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One more definition of the curvature and torsion of smooth curves

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It is the aim of this paper to introduce Frenet's trihedron for curves of the D^1 type (differential curves). The classical definitions cannot be applied, of course, to these curves, as they assume the existence of continuous derivatives of the second or third order of a vectorial function representing this curve. Thus it has been necessary to introduce new definitions, and I am going to prove that in the case of curves belonging to the C^2 or C^3 type they conform to the classical ones.

Various suggestions have been made of introducing the curvature for the curves:

$$r = r(s) \in D^1. \tag{1}$$

In [3], for instance, definition (1) has been applied to a flat case. As the curvature of the curve $y = y(x) \in D^1$ at the point P the expression $\mathscr{K} = \lim\limits_{d\to 0} \dfrac{12S}{l^3}$ has been taken, where S is the area between this curve and the straight line L, which runs parallely to the tangent at the point P at a distance d; l is the distance between the two points of intersection of this curve by the straight line L. This definition does not comprise all the points of curve (1), at which, according to the classical definition, the curvature is equal to 0.

I am going, therefore, to introduce a somewhat different definition,

Let curve (1) be flat, while $|r'(s)| = 1$ (Fig. 1). Through a given point P there are plotted a tangent T and an oriented normal N; to the

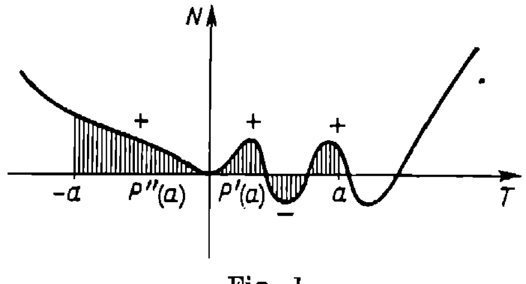

Fig. 1

tangent T we plot two perpendiculars p' and p'' at a distance a from the point P. For a sufficiently small a we can consider two algebraic areas $P'(a)$ and $P''(a)$, confined by the tangent T, by curve (1) and by the straight lines p' and p'' respectively. The areas are regarded as positive (negative) if they are located on the positive (negative) side of T, as defined by the orientation of N.

Let $P(a) = |P'(a)+P''(a)|$ (1).

Definition 1. By a *curvature of the order* $\alpha \geqslant 2$ we understand the value

$$\mathscr{H}^{(\alpha)} = \lim_{a\to 0} \frac{\alpha P^{(\alpha)}}{a^{\alpha}} \tag{2}$$

if such a limit exists. Instead of $\mathscr{H}^{(3)}$ we shall simply write $\mathscr{H}$. If, for example, the curve K consists of two semicircles with the radii b and c, joined together as in Fig. 2, $\mathscr{H} = \frac{1}{2}\left|\frac{1}{b}-\frac{1}{c}\right|$ will exist at the point P,

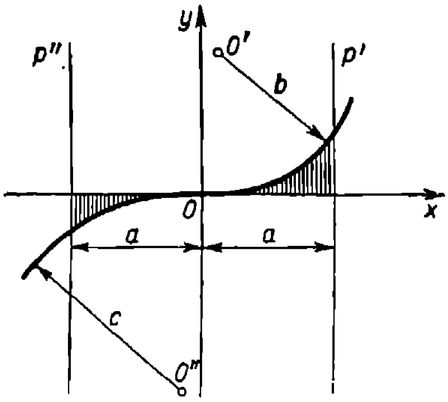

Fig. 2

while K is a curve of the type C^1 only, whereas $\mathscr{H}^{(2)} = 0$ at every point. $\mathscr{H}^{(\alpha)}$ does not exist in the case of $\alpha > 3$.

Theorem 1. *The definition coincides with the classical definition when* $r = r(S) \in C^2$ *and* $\alpha = 3$.

