ANNALES
POLONICI MATHEMATICI
XXXIX (1981)

A generalization of Pick's theorem and its applications to intrinsic metrics

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Dedicated to the memory of my teacher Stefan Bergman

Abstract. A famous theorem of Pick is generalized in several directions. This leads to various distortion theorems, extending earlier results of us and that of Skwarczyński. Other results related to the Bergman and the Carathéodory metrics are proven.

1. Introduction. A well-known theorem of Pick states that if f is a holomorphic mapping of the unit disk Δ into itself, then the kernel $(1-z\bar{\xi})^{-1}[1-f(z)\overline{f(\xi)}]$ is positive definite. This means that

$$\det\left[\frac{1-f(z_m)\overline{f(z_n)}}{1-z_m\bar{z}_n}\right]_{m,n=1}^{N} \geq 0$$

for any $z_1, \ldots, z_N \in \Delta$ (see Ahlfors [1], p. 3–4). This fact enables one to deduce various distortion theorems on the unit disk as that of the lemma of Schwarz–Pick and alike. We note that here $K(z, \bar{\xi}) = (2\pi)^{-1}(1-z\bar{\xi})^{-1}$ is precisely the Szegö reproducing kernel of the Hardy–Szegö space $H_2(\partial\Delta)$.

In this paper we provide a generalization of this theorem in two directions: (i) the above theorem holds when $K(z, \bar{\xi})$ is the *generalized Szegö kernel* for any plane region $D \notin O_{AB}$ (Theorem 5), and, (ii) that the above theorem holds true for any reproducing kernel $K(z, \bar{\xi})$ of holomorphic forms in a complex manifold M in C^n (Theorem 3 or Corollary 1). The present proof is even simpler than its special case, appearing, for example, in [1], p. 3–4.

With the aid of this generalization we will be able to relate various pseudo-distances, on the manifold M, with the Bergman and the Carathéodory pseudo-distances of M. In a way, this paper constitutes an extension of our earlier work [3], [4]. For example, it was shown in [3] that the Carathéodory pseudo-distance is always dominated by any generalized Bergman pseudo-

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distance. Here, however, this result is strengthened by showing, for example, that $\delta_M = \sqrt{2}\varrho_M$ (Corollary 2), where δ_M is the so-called *Möbius pseudo-distance* of M and ϱ_M is the *Skwarczyński pseudo-distance* of M. A special instance of the pseudo-distance ϱ_M was first studied by Skwarczyński [9] but, here we were able to extend the results of [9] and by so doing, to also find an intimate relation between ϱ_M and the Carathéodorý pseudo-distance of M. This leads to several interesting distortion theorems and statements on the completeness of certain metrics.. Moreover, many of the results of our previous work [3], [4] are included in this paper as special cases.

Section 2 provides a brief review of certain pseudo-distances needed in our work. In Section 3 we study metrics that are induced by reproducing kernels of certain Hilbert spaces of holomorphic forms on M. Here, we generalize the results of [9] in the form of Theorems 1 and 2. The section is concluded with an example concerned with an interesting family of reproducing kernels. Our main results appear in Section 4 where, the generalization of Pick's theorem is stated and proved (Theorem 3). This yields a crucial distortion theorem (Theorem 4) needed for relating ϱ_M with δ_M (Corollary 2). The applications of these results are given in Corollaries 1, 2, 3 and 4. Section 5 is devoted to specializing the results of the previous section to the Szegö reproducing kernel. By means of a canonical exhausition, this implies a yet another generalization of Pick's theorem (Theorem 5) on the generalized Szegö kernel of any plane region $D \notin O_{AB}$.

This paper is dedicated to the memory of my teacher, Professor Stefan Bergman. His pioneering contributions to the general theory of the geometric aspects of several complex variables can hardly be overstated.

