stands for the real line, $-\infty < x < +\infty$.

3. In paper [6] Mukhopadhyay has shown that if f is approximately continuous on $[a, b]$ and if $f_{\mathrm{ap}}^{(1)}$ is non-negative everywhere on $[a, b]$, then f is non-decreasing on $[a, b]$, and that for a monotone function f, if $f_{\mathrm{ap}}^{(1)}$ exists on $[a, b]$, then $f^{(1)}$ also exists on $[a, b]$. In this section these results are strengthened.

LEMMA 1. *Let $[a, b] \subset I$, where I is an open interval, and let $f\colon I \to R$ be approximately continuous on $[a, b]$. If $\underline{f}_{\mathrm{ap}}^{(1)} > 0$ on $[a, b]$ and if there is a point x_0, $a < x_0 < b$, such that $f(a) < f(x_0)$, then $f(a) \leqslant f(b)$.*

Proof. Let $E = \{x\colon x \in [a, b]; f(a) \leqslant f(x)\}$. We show that $b \in E$. Let C be a set such that

(i) $$C \subset E,$$

(ii) $$\frac{|E \cap [x', x'']|}{x'' - x'} \geqslant \frac{1}{2},$$

whenever $x', x'' \in C$ and $x' < x''$.

Let S be the collection of all such sets C. Since f is approximately continuous at x_0, S is non-empty. Let S be partially ordered by set inclusion. Then one can verify that every linearly ordered subset of S has an upper bound in S. So, by Zorn's Lemma, S has a maximal element K. If $m = \sup K$, then $m \in E$. For if $x < m$, $x \in K$, then there is a sequence $\{x_n\} \subset K$ such that $x < x_n \leqslant x_{n+1}$, $\lim x_n = m$ and

$$\frac{|E \cap [x, m]|}{m - x} = \lim \frac{|E \cap [x, x_n]|}{x_n - x} \geqslant \frac{1}{2}.$$

Thus m is a point of positive upper density of E and since f is approximately continuous, $m \in E$ [2].

We shall now show that $m = b$. If possible, let $m < b$. Then, since $\underline{f}_{\mathrm{ap}}^{(1)}(m) > 0$, by the above conclusion there exists $y \in E$ such that $y > m$ and

$$\frac{|E \cap [m, y]|}{y - m} \geqslant \frac{1}{2}.$$

This shows that $y \in K$ and $y > m$, which is a contradiction, for $m = \sup K$. This completes the proof.

THEOREM 1. *Let $[a, b] \subset I$, where I is an open interval, and let $f\colon I \to R$ be approximately continuous on $[a, b]$. If $\underline{f}_{\mathrm{ap}}^{(1)} \geqslant 0$ on $[a, b]$, then f is non-decreasing on $[a, b]$.*

Proof. Let us first assume that $\underline{f}_{\mathrm{ap}}^{(1)} > 0$ on $[a, b]$. If possible, we suppose $f(a) > f(b)$. Since f is approximately continuous, it satisfies Darboux property [1]. So for any two reals η_1 and η_2 satisfying $f(b) < \eta_2 < \eta_1 < f(a)$ there are points x_1 and x_2, $a < x_1 < x_2 < b$, such that $f(x_1) = \eta_1$ and $f(x_2) = \eta_2$. We assert that $f(x) \geqslant \eta_1$ for all $x \in [a, x_1]$. For, if possible, suppose there is a point $x' \epsilon (a, x_1)$ for which $f(x') < \eta_1$. Since f satisfies Darboux property, we may suppose $f(x') > \eta_2$. But applying Lemma 1 to $[x', x_2]$ we conclude that $f(x') \leqslant \eta_2$, which is a contradiction. Similarly, $f(x) \leqslant \eta_1$ for all $x \epsilon [x_1, x_2]$. For if there is $x'' \epsilon (x_1, x_2)$ such that $f(x'') > \eta_1$, then applying Lemma 1 to $[x_1, x_2]$ we have $\eta_1 \leqslant \eta_2$, which is a contradiction. Thus $f(x) \geqslant \eta_1$ for all $x \epsilon [a, x_1]$ and $f(x) \leqslant \eta_1$ for all $x \epsilon [x_1, x_2]$. Hence $\bar{f}^{(1)}_{\mathrm{ap}}(x_1) \leqslant 0$; but this is a contradiction, because $\underline{f}^{(1)}_{\mathrm{ap}} > 0$ on $[a, b]$.