Proof. If an axis x is tangent to a curve at the point $x = 0$, this curve may be locally represented as $y = y(x)$, and when developed, it gives

$$y = \tfrac{1}{2}y''(O)x^2 + O(x^2). \tag{3}$$

By (2) and (3)

$$\frac{\mathscr{H}}{3} = \lim_{a\to 0}\frac{|P(a)|}{a^3}\left|\lim_{a\to 0}\frac{\int_{-a}^{a} y\,dx}{a^3}\right| = \left|\lim_{a\to 0}\frac{\frac{y''(0)}{2}\cdot\frac{x^3}{3}\Big|_{-a}^{a}}{a^3}\right| = \frac{\mathscr{H}^*}{3} \tag{4}$$

(1) Thus $P(a)$ does not depend on the orientation of N.

because $\lim\limits_{a\to 0}\frac{\int_{-a}^{a} O(x^2)\,dx}{a^3} = 0$ and thus $\mathscr{H} = \mathscr{H}^*$, $\mathscr{H}^*$ is the curvature according to the classical definition.

Note. The relative curvature $\mathscr{H} = \pm\mathscr{H}^*$ may also be introduced in quite the same way as in the case of the curves C^2 (cf. [2]).

Let us assume now that (1) represents a spatial curve. Through the tangent we plot at point P a pencil of planes, projecting on each of them the given curve (orthogonal projection).

DEFINITION 2. By a *curvature of the order* a of curve (1) at the point P we understand the finite maximum of $\mathscr{H}^{(a)}(\theta)$, where the symbol $\mathscr{H}^{(a)}(\theta)$ denotes the curvature of the order a in projection (1) on to a plane forming $\sphericalangle\ \theta$ with an optionally determined plane in the pencil of planes $0 \leqslant \theta < \pi$, if such a maximum exists.

DEFINITION 3. By a *curvature deflection* $K^{(a)}$ *of the order* a of a flat curve (1) we understand the expression $K^{(a)} = a\lim\limits_{a\to 0}\frac{|P'(a)-P''(a)|}{a^a}$, (4), where a, $P'(a)$ and $P''(a)$ mean the same as in Definition 1.

DEFINITION 4. By a *curvature deflection of the order* a of curve (1) at the point P we understand the maximum $K^{(a)}(\theta)$, where $K^{(a)}(\theta)$ denotes a curvature deflection of the order a of projection (1) on to a plane forming $\sphericalangle\ \theta$ with a determined plane in the pencil of planes, if such a maximum exists.

DEFINITION 5. An *osculating plane at point* P of curve (1) is a plane π that passes through a tangent at P with one of the following properties: (i) There is such an a that $0<\mathscr{H}^{(a)}<\infty$ and projection (1) on π has the flat curvature $\mathscr{H}^{(a)}$; (ii) There does not exist such an a that $0<\mathscr{H}^{(a)}<\infty$, but for an a we have $0<K^{(a)}<\infty$ and $K^{(a)}$ is realised on π.

THEOREM 2. *If one of the two conditions* (i) *or* (ii) *is fulfilled, the osculating plane is determined uniquely.*

Proof. Let P be the origin of a coordinate system, where the axis x is tangent to the curve at P. We may, then, represent curve (1) locally in the following way:

$$y = y(x), \quad z = z(x). \tag{6}$$

Let us rotate the coordinate system about the x-axis and project (1) on to the resulting planes $(x, \bar{y})$.

Considering the fact that between the coordinates of the points in various systems there occur the relations

$$\bar{y} = y\cos\theta + z\sin\theta, \quad \bar{z} = -y\sin\theta + z\cos\theta, \quad \bar{x} = x, \tag{7}$$

the projections of the curve on various planes assume the following shape:

(8) $$\bar{y} = y(x)\cos\theta + z(x)\sin\theta.$$

Hence

$$\mathscr{H}^{(\alpha)}(\theta) = \lim_{a\to 0}\left|\frac{\alpha\int_{-a}^{a}\bar{y}(x)\,dx}{a^{\alpha}}\right|$$

$$= |A_{\alpha}\cdot\cos\theta + B_{\alpha}\cdot\sin\theta|,$$

where

$$|A_{\alpha}| = \left|\lim_{a\to 0}\frac{\alpha\int_{-a}^{a}y(x)\,dx}{a^{\alpha}}\right| = \mathscr{H}^{\alpha}(0), \qquad |B_{\alpha}| = \lim_{a\to 0}\left|\frac{\alpha\int_{-a}^{a}z(x)\,dx}{a^{\alpha}}\right| = \mathscr{H}^{(\alpha)}\left(\frac{\pi}{2}\right).$$

Assuming (i), we have $A_{\alpha}^2+B_{\alpha}^2 > 0$ and $\mathscr{H}^{\alpha}(\theta)$ takes its maximum exactly for

$$\cos\theta = \frac{A_{\alpha}}{\sqrt{A_{\alpha}^2+B_{\alpha}^2}}, \qquad \sin\theta = \frac{B_{\alpha}}{\sqrt{A_{\alpha}^2+B_{\alpha}^2}}.$$

Then

(9) $$H^{(\alpha)} = \sqrt{A_{\alpha}^2+B_{\alpha}^2} \ \text{(}^2\text{)}$$

and this is a formula for $\mathscr{H}^{(\alpha)}$ in the three-dimensional case.

Attention should be drawn to the fact that if for a given α, $0 < \mathscr{H}^{(\alpha)} < \infty$, then $\mathscr{H}^{(\beta)} = 0$ for $\beta < \alpha$, whereas $\mathscr{H}^{(\gamma)} = \infty$ for $\gamma > \alpha$. Thus there is a unique osculating plane, and it forms $\sphericalangle\,\theta = \operatorname{arctg}\dfrac{B_{\alpha}}{A_{\alpha}}$ with the plane (x, y). If $\mathscr{H}^{(\alpha)} = 0$, but $0 < K^{(\alpha)} < \infty$, the theorem may be proved in the same way, where

$$\bar{A}_{\alpha} = \lim_{a\to 0}\frac{\alpha\left(\int_0^a y\,dx + \int_0^{-a} y\,dx\right)}{a^{\alpha}}, \qquad \bar{B}_{\alpha} = \lim_{a\to 0}\frac{\alpha\left(\int_0^a z\,dx + \int_0^{-a} z\,dx\right)}{a^{\alpha}}.$$

Example. Let us take a circle with the radius R. We shall cut it through at point B and draw a straight tangent p to the circle at point A, which is situated on the straight line OB. Next we shall deflect one half of the circle in such a way that the part which is located on the right

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[2] It may be easily proved that $\sqrt{A_{\alpha}^2+B_{\alpha}^2}$ does not depend on the choice of the initial plane (x, y).

of point A will lie in the plane which is perpendicular to the plane presented in the drawing and passes through the tangent p (Fig. 3).

Let us calculate $\mathscr{K}$ at point A.

Because the curvature according to the classical definition of the circle is equal to $\frac{1}{R}$, the semicircle will be equal — according to this definition — to $\frac{1}{2R}$, so that $A_3 = \frac{1}{2R}$; $B_3 = \frac{1}{2R}$.

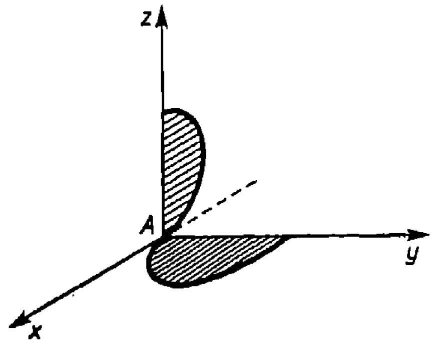

Fig. 3

Hence $\mathscr{K} = \sqrt{A_3^2+B_3^2} = \frac{\sqrt{2}}{2R}$, where as $\theta = \operatorname{arctg}\frac{B_3}{A_3} = 1$, i.e. the osculating plane is the plane $y = z$. Let us note the fact that, at point A, $\mathscr{K}^{(a)} = 0$ and $K^3 = \mathscr{K}$, and that $\mathscr{K}^{(a)}$ do not exist for the case $a > 3$.