2. Preliminaries. Let $\Delta = \{z\colon |z| < 1\}$ be the open unit disk in $\boldsymbol{C}$ and let $G_\Delta = \operatorname{Aut}(\Delta)$ denote the group of holomorphic automorphisms of Δ. Here, any $\varphi \in G_\Delta$ is given by

$$\varphi(z) = e^{i\theta}\frac{z-\xi}{1-\bar{\xi}z}, \qquad z \in \Delta,$$

for some $\xi \in \Delta$ and some $\theta \in [0, 2\pi)$. On Δ we can introduce the distance function

$$\delta_\Delta(z, \xi) = \left|\frac{z-\zeta}{1-\bar{\xi}z}\right|; \qquad z, \zeta \in \Delta. \tag{2.1}$$

It is well known that this is indeed a distance on Δ and that it is invariant under the action of the group G_Δ on Δ. We also note the useful identity

$$1-\delta_\Delta^2(z, \xi) = \frac{(1-|z|^2)(1-|\xi|^2)}{|1-\bar{\xi}z|^2}. \tag{2.2}$$

The *Poincaré metric* $c_\Delta(z\colon dz) = (1-|z|^2)^{-1}|dz|$ has a constant Gaussian curvature, equal to -4 and the *Poincaré destance* induced by this metric is given by

$$(2.3)\qquad c_\Delta(z,\xi) = \tfrac{1}{2}\log\frac{1+\delta_\Delta(z,\xi)}{1-\delta_\Delta(z,\xi)};\quad z,\xi\in\Delta.$$

The following is the celebrated classical *Schwarz–Pick lemma*:

PROPOSITION 1. *Let* $\varphi\colon \Delta\to\Delta$ *be a holomorphic mapping of* Δ *into* Δ. *Then*

$$\delta_\Delta(\varphi(z),\varphi(\xi)) \leqslant \delta_\Delta(z,\xi)$$

and

$$C_\Delta(\varphi(z),\varphi(\xi)) \leqslant C_\Delta(z,\xi)$$

for any z, $\xi\in\Delta$. *Moreover, for a fixed pair* $(z,\xi)\in\Delta\times\Delta$, $z\neq\xi$, *equality in any of the above inequalities occurs if and only if* $\varphi\in G_\Delta$. *Furthermore,*

$$c_\Delta(\varphi(z)\colon d\varphi(z)) \leqslant c_\Delta(z\colon dz)$$

for all $z\in\Delta$ *and equality, at one fixed point* $z\in\Delta$, *occurs if and only if* $\varphi\in G_\Delta$.

Let M be a complex manifold in C^n and let $H(M\colon \Delta)$ denote the family of holomorphic functions from M into Δ. One introduces the *Möbius pseudo-distance*

$$(2.4)\qquad \delta_M(z,\xi) = \sup\{\delta_\Delta(f(z),f(\xi))\colon f\in H(M\colon \Delta)\}$$

of two points z, $\xi\in M$. A normal family argument shows that the supremum is indeed attained by a member of $H(M\colon \Delta)$. This pseudo-distance satisfies all axioms of a distance except that $\delta_M(z,\xi)$ can be zero even if $z\neq\xi$. M is said to belong to class $\mathscr{B}$ if for any two distinct points z, $\xi\in M$ there exists an $f\in H(M\colon \Delta)$ with $f(z)\neq f(\xi)$. Clearly, δ_M is a distance on M if and only if $M\in\mathscr{B}$.

The *Carathéodory pseudo-distance* (cf. [7], p. 49–57) is defined by

$$(2.5)\qquad C_M(z,\xi) = \sup\{C_\Delta(f(z),f(\xi))\colon f\in H(M\colon \Delta)\}:$$

Again, a normal family argument shows that the supremum is attained. Therefore, using the monotonicity of $\log\frac{1+r}{1-r}$, $0\leqslant r<1$, (2.1), (2.2), (2.3) and (2.4) we obtain

$$(2.6)\qquad C_M(z,\xi) = \tfrac{1}{2}\log\frac{1+\delta_M(z,\xi)}{1-\delta_M(z,\xi)};\quad z,\xi\in M.$$

Obviously, the Carathéodory pseudo-distance C_M is a distance if and only if $M\in\mathscr{B}$.

Since

$$\tfrac{1}{2}\log\frac{1+r}{1-r} = \sum_{k=0}^{\infty} \frac{1}{2k+1} r^{2k+1} \geqslant r, \qquad r \in [0, 1),$$

it follows, by use of (2.6), that

$$\delta_M(z, \xi) \leqslant C_M(z, \xi); \quad z, \xi \in M. \tag{2.7}$$

The natural pairing between a cotangent vector α and a tangent vector v is denoted by $\langle \alpha, v \rangle$. Especially, if f is a C^1-function near the point $z = (z_1, \ldots, z_n) \in C^n$, and $v = (v_1, \ldots, v_n) \in C^n$, then

$$\partial_v f(z) = \langle \partial f(z), v \rangle = \sum_{j=1}^{n} \frac{\partial f}{\partial z_i} v_i.$$