Now if $\underline{f}^{(1)}_{\mathrm{ap}} \geqslant 0$ on $[a, b]$, then, for arbitrary $\varepsilon > 0$, consider the function φ_ε, where

$$\varphi_\varepsilon(x) = f(x) + \varepsilon x.$$

Applying the above conclusion to φ_ε, we have $\varphi_\varepsilon(a) \leqslant \varphi_\varepsilon(b)$. This gives $f(a) + \varepsilon a \leqslant f(b) + \varepsilon b$. Since ε is arbitrary, $f(a) \leqslant f(b)$. This completes the proof.

The following theorem generalizes Theorem 1:

THEOREM 2. *Let $[a, b] \subset I$, where I is an open interval, and let $f: I \to R$ be approximately continuous on $[a, b]$. Let*

(i) $\underline{f}^{(1)}_{\mathrm{ap}} \geqslant 0$ *almost everywhere on* $[a, b]$;

(ii) $\underline{f}^{(1)}_{\mathrm{ap}} > -\infty$ *everywhere on* $[a, b]$.

Then f is non-decreasing on $[a, b]$.

Proof. Let $E = \{x: x \epsilon [a, b];\ \underline{f}^{(1)}_{\mathrm{ap}}(x) < 0\}$. Then $m(E) = 0$. Hence, by [8], p. 214, there exists a continuous non-decreasing function $\sigma(x)$ on $[a, b]$ such that

$$\sigma'(x) = +\infty \quad \text{for each point } x \epsilon E.$$

Let $\varepsilon > 0$ be arbitrary. Consider the function ψ defined by

$$\psi(x) = f(x) + \varepsilon\sigma(x).$$

Function ψ is approximately continuous on $[a, b]$ and $\underline{\psi}^{(1)}_{\mathrm{ap}}(x) \geqslant 0$ for all x on $[a, b]$. Hence, by Theorem 1, ψ is non-decreasing on $[a, b]$. Since ε is arbitrary, f is non-decreasing on $[a, b]$. This proves the theorem.

THEOREM 3. *Let $[a, b] \subset I$, where I is an open interval, and let $f: I \to R$ be approximately continuous on $[a, b]$. Let*

(i) $\bar{f}^{(1)}_{\mathrm{ap}} \leqslant 0$ *almost everywhere on* $[a, b]$;

(ii) $\bar{f}^{(1)}_{\mathrm{ap}} < +\infty$ *everywhere on* $[a, b]$.

Then f is non-increasing on $[a, b]$.

Proof. This can be proved by putting $f(x) = -g(x)$ and applying Theorem 2.

From Theorems 2 and 3 we get the following

THEOREM 4. *Let $[a, b] \subset I$, where I is an open interval, let $f: I \to R$ be approximately continuous on $[a, b]$ and let $-\infty < \underline{f}^{(1)}_{\mathrm{ap}} \leqslant \bar{f}^{(1)}_{\mathrm{ap}} < +\infty$ hold everywhere on $[a, b]$. If $f^{(1)}_{\mathrm{ap}}$ exists and equals to zero almost everywhere on $[a, b]$, then f is constant.*

THEOREM 5. *If f is monotone in any open interval I, then* (i) $\underline{f}^{(1)}_{\mathrm{ap}} = \underline{f}^{(1)}$ *and* (ii) $\bar{f}^{(1)}_{\mathrm{ap}} = \bar{f}^{(1)}$ *everywhere in I.*

Proof. Suppose that f is monotonically increasing in the open interval I and that there is a point $\xi \in I$ such that

$$m = \underline{f}^{(1)}(\xi) < \underline{f}^{(1)}_{\mathrm{ap}}(\xi) = k.$$

Choose ε such that $0 < \varepsilon < (k-m)/2$. Since $m < k-2\varepsilon$, there is a sequence $h_n \to 0+$ such that $\xi \pm h_n \in I$ and

$$\frac{f(\xi+h_n)-f(\xi-h_n)}{2h_n} < k-2\varepsilon \quad \text{for all } n. \tag{1}$$