THEOREM 3. *When the curve* $r = r(S) \in C^2$ *and* $a = 3$, *Definitions* 2 *and* 3 *conform with the corresponding classical definitions.*

If $\mathscr{K}^* \neq 0$, there exists a plane, called an *osculating* one, which has the property that, from all the orthogonal projections on to the pencil of planes passing through the tangent at the given point, the curvature of the projection is the largest, being equal to the curvature $\mathscr{K}^*$ of the given curve ([1], p. 201, 202). And because $\mathscr{K}^* = \mathscr{K}$ in the case of a flat curve, this plane is also an osculating plane according to Definition 5. Let us determine such a coordinate system that the plane (x, y) will be osculating to $r = r(s)$ and the x-axis tangential at the given point. Let t_0; u_0 be the elementary vectors on the x-axis and the y-axis, respectively. In such a case

$$r = s \cdot t_0 + \frac{\mathscr{K}^*}{2} \cdot s^2 \cdot n_0 + O(s^2)$$

in a neighbourhood of point $r(0)$, [2]. This curve behaves like an even function, and $K^{(3)}$, being the limit of the difference of two areas with the same symbol devided by a^3, tends towards 0, as the factor $\int O(s^2)\,ds$

is of a higher order. It might be said that $K^{(3)}$ is the "measure" for the deflection of the given curve (1) from its "C^2 standard".

Let us take $y = bx^2$ and calculate $\mathscr{H}$ and $K^{(3)}$ at point O

$$\mathscr{H} = 3 \lim_{a\to 0} \frac{2b \int_0^a x^2 dx}{a^3} = 2b; \qquad K^{(3)} = 0.$$

Let us take one branch, e.g. $x < 0$ and rotate it about the x-axis; then $\mathscr{H} = 0$ but $K^{(3)} = 2b$.

Definition 6. Let us assume $\boldsymbol{r} = \boldsymbol{r}(s) \in C^1$, $\infty > \mathscr{H}^{(\alpha)} \neq 0$. As a normal vector of the order $\alpha(\boldsymbol{n})$ at point P we denote a unit vector lying in the osculating plane which is perpendicular to the tangent vector and directed towards the projection of the curve on this plane, if the projection is situated entirely on one side of the tangent.

As a binormal vector we assume $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$.

Definition 7. By a *rectifying plane* of curve (1) we understand a plane which is perpendicular to the osculating plane and passes through the tangent.

Definition 8. By the *torsion of the order* β of curve (1) at point P we understand the expression $\tau^{(\beta)}$ for which the module (9)

$$|\tau^{(\beta)}| = \lim_{a\to 0} \frac{Q(a)}{a^{\beta-3} P(a)}; \qquad Q(a) = |Q'(a) - Q''(a)|; \qquad \beta \geqslant 3,$$

where $Q'(a)$ and $Q''(a)$ are defined for projection (1) on the rectifying plane in the same way as $P'(a)$ and $P''(a)$ in Definition 1. This also concerns $P(a)$, but it is necessary to take projection (1) on the osculating plane. Instead of $\tau^{(4)}$ we shall simply write τ. The sign of $\tau^{(\beta)}$ is to be determined in the following way:

We take point B which does not coincide with point P, at which we are going to determine the sign of $\tau^{(\beta)}$. Next we take the osculating planes to (1) at the points B and P. The edge of the intersection of the planes is turned towards the semispace, to which is also directed the vector $\boldsymbol{t}$, tangential to the curve at point P. $\tau^{(\beta)}$ is provided with sign $(+)$ if the turn at an acute angle from the plane at point B with a smaller value of the parameter to the osculating plane at point P with a larger parameter value occurs in such a direction that a right-handed screw turned in the same direction would move in conformity to the twist of the edge.