For a fixed $\xi \in M$ we write $H_\xi(M\colon \Delta) = \{f \in H(M\colon \Delta)\colon f(\xi) = 0\}$. The *Carathéodory–Reiffen metric* (cf. Reiffen [8]) is given by

$$c_M(\xi\colon v) = \sup\{|\partial_v f(\xi)|\colon f \in H(M\colon \Delta)\}.$$

Evidently,

$$C_M(\xi\colon v) = \max\{|\partial_v f(\xi)|\colon f \in H_\xi(M\colon \Delta)\}, \tag{2.8}$$

and there exists an $F \in H_\xi(M\colon \Delta)$ such that $\partial_v F(\xi) = C_M(\xi\colon v)$. Here $F(z) = F(z\colon \xi, v)$ and $C_M(\xi\colon v) \geqslant 0$. Let $\tilde{C}_M$ denote the integrated metric on M of C_M. Then

$$C_M(z, \xi) \leqslant \tilde{C}_M(z, \xi); \quad z, \xi \in M. \tag{2.9}$$

Clearly, $\tilde{C}_M$ is a pseudo-distance on M and when $M \in \mathscr{B}$ it is also a distance.

Obviously, in the case that M is Δ we have

$$\delta_M = \delta_\Delta, \quad C_M = \tilde{C}_M = C_\Delta, \quad c_M = c_\Delta,$$

which means that the Poincaré distance on Δ is the Carathéodory distance and so on. This, with Proposition 1, leads to the following well-known proposition:

PROPOSITION 2. *Let $\varphi\colon M \to M^*$ be a holomorphic mapping of a complex manifold M of C^n into another complex manifold M^* of C^m. Then*

$$\delta_{M^*}(\varphi(z), \varphi(\xi)) \leqslant \delta_M(z, \xi), \quad C_{M^*}(\varphi(z), \varphi(\xi)) \leqslant C_M(z, \xi),$$
$$\tilde{C}_{M^*}(\varphi(z), \varphi(\xi)) \leqslant \tilde{C}_M(z, \xi),$$

where $z, \xi \in M$. Also,

$$c_{M^*}(\varphi(z); \varphi_*(v)) \leqslant c_M(z\colon v),$$

where $\varphi_*(v) = (\partial_v \varphi_1, \ldots, \partial_v \varphi_m)$. Here $\varphi = (\varphi_1, \ldots, \varphi_m)$ and $\varphi_j(z) = \varphi_j(z_1, \ldots, z_n)$, $1 \leqslant j \leqslant m$.

An immediate consequence of this proposition is that the above pseudo-distances and metric are biholomorphically invariant.

When d_M is a distance function on M, M becomes a metric space (M, d_M). This metric space is said to be complete (see also [7], p. 53) if for each point $\xi \in M$ and each $r > 0$, the closed ball $\{z \in M: d_M(z, \xi) \leqslant r\}$ is a compact subset of M. If (M, d_M) is complete in this sense then it is complete in the usual sense. The converse is not true in general, but it is true if d_M is induced from a Riemannian metric.

According to (2.7) and (2.9), $\delta_M \leqslant C_M \leqslant \tilde{C}_M$, and, therefore, if $M \in \mathscr{B}$, then M is a metric space with respect to each of these three distances. Because of the continuity of these distances we also have:

PROPOSITION 3. *Let* $M \in \mathscr{B}$. *Then*

$$(M, \delta_M) \text{ complete} \Rightarrow (M, C_M) \text{ complete} \Rightarrow (M, \tilde{C}_M) \text{ complete.}$$

We note that (2.4) can be also written as

$$\delta_M(z, \xi) = \max\{|f(z)|: f \in H_\xi(M: \Delta)\}. \tag{2.10}$$

Indeed, by (2.4),

$$\delta_M(z, \xi) = \max\left\{|G(z)|: G(w) = \frac{f(w)-f(\xi)}{1-\overline{f(\xi)}f(w)}, f \in H(M: \Delta)\right\}$$
$$\leqslant \max\{|G(z)|: G \in H_\xi(M: \Delta)\}.$$

On the other hand, if $G \in H_\xi(M: \Delta)$, then

$$\delta_\Delta(G(z), G(\xi)) = \delta_\Delta(G(z), 0) = |G(z)|$$

and (2.10) follows.