Hence

$$\frac{f(\xi+h_n)-f(\xi-h_n)}{2(k-\varepsilon)} < h_n \quad \text{for all } n.$$

For a given n we denote by J_n and I_n the intervals

$$[0, h_n] \quad \text{and} \quad \left[\frac{f(\xi+h_n)-f(\xi-h_n)}{2(k-\varepsilon)}, h_n\right],$$

respectively. Then

$$m(J_n) = h_n \quad \text{and} \quad m(I_n) = h_n - \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k-\varepsilon)}.$$

So, from (1),

$$\frac{m(I_n)}{m(J_n)} = 1 - \frac{f(\xi+h_n)-f(\xi-h_n)}{2h_n(k-\varepsilon)} > 1 - \frac{k-2\varepsilon}{k-\varepsilon}. \tag{2}$$

Since $\underline{f}^{(1)}_{\mathrm{ap}}(\xi) = k$, there is an N such that the set

$$E = \left\{h : h > 0; \frac{f(\xi+h)-f(\xi-h)}{2h} - k > -\varepsilon\right\}$$

satisfies the inequality

$$\frac{m(E \cap J_n)}{m(J_n)} > 1 - \frac{\varepsilon}{k-\varepsilon} \quad \text{for all } n \geqslant N. \tag{3}$$

Now, if $h \in I_n$, we have

$$\xi + \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k-\varepsilon)} \leqslant \xi + h \leqslant \xi + h_n$$

and

$$f(\xi+h) \leqslant f(\xi+h_n) \leqslant f(\xi-h_n) + 2h(k-\varepsilon) \leqslant f(\xi-h) + 2h(k-\varepsilon).$$

So,

$$\frac{f(\xi+h)-f(\xi-h)}{2h} \leqslant k-\varepsilon$$

and hence $h \notin E$.

Therefore $E \cap I_n = 0$. Hence from (2)

$$\text{(4)} \qquad \frac{m(E\cap J_n)}{m(J_n)} = \frac{m\big(E\cap(J_n-I_n)\big)}{m(J_n)} \leqslant \frac{m(J_n)-m(I_n)}{m(J_n)}$$

$$= 1-\frac{m(I_n)}{m(J_n)} < \frac{k-2\varepsilon}{k-\varepsilon}.$$

Since (3) and (4) are contradictory for $n \geqslant N$, we conclude

$$\underline{f}^{(1)}_{\mathrm{ap}} = \underline{f}^{(1)} \quad \text{for all } x \text{ in } I.$$

To prove the second condition, suppose that f is monotonically increasing in the open interval I and that there is a point $\xi \in I$ such that

$$k_1 = \bar{f}^{(1)}_{\mathrm{ap}}(\xi) < \bar{f}^{(1)}(\xi) = k_2.$$

Choose ε such that $0 < \varepsilon < (k_2-k_1)/2$. Since $k_2 > k_1+2\varepsilon$, there is a sequence $h_n \to 0+$ such that $\xi \pm h_n \in I$ and

$$\text{(5)} \qquad \frac{f(\xi+h_n)-f(\xi-h_n)}{2h_n} > k_1+2\varepsilon \quad \text{for all } n.$$

Hence

$$\frac{f(\xi+h_n)-f(\xi-h_n)}{2(k_1+\varepsilon)} > h_n \quad \text{for all } n.$$

For a given n we denote by J_n and I_n the intervals

$$\left[0, \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k_1+\varepsilon)}\right] \quad \text{and} \quad \left[h_n, \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k_1+\varepsilon)}\right],$$

respectively. Then

$$m(J_n) = \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k_1+\varepsilon)} \quad \text{and} \quad m(I_n) = \frac{f(\xi+h_n)-f(\xi-h_n)}{2(k_1+\varepsilon)} - h_n.$$

Hence, from (5)

$$\text{(6)} \qquad \frac{m(I_n)}{m(J_n)} = 1-\frac{2h_n(k_1+\varepsilon)}{f(\xi+h_n)-f(\xi-h_n)} > 1-\frac{k_1+\varepsilon}{k_1+2\varepsilon}.$$