Theorem 6. *This definition conforms to the classical definition when* $\beta = 4$, $r(s) \in C^3$ *and* $\mathscr{H} \neq 0$.

Proof. We may write [2]

$$r = \left(S - \frac{\mathscr{H}^2}{6} S^3\right) t + \left(\frac{\mathscr{H}}{2} S^2 + \frac{1}{6} \mathscr{H}' S^3\right) n_0 + \left(\frac{1}{6} \mathscr{H}\tau^* \cdot S^3\right) b + O(S^3).$$

Let the x-axis be a tangent at point P and let it be directed according to the rising s, let the y-axis be parallel to $\boldsymbol{n}$ and the z-axis parallel to $\boldsymbol{b}$, and we shall obtain around point P ([3])

$$(11) \qquad y = \tfrac{1}{2}\mathscr{H}x^2 + O(x^2); \qquad z = \tfrac{1}{6}H \cdot \tau^* \cdot x^3 + O(x^3), \qquad H = H^*,$$

where τ^* denotes the torsion according to the classical definition. We multiply the second equation (11) by dx and integrate from 0 to a. Next we divide both sides by a^4 and pass to the limit; when $a \to 0$, we get

$$\lim_{a\to 0} \frac{4\left|\int_0^a z\,dx + \int_0^{-a} z\,dx\right|}{a^4} = \frac{1}{3}\mathscr{H}^* \cdot \tau^*; \qquad \lim_{a\to 0} \frac{a^3}{\left|\int_{-a}^{a} y\,dx\right|} = \frac{3}{\mathscr{H}^*}.$$

Hence the limit of the product is equal to τ^*, i.e. $\tau = \tau^*$ (note the fact that the projection on to the osculating plane behaves like an even function, whereas a projection on to the rectifying plane behaves like

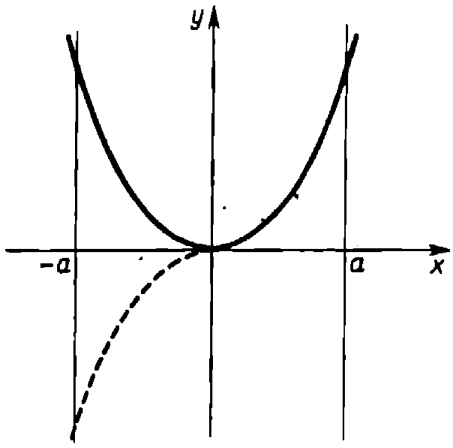

Fig. 4

an odd function around point p). The signs will also be the same as those resulting from their determination [2]. The curves of type C^3 have $\tau < \infty$ at the points at which $\mathscr{H} \neq 0$. The curve shown in Fig. 3 has, at point A, $\tau = \infty$, whereas $\tau^{(3)} = 3$. In the case of curves of the C^3 type, $\tau^{(3)} = 0$ if $\mathscr{H} \neq 0$.

([3]) P is the origin of the coordinate system.

Definition 9. By a *torsion deflection of the order* β we understand the expression

$$\gamma^{(\beta)} = \lim_{a\to 0} \frac{\beta\cdot|Q'(a)-Q'(a)|}{a^{\beta-3}\cdot P(a)}; \quad \beta \geqslant 3, \tag{12}$$

$Q'(a)$, $Q''(a)$ and $P(a)$ have the same meaning as in Definition 8. $\gamma^{(4)}$ for a curve of the C^3 type, for which $\mathscr{H} \neq 0$ is equal to 0, as its projection on to a rectifying plane, behaves like and odd function.

References

[1] F. Biernacki, *Geometria różniczkowa*, Vol. 1, Warszawa 1954.
[2] A. Goetz, *Geometria różniczkowa*, 1959.
[3] Morytko and Marszał, Zeszyty Naukowe WSP, Katowice 1963.

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