In the special case that M is an open Riemann surface (with respect to a local parameter z) then the Carathéodory–Rieffen metric becomes $c_M(z)|dz|$. Here $c_M(\xi) = \max\{|f'(\xi)|: f \in H_\xi(M: \Delta)\}$ is the *analytic capacity* of M at ξ. Assume that $M \notin O_{AB}$, i.e., M admits a non-constant bounded holomorphic function. Then, there exists a unique $F \in H_\xi(M: \Delta)$, called the *Ahlfors function* $F(z) = F(z: \xi)$, with $F'(\xi: \xi) = c_M(\xi) > 0$. Moreover, for two distinct points $\omega, \xi \in M$ there exists a unique $G \in H_\xi(M: \Delta)$, called also the *Ahlfors function* $G(z) = G(z: \xi, \omega)$ with $G(w: \xi, \omega) = \delta_M(\omega, \xi) > 0$.

3. Kernel functions and Bergman distances. Let M be a complex manifold in C^n. We shall assume that $d\mu = d\mu_M$ is a positive measure on M or any other set determining the holomorphic forms on M, for example the *Šilov boundary* of M. With $d\mu$ we consider the usual Hilbert space $H_2(M: \Delta)$ of holomorphic forms on M. In what follows there is no loss of generality in assuming global coordinates z in M (see also [3] for more details). Therefore, for sake of simplicity, we shall assume that $H_2(M) = H_2(M: \mu)$ is in fact a space of holomorphic functions on M with the inner product

$$(f, g) = \int f(z)\overline{g(z)}\, d\mu(z), \qquad \|f\|^2 = (f, f).$$

Some modifications of the argument are required when $d\mu$ does not act on M but, rather, on its Šilov boundary. We shall discuss such a situation in Section 5 when M is a plane region.

We also assume that point evaluations are bounded linear functionals on $H_2(M)$. Therefore, $H_2(M)$ possesses a reproducing kernel $k_\xi(z)$ and convergence in the norm implies uniform convergence on compacta of M. It follows from the usual properties of reproducing kernels that for $\xi \in M$

$$f(\xi) = (f, k_\xi)$$

for each $f \in H_2(M)$. We also write $K(z, \bar{\xi}) = k_\xi(z)$ and note that $K(z, \bar{z})$ is real analytic and non-negative. M is said to belong to class $\mathcal{M}$ if for any $\xi \in M$ there exists an $f \in H_2(M\colon \mu)$ so that $f(\xi) \neq 0$. Clearly, $M \in \mathcal{M}$ if and only if $K(z, \bar{z}) > 0$ for every $z \in M$. We shall also assume that, indeed, $M \in \mathcal{M}$.

Let $\xi \in M$ and consider the circle

$$U_M(\xi) = \left\{ f \in H_2(M\colon \mu)\colon f = e^{i\theta} \frac{k_\xi}{\|k_\xi\|}, 0 \leqslant \theta \leqslant 2\pi \right\}.$$

Here, $\|k_\xi\| = \sqrt{K(\xi, \bar{\xi})} > 0$. The following pseudo-distance on M was first studied by Skwarczyński [9] when $d\mu$ is the usual volume element of M. We write

$$\varrho_M(t, \xi) = \frac{1}{\sqrt{2}} \operatorname{dist}\big(U_M(t), U_M(\xi)\big); \qquad t, \xi \in M.$$

This is clearly a pseudo-distance on M and we label it as the *Skwarczyński pseudo-distance* of M. As was mentioned in [9] this pseudo-distance can be expressed in terms of the reproducing kernel of M. In fact,

$$\varrho_M(t, \xi) = \frac{1}{\sqrt{2}} \min_{\varphi, \theta} \left\| e^{i\varphi} \frac{k_t}{\|k_t\|} - e^{i\theta} \frac{k_\xi}{\|k_\xi\|} \right\| = \frac{1}{\sqrt{2}} \left\{ 2 - 2 \left| \left(\frac{k_t}{\|k_t\|}, \frac{k_\xi}{\|k_\xi\|} \right) \right| \right\}^{\frac{1}{2}}.$$

Therefore,

$$\varrho_M(t, \xi) = \left[1 - \left(\frac{|K(t, \bar{\xi})|^2}{K(t, \bar{t})\, K(\xi, \bar{\xi})} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \tag{3.1}$$

which is the desired expression. We now prove:

THEOREM 1. *Let $M \in \mathcal{M} \cap \mathcal{B}$. Then ϱ_M is a distance on M.*

Proof. Assume that $\varrho_M(z, \xi) = 0$ for $z, \xi \in M$. Therefore, using (3.1) we have that $|K(z, \bar{\xi})|^2 = K(z, \bar{z}) K(\xi, \bar{\xi})$. Using the properties of reproducing kernels this is equivalent to

$$K(t, \bar{\xi}) = \lambda K(t, \bar{z}) \tag{3.2}$$

for some $\lambda \in C$ and all $t \in M$. This shows that $g(\xi) = \bar{\lambda} g(z)$ for every $g \in H_2(M: \mu)$. However, for every $s \in M$ and each $f \in H(M: \Delta)$ the functions

$$g_s(t) = f(t) K(t, \bar{s})$$

are in $H_2(M: \mu)$. In fact, $\|g_s\|^2 \leqslant K(s, \bar{s})$. Therefore,

$$f(\xi) K(\xi, \bar{s}) = \bar{\lambda} f(z) K(z, \bar{s})$$

and, in view of (3.2), $f(\xi) = f(z)$. Consequently, $f(\xi) = f(z)$ for each $f \in H(M: \Delta)$ and since $M \in \mathscr{B}$ this implies that $z = \xi$. This concludes the proof.

The second differential of ϱ_M^2 yields a semidefinite Kähler metric which we also label as the *Bergman metric*. In fact, for a tangent vector $v = (dt_1, \ldots, dt_n) \in C^n$ we have

$$d^2 \varrho_M^2(t, \xi)|_{t=\xi} = \partial_v \bar{\partial}_v \log K, \tag{3.3}$$

where $K = K(\xi, \bar{\xi})$ and $\bar{\partial}_v = \sum_{j=1}^{n} \bar{v}_i \frac{\partial}{\partial \bar{z}_j}$. When $d\mu$ is the usual volume element, (3.3) is, precisely, the classical Bergman metric of M. We write

$$b_M^2(\xi: v) = \partial_v \bar{\partial}_v \log K, \quad K = K(\xi, \bar{\xi}), \tag{3.4}$$

and note that in analogy with (2.8) we have

$$b_M(\xi: v) = \sqrt{K(\xi, \bar{\xi})} \max \{|\partial_v f(\xi)|: f \in H_\xi(M: U)\},$$

where

$$H_\xi(M: U) = \{f \in H_2(M: \mu): \|f\| \leqslant 1, f(\xi) = 0\}.$$

We note that $d\varrho_M^2(t, \xi)|_{t=\xi} = 0$ and that the critical points of $\varrho_M(t, \xi)$ are intimately related to the so-called *Lu Qi-Keng conjecture* and the *Bergman representative domain* (see [9] for more details).

Let B_M denote the integrated metric on M of b_M. B_M is called the *Bergman pseudo-distance* of M. Again, if $d\mu$ is the volume element of M, then B_M is the usual Bergman pseudo-distance of M.

We should emphasize here that, in general, ϱ_M, B_M and b_M are not necessarily biholomorphically invariant. We, however, make the following observation: Let $\varphi: M \to M^*$ be a biholomorphic mapping of M onto M^* with a non-vanishing Jacobian $J_\varphi = \partial w / \partial z$, $w = \varphi(z)$. Assume that there is an $\alpha > 0$ such that $[J_\varphi(z)]^\alpha$ is holomorphic in M and that

$$\mu_{M^*}(\varphi(N)) = \int_N |J_\varphi(z)|^{2\alpha} d\mu_M(z)$$

for each Borel subset N of M. Then $f \to (f_0 \varphi) J_\varphi^\alpha$ is an isometry of $H_2(M^*: \mu_{M^*})$ onto $H_2(M: \mu_M)$ and therefore

$$K_M(z,\bar{\xi}) = K_{M^*}(\varphi(z), \overline{\varphi(\xi)})\,[J_\varphi(z)]^\alpha\,[\overline{J_\varphi(\xi)}]^\alpha \tag{3.5}$$

for z, $\xi \in M$. Using (3.1) and (3.4) we obtain (see also [3]):

THEOREM 2. *Let the assumptions of formula* (3.5) *prevail. Then* ϱ_M, B_M *and* b_M *are biholomorphically invariant.*