Since $\bar{f}^{(1)}_{\mathrm{ap}}(\xi) = k_1$, there is an N such that the set

$$E = \left\{h : h > 0; \frac{f(\xi+h)-f(\xi-h)}{2h} - k_1 < \varepsilon\right\}$$

satisfies the inequality

$$\text{(7)} \qquad \frac{m(E\cap J_n)}{m(J_n)} > 1-\frac{\varepsilon}{k_1+2\varepsilon} \quad \text{for all } n \geqslant N.$$

Now, if $h \in I_n$, we have

$$\xi + h_n \leqslant \xi + h \leqslant \xi + \frac{f(\xi + h_n) - f(\xi - h_n)}{2(k_1 + \varepsilon)}$$

and

$$f(\xi + h) \geqslant f(\xi + h_n) \geqslant f(\xi - h_n) + 2(k_1 + \varepsilon)h \geqslant f(\xi - h) + 2(k_1 + \varepsilon)h.$$

So

$$\frac{f(\xi + h) - f(\xi - h)}{2h} > k_1 + \varepsilon$$

and hence $h \notin E$. So, $E \cap I_n = 0$. Hence, by (6),

$$\text{(8)} \qquad \frac{m(E \cap J_n)}{m(J_n)} = \frac{m\big(E \cap (J_n - I_n)\big)}{m(J_n)} \leqslant \frac{m(J_n) - m(I_n)}{m(J_n)}$$
$$= 1 - \frac{m(I_n)}{m(J_n)} < \frac{k_1 + \varepsilon}{k_1 + 2\varepsilon}.$$

Since (7) and (8) are contradictory for $n \geqslant N$ we conclude that $\bar{f}^{(1)}_{\mathrm{ap}} = \bar{f}^{(1)}$ for all x in I.

Similar results can be proved if f is monotonically decreasing. This completes the proof.

COROLLARY 1. *If f is monotone in I and if $f^{(1)}_{\mathrm{ap}}$ exists in I, then $f^{(1)}$ also exists in I.*

COROLLARY 2. *Let $f: I \to R$ be approximately continuous and let there exist a continuous function $\varphi: I \to R$ such that $\varphi^{(1)}$ exists in I and satisfies any one of the following relations for all x in I:* (i) $\bar{f}^{(1)}_{\mathrm{ap}}(x) \leqslant \varphi^{(1)}(x)$; (ii) $\underline{f}^{(1)}_{\mathrm{ap}}(x) \geqslant \varphi^{(1)}(x)$.

Then f is continuous and $\bar{f}^{(1)}_{\mathrm{ap}} = \bar{f}^{(1)}$ and $\underline{f}^{(1)}_{\mathrm{ap}} = \underline{f}^{(1)}$ in I.

Proof. $\bar{f}^{(1)}_{\mathrm{ap}} \leqslant \varphi^{(1)}$ in I, then the function $g = f - \varphi$ is approximately continuous in I and $\bar{g}^{(1)}_{\mathrm{ap}} \leqslant 0$ in I and hence g is non-increasing in I. So g is continuous and $\bar{g}^{(1)}_{\mathrm{ap}} = \bar{g}^{(1)}$, $\underline{g}^{(1)}_{\mathrm{ap}} = \underline{g}^{(1)}$ in I, which implies that f is continuous and $\bar{f}^{(1)}_{\mathrm{ap}} = \bar{f}^{(1)}$ and $\underline{f}^{(1)}_{\mathrm{ap}} = \underline{f}^{(1)}$ in I.

4. In a recent paper [11] Weil proved that if f has an approximate derivative f'_{ap} and if for an open interval (α, β) the set $f'^{-1}_{\mathrm{ap}}\big((\alpha, \beta)\big)$ is non--empty, then the set $\{x: f'(x)$ exists and $\alpha < f'(x) < \beta\}$ is of positive measure. In this section we obtain an analogue for approximate symmetric derivative. In our proof we shall use an analogue of Denjoy's Theorem for symmetric derivatives which informs that if f is continuous in an open interval I and if $f^{(1)}$ exists and possesses Darboux property on $[a, b] \subset I$, then for any open interval (α, β) the set $\{x: x \in [a, b];\ \alpha < f^{(1)}(x) < \beta\}$ is either void or of positive measure [5].

Finally, with the help of the above theorem we obtain the analogue of Denjoy's theorem for approximate symmetric derivatives. (It may be of interest to note that Marcus [4] obtained the analogue of Denjoy's theorem for approximate derivative.) Also it is shown that under certain conditions the approximate symmetric derivatives belong to Zahorski's class M_2. In our proof we require $f_{\mathrm{ap}}^{(1)}$ to be in Baire class 1. Since this property of $f_{\mathrm{ap}}^{(1)}$ is not known, we give certain information regarding it.

LEMMA 2. *Let g be defined on an open interval I and be finite on a dense subset of I. Then*

$$\{x\colon x\in I;\, g(x) = \pm\infty\} \subset \{x\colon x\in I;\, \omega(x,g) = \infty\}$$

and the set

$$\{x\colon x\in I;\, \omega(x,g) = \infty\}$$

is closed in I.

The proof is simple and hence omitted.

LEMMA 3. *Let the set G be open in I, where I is an open interval, let f be approximately continuous, let $f_{\mathrm{ap}}^{(1)}$ exist finitely in G and let $\omega(x, f_{\mathrm{ap}}^{(1)}) < \infty$ in G.*

Then for all real numbers a the sets $\{x\colon f_{\mathrm{ap}}^{(1)}(x) < a\}\cap G$ and $\{x\colon f_{\mathrm{ap}}^{(1)}(x) > a\}\cap G$ are F_σ.

Proof. Let (α, β) be a component interval of G and let

$$I_n = \left[\alpha+\frac{1}{n},\, \beta-\frac{1}{n}\right],$$

where n is chosen such that $\alpha < \alpha+1/n < \beta-1/n < \beta$. Since $\omega(x, f_{\mathrm{ap}}^{(1)}) < \infty$ on I_n and $f_{\mathrm{ap}}^{(1)}$ is finite on I_n, $f_{\mathrm{ap}}^{(1)}$ is bounded on I_n. Hence, by Corollary 2, f is continuous and $f^{(1)}$ exists on I_n. Hence $f^{(1)}$ is of Baire class 1 on I_n. So the sets $\{x\colon f_{\mathrm{ap}}^{(1)}(x) < a\}\cap I_n$ and $\{x\colon f_{\mathrm{ap}}^{(1)}(x) > a\}\cap I_n$ are F_σ ([9], p. 141). Consequently,

$$\{x\colon f_{\mathrm{ap}}^{(1)}(x) < a\}\cap G = \bigcup\bigcup_n \left(\{x\colon f_{\mathrm{ap}}^{(1)}(x) < a\}\cap I_n\right),$$

where the last union is taken over all components of G, is an F_σ. Similarly, $\{x\colon f_{\mathrm{ap}}^{(1)}(x) > a\}\cap G$ is an F_σ.

THEOREM 6. *Let I be an open interval, let $f\colon I\to R$ be approximately continuous, and let $f_{\mathrm{ap}}^{(1)}$ exist on I and be finite on a dense subset of I. If $\omega(x, f_{\mathrm{ap}}^{(1)}) < \infty$ holds except for a countable subset of I, then $f_{\mathrm{ap}}^{(1)}$ is of Baire class 1.*

Proof. Let

$$E = \{x\colon x\in I;\, \omega(x, f_{\mathrm{ap}}^{(1)}) = \infty\}, \qquad E_1 = \{x\colon x\in I;\, f_{\mathrm{ap}}^{(1)}(x) = \pm\infty\}.$$

Then, by Lemma 2, $E_1 \subset E$ and the set E is closed in I. Also, by hypothesis, E is countable. Let $G = I-E$. Then G is open in I. Also $f_{\mathrm{ap}}^{(1)}$ is finite in G and $\omega(x, f_{\mathrm{ap}}^{(1)}) < \infty$ in G. Hence, by Lemma 3, the sets $\{x\colon f_{\mathrm{ap}}^{(1)}(x) < a\}\cap G$ and $\{x\colon f_{\mathrm{ap}}^{(1)}(x) > a\}\cap G$ are F_σ.

Since E is countable, for any closed interval $[a, b] \subset I$ the sets $\{x: f_{\mathrm{ap}}^{(1)}(x) < a\} \cap [a, b]$ and $\{x: f_{\mathrm{ap}}^{(1)}(x) > a\} \cap [a, b]$ are F_σ. Hence $f_{\mathrm{ap}}^{(1)}$ is of Baire class 1 [9].