EXAMPLE. Let D be a simply connected plane region and let $K(z,\bar{\xi})$ be its usual Bergman kernel function. Let $q > 0$; then

$$K_q(z,\bar{\xi}) = \Gamma(2q)\pi^q[K(z,\bar{\xi})]^q; \quad z, \xi \in D,$$

is the reproducing kernel of a certain Hilbert space of holomorphic functions $\mathscr{D}_q(D)$ called *Hardy space* (cf. [5]). We then have that

$$\varrho_D^{(q)}(z,\xi) = \left[1 - \left(\frac{|K(z,\xi)|^2}{K(z,\bar{z})K(\xi,\bar{\xi})}\right)^{q/2}\right]^{1/2} \tag{3.6}$$

is a distance function on D for any $q > 0$. Similarly, $b_D^{(q)} = \sqrt{q}\, b_D^{(1)}$ and $B_D^{(q)} = \sqrt{q}\, B_D^{(1)}$, where $b_D^{(1)}$ and $B_D^{(1)}$ are the usual Bergman–Poincaré metric and distance, respectively, of D. Using (3.6) and identity (2.2) we obtain, in the case that D is the unit disk Δ,

$$\varrho_\Delta^{(q)}(z,\xi) = [1-(1-\delta_\Delta^2(z,\xi))^q]^{1/2}$$

and thus $\varrho_\Delta^{(1)} = \delta_\Delta$. Other interesting examples can be obtained by using different reproducing kernels.

4. A generalization of Pick's theorem. Let M be a complex manifold as before and let $f \in H(M\colon \Delta)$. We now introduce another Hermitian kernel

$$B(z,\bar{\xi}) = K(z,\bar{\xi})[1 - f(z)\overline{f(\xi)}]; \quad z, \xi \in M.$$

Clearly, for any $\xi \in M$, $B(z,\bar{\xi})$ is in $H_2(M\colon \mu)$ and, in fact,

$$\|B(\ ,\bar{\xi})\|^2 \leqslant 4K(\xi,\bar{\xi}).$$

Also, $B(z,\bar{z}) \geqslant 0$ for $z \in M$. We also write

$$R(z,\xi) = K(\xi,\bar{z})[f(z) - f(\xi)]$$

and

$$S(z,\xi) = K(\xi,\bar{z})[1 - |f(z)|^2]^{1/2}.$$

For any $\xi \in M$ these functions belong to $L_2(M\colon \mu)$ and we have

$$\|R(\ ,\xi)\|^2 \leqslant 4K(\xi,\bar{\xi}),$$

and,

$$\|S(\ ,\xi)\|^2 \leqslant 2K(\xi,\bar{\xi}).$$

Again, some minor modifications of the argument are required when $d\mu$ acts on the Šilov boundary of M. This, along with other possible extensions, will be described in the next section in the case of plane regions. We now prove:

LEMMA 1. *For $z, \xi \in M$ we have*

$$B(z,\bar{\xi}) = \int [R(t,z)\overline{R(t,\xi)} + S(t,z)\overline{S(t,\xi)}]\,d\mu(t).$$

Proof. In view of the reproducing property,

$$B(z,\bar{\xi}) = \int B(t,\bar{\xi})\overline{K(t,\bar{z})}\,d\mu(t) = \int K(t,\bar{\xi})[1-f(t)\overline{f(\xi)}]\,\overline{K(t,\bar{z})}\,d\mu(t).$$

But

$$1-f(t)\overline{f(\xi)} = 1-|f(t)|^2+f(t)[\overline{f(t)}-\overline{f(\xi)}].$$

Also

$$f(t)[\overline{f(t)}-\overline{f(\xi)}] = [f(t)-f(z)][\overline{f(t)}-\overline{f(\xi)}]+f(z)[\overline{f(t)}-\overline{f(\xi)}].$$

We observe, however, that

$$\int K(t,\bar{\xi})\{f(z)[\overline{f(t)}-\overline{f(z)}]\}\,\overline{K(t,\bar{z})}\,d\mu(t)$$

is precisely zero by the very definition of a reproducing kernel. Consequently,

$$B(z,\bar{\xi}) = \int K(t,\bar{\xi})\{1-|f(t)|^2+[f(t)-f(z)][\overline{f(t)}-\overline{f(\xi)}]\}\,\overline{K(t,\bar{z})}\,d\mu(t)$$

which concludes the proof.