LEMMA 4. *Let $[a, b] \subset I$, where I is an open interval, let $f: I \to R$ be approximately continuous, and let $f_{\mathrm{ap}}^{(1)}$ exist, possess Darboux property and be of Baire class 1 on $[a, b]$. If, for an open interval (α, β), the set*

$$\{x: x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)}(x) < \beta\}$$

is non-empty, then there is a point on $[a, b]$ for which $f^{(1)}$ exists and lies between α and β.

Proof. Suppose to the contrary that there is an open interval (α, β) such that the set $\{x: x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)}(x) < \beta\}$ is non-empty but contains no point where $f^{(1)}$ exists.

Let $E_\alpha = \{x: x \in [a, b]; f^{(1)}$ exists and $f^{(1)}(x) \leqslant \alpha\}$ and $E^\beta = \{x: x \in [a, b]; f^{(1)}$ exists and $f^{(1)}(x) \geqslant \beta\}$. Let Λ_k denote non-degenerate components of E_α. We shall prove that their interiors (relative to $[a, b]$) form the complement of a perfect set. Since $f_{\mathrm{ap}}^{(1)}$ is bounded from above by α in the interiors of Λ_k and $f_{\mathrm{ap}}^{(1)}$ has Darboux property, the values of $f_{\mathrm{ap}}^{(1)}$ at end points of Λ_k are not greater than α. Hence on the closed intervals $\bar{\Lambda}_k$, $f_{\mathrm{ap}}^{(1)}$ is bounded from above by α and, by Corollary 2, $f_{\mathrm{ap}}^{(1)} = f^{(1)}$, and two different $\bar{\Lambda}_k$ cannot have a common point. Similar is the case for E^β.

Let P be the perfect set in $[a, b]$ whose complement is the union of the interiors (relative to $[a, b]$) of all non-degenerate components of E_α and E^β. Let $x_0 \in P$ and let I_0 be any open interval containing x_0. Then there are points x' and x'' in $I_0 \cap [a, b]$ such that $f_{\mathrm{ap}}^{(1)}(x') \geqslant \beta$ and $f_{\mathrm{ap}}^{(1)}(x'') \leqslant \alpha$; for, if $f_{\mathrm{ap}}^{(1)}(x) < \beta$ for all $x \in I_0 \cap [a, b]$, then, by Corollary 2, $f^{(1)}$ exists on $I_0 \cap [a, b]$. Since the set $\{x: x \in [a, b]; \alpha < f_{\mathrm{ap}}^{(1)} < \beta\}$ contains no point where $f^{(1)}$ exists, $f^{(1)}(x) \leqslant \alpha$ on $I_0 \cap [a, b]$. Hence $I_0 \cap [a, b]$ is contained in the interior (relative to $[a, b]$) of some components of E_α which is contrary to the fact that $x_0 \in I_0 \cap P$. The other case follows similarly. We assert that there is a point x_0' in $I_0 \cap P$ such that $f_{\mathrm{ap}}^{(1)}(x_0') \geqslant \beta$ and a point x_0'' in $I_0 \cap P$ such that $f_{\mathrm{ap}}^{(1)}(x_0'') \leqslant \alpha$. For if $x' \in P$, the assertion follows. If $x' \notin P$, then x' is in the interior (relative to $[a, b]$) of a component of E^β. We take x_0' to be that end point of this component which lies between x_0 and x'. The other case can be proved similarly. Hence we conclude that

$$\sup_{x \in I_0 \cap P} f_{\mathrm{ap}}^{(1)}(x) \geqslant \beta, \qquad \inf_{x \in I_0 \cap P} f_{\mathrm{ap}}^{(1)}(x) \leqslant \alpha.$$

So, the saltus of $f_{\mathrm{ap}}^{(1)}$ at each point of P relative to P is at least $\beta - \alpha$. Hence the function $f_{\mathrm{ap}}^{(1)}$ is discontinuous at each point of P relative to P. Since, by hypothesis, $f_{\mathrm{ap}}^{(1)}$ is of Baire class 1, this is a contradiction.

THEOREM 7. *Let $[a, b] \subset I$, where I is an open interval, let $f: I \to R$ be approximately continuous, and let $f_{\mathrm{ap}}^{(1)}$ exist, possess Darboux property* *and be of Baire class* 1 *on* $[a, b]$. *If for an open interval* (α, β) *the set* $\{x: x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)} < \beta\}$ *is non-empty, then the set*

$$\{x: x \in [a, b];\ f^{(1)}(x) \text{ exists and } \alpha < f^{(1)}(x) < \beta\}$$

is of positive measure.

Proof. Suppose to the contrary that the set $\{x: x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)}(x) < \beta\}$ is non-empty but the set $E = \{x: x \in [a, b];\ f^{(1)}$ exists and $\alpha < f^{(1)}(x) < \beta\}$ is of measure zero for an open interval (α, β).

Let

$$E_\alpha = \{x: x \in [a, b]; f_{\mathrm{ap}}^{(1)} \leqslant \alpha\}$$

and

$$E^\beta = \{x: x \in [a, b]; f_{\mathrm{ap}}^{(1)} \geqslant \beta\}.$$

By Lemma 4 the set E is dense in itself; for if $x_0 \in E \cap I_0$, where I_0 is an open interval, then since $f_{\mathrm{ap}}^{(1)}$ possesses Darboux property, there is a point x_1 $(x_1 \neq x_0) \in I_0 \cap [a, b]$ such that $f_{\mathrm{ap}}^{(1)}(x_1)$ lies in (α, β). Choose a closed interval $J \subset I_0 \cap [a, b]$ which contains x_1 but not x_0. Then, applying Lemma 4, there is a point $x_2 \in J$ such that $f^{(1)}(x_2)$ exists and lies in (α, β). This shows that $\bar{E}$ is a perfect set, where $\bar{E}$ is the closure of E.

Let $x_0 \in E \cap I_0$, where I_0 is an open interval. If $E_\alpha \cap I_0 = 0$, then $f_{\mathrm{ap}}^{(1)}(x) > \alpha$ for all x in I_0 and, by Corollary 2, f is continuous, $f^{(1)}$ exists and $f_{\mathrm{ap}}^{(1)} = f^{(1)}$ in I_0. Hence, by a result of [5], we have $m(E \cap I_0) > 0$. This contradicts the hypothesis that $m(E) = 0$. Hence $E_\alpha \cap I_0 \neq 0$.

Similarly, it can be shown that $E^\beta \cap I_0 \neq 0$. Thus $E \subset E'_\alpha \cap E'^\beta$, where E'_α and E'^β are the sets of all limiting points of E_α and E^β, respectively. Let now $x_0 \in \bar{E} \cap I_0$, where I_0 is an open interval. Then there are points x_1, x_2 and x_3 such that $x_1 \in E \cap I_0$, $x_2 \in E^\beta \cap I_0$ and $x_3 \in E_\alpha \cap I_0$. Since by hypothesis $f_{\mathrm{ap}}^{(1)}$ satisfies Darboux property, for any ε, $0 < \varepsilon < (\beta - \alpha)/2$, there are points x'_2, x'_3 in $I_0 \cap [a, b]$ such that

$$\beta - \varepsilon < f_{\mathrm{ap}}^{(1)}(x'_2) < \beta, \qquad \alpha < f_{\mathrm{ap}}^{(1)}(x'_3) < \alpha + \varepsilon.$$

Hence, by Lemma 4, there are points x''_2 and x''_3 in $I_0 \cap [a, b]$ such that $f^{(1)}(x''_2)$ and $f^{(1)}(x''_3)$ exist and

$$\beta - \varepsilon < f^{(1)}(x''_2) < \beta, \qquad \alpha < f^{(1)}(x''_3) < \alpha + \varepsilon.$$

So x''_2 and x''_3 are in $E \cap I_0$. Since ε is arbitrary,

$$\sup_{x \in \bar{E} \cap I_0} f_{\mathrm{ap}}^{(1)}(x) \geqslant \beta, \qquad \inf_{x \in \bar{E} \cap I_0} f_{\mathrm{ap}}^{(1)}(x) \leqslant \alpha.$$

Thus the saltus of $f_{\mathrm{ap}}^{(1)}(x)$ at x_0 relative to $\bar{E}$ is at least $\beta - \alpha$ and hence $f_{\mathrm{ap}}^{(1)}(x)$ is not continuous at x_0 relative to $\bar{E}$. Since x_0 is any arbitrary point of $\bar{E}$, no point of $\bar{E}$ is a point of continuity of $f_{\mathrm{ap}}^{(1)}$ relative to $\bar{E}$. But since $f_{\mathrm{ap}}^{(1)}$ is of Baire class 1, this is a contradiction. This completes the proof.

From Theorem 7 one easily deduce the following analogue of Denjoy's theorem for approximate symmetric derivatives:

THEOREM 8. *Let* $[a, b] \subset I$, *where* I *is an open interval, and let* $f\colon I \to R$ *be approximately continuous on* $[a, b]$. *Let* $f_{\mathrm{ap}}^{(1)}$ *exist, possess Darboux property and be of Baire class* 1 *on* $[a, b]$.

Then for any two reals α, β $(\alpha < \beta)$ *the set* $\{x\colon x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)}(x) < \beta\}$ *is either empty or of positive measure.*

COROLLARY 3. *Under the hypotheses of Theorem* 8, *if* $f_{\mathrm{ap}}^{(1)} \geqslant 0$ *for almost all* x *on* $[a, b]$, *then* f *is non-decreasing on* $[a, b]$.

Proof. Suppose to the contrary that there is a point $\xi \in [a, b]$ such that $f_{\mathrm{ap}}^{(1)}(\xi) < 0$. Choose α, β such that $\alpha < f_{\mathrm{ap}}^{(1)} < \beta < 0$. Then the set $\{x\colon x \in [a, b];\ \alpha < f_{\mathrm{ap}}^{(1)} < \beta\}$ is non-empty and hence it is of positive measure. But this contradicts the hypotheses. So $f_{\mathrm{ap}}^{(1)}(x) \geqslant 0$ everywhere on $[a, b]$ (see also [7]). Applying Theorem 1, we have f is non-decreasing on $[a, b]$.

COROLLARY 4. *Under hypotheses of Theorem* 8, $f_{\mathrm{ap}}^{(1)} \in M_2$.

Proof. Since $f_{\mathrm{ap}}^{(1)}$ is a function of Baire class 1, the sets $E^\alpha = \{x\colon x \in [a, b];\ f_{\mathrm{ap}}^{(1)}(x) > \alpha\}$ and $E_\beta = \{x\colon x \in [a, b];\ f_{\mathrm{ap}}^{(1)} < \beta\}$ are F_σ for arbitrary α and β [9].

Let $\xi \in E^\alpha$ and let J be any one-sided neighbourhood of ξ. Then $\xi \in J$ and $f_{\mathrm{ap}}^{(1)}(\xi) > \alpha$. Hence $E^\alpha \cap J \neq 0$. So, by Theorem 8, $m(E^\alpha \cap J) > 0$. Thus $E^\alpha \in M_2$. Similarly, $E_\beta \in M_2$. Hence $f_{\mathrm{ap}}^{(1)} \in M_2$.

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REFERENCES

[1] A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématique de France 43 (1915), p. 161-248.

[2] C. Goffman and C. J. Neugebauer, *On approximate derivatives*, Proceedings of the American Mathematical Society 11 (1960), p. 962-966.

[3] A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fundamenta Mathematicae 9 (1927), p. 212-279.

[4] S. Marcus, *On a theorem of Denjoy and on approximate derivative*, Monatshefte für Mathematik 66 (1962), p. 435-440.

[5] S. N. Mukhopadhyay, *On Schwarz differentiability, IV*, Acta Mathematica Academiae Scientiarum Hungaricae 17 (1966), p. 129-136.

[6] — *On approximate Schwarz differentiability*, Monatshefte für Mathematik 70 (1966), p. 454-460.

[7] — *On a certain property of the derivative*, Fundamenta Mathematicae 67 (1970), p. 279-284.

[8] I. P. Natanson, *Theory of functions of a real variable*, vol. 1, New York 1960.

[9] — *Theory of functions of a real variable*, vol. 2, New York 1960.

[10] S. Saks, *Theory of the integral*, Warszawa 1937.

[11] C. E. Weil, *On approximate and Peano derivatives*, Proceedings of the American Mathematical Society 20 (1969), p. 487-490.

[12] Z. Zahorski, *Sur la première dérivée*, Transactions of the American Mathematical Society 69 (1950), p. 1-54.

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