The following theorem is a generalization of the theorem of Pick (see, for example, Ahlfors [1], p. 3–4) on the Szegö kernel of the unit disk $K(z,\bar{\xi}) = (2\pi)^{-1}(1-z\bar{\xi})^{-1}$. Our theorem holds for any reproducing kernel of $H_2(M\colon \mu)$ with M being any complex manifold. Our proof is also simpler.

THEOREM 3. *Let $f \in H(M\colon \Delta)$. Then the kernel $B(z,\bar{\xi})$ is positive definite on M. That is, given any finite number of points $z_1, \ldots, z_N \in M$ and corresponding numbers $\alpha_1, \ldots, \alpha_N \in C$ we have*

$$\sum_{m,k=1}^{N} B(z_m,\bar{z}_k)\alpha_m\overline{\alpha_k} \geqslant 0.$$

Proof. We use Lemma 1. Then

$$\sum_{m,k=1}^{N} B(z_m,\bar{z}_k)\alpha_m\overline{\alpha_k} = \sum_{m,k=1}^{N} \alpha_m\overline{\alpha_k}\int\{R(t,z_m)\overline{R(t,z_k)}+S(t,z_m)\overline{S(t,z_k)}\}\,d\mu(t)$$

$$= \int\Big\{\Big|\sum_{m=1}^{N}\alpha_m R(t,z_m)\Big|^2+\Big|\sum_{m=1}^{N}\alpha_m S(t,z_m)\Big|^2\Big\}\,d\mu(t) \geqslant 0$$

and the theorem is proved.

COROLLARY 1. *Let $f \in H(M\colon \Delta)$. Then*

$$\det\,[B(z_m,\bar{z}_k)]_{m,k=1}^{N} \geqslant 0$$

for any $z_1, \ldots, z_N \in M$.

THEOREM 4. *Let $M \in \mathscr{M}$ and let $f \in H(M\colon \Delta)$. Then*

$$\frac{|K(z,\bar{\xi})|^2}{K(z,\bar{z})K(\xi,\bar{\xi})} \leqslant \frac{(1-|f(z)|^2)(1-|f(\xi)|^2)}{|1-f(z)\overline{f(\xi)}|^2}$$

for any $z, \xi \in M$.

Proof. This follows from Corollary 1 by taking $N = 2$ and $z_1 = z$, $z_2 = \xi$.

COROLLARY 2. *Let* $M \in \mathcal{M}$. *Then*

$$2\varrho_M^2(z,\xi) \geqslant \varrho_M^4(z,\xi)+\delta_M^2(z,\xi) \geqslant \delta_M^2(z,\xi)$$

for any $z, \xi \in M$.

Proof. Let $\varrho_M = \varrho_M(z,\xi)$ and $\delta_M = \delta_M(z,\xi)$. Using (2.2), (2.4), (3.1) and Theorem 2 we obtain

$$(2-\varrho_M^2)\varrho_M^2 \geqslant \delta_M^2.$$

This concludes the proof.

In our previous work [3] (see also [6]) we have shown that for $M \in \mathcal{M}$, $C_M \leqslant b_M$. Theorem 4 and Corollary 2 constitute an improvement of this result as the following corollary shows.

COROLLARY 3. *Let* $M \in \mathcal{M}$. *For each* $\xi \in M$ *and each tangent vector* $v \in C^n$ *we have*

$$c_M(\xi\colon v) \leqslant b_M(\xi\colon v).$$

Proof. Let $f \in H_\xi(M\colon \Delta)$ and consider the function

$$g(z,\bar{z}) = \log(1-|f(z)|^2)-\log\frac{|K(z,\bar{\xi})|^2}{K(z,\bar{z})K(\xi,\bar{\xi})}.$$

According to Theorem 4, $g \geqslant 0$ and, moreover, g assumes a local minimum at $z = \xi$. Therefore, for each direction $v \in C^n$ the Hessian of g

$$\Delta_v g \equiv 4\partial_v\bar{\partial}_v g$$

is non-negative at $z = \xi$. However, by a direct computation,

$$\Delta_v g = \Delta_v \log(1-|f(z)|^2)+\Delta_v \log K(z,\bar{z}),$$

or

$$\Delta_v g = -4\frac{1}{(1-|f(z)|^2)^2}|\partial_v f|^2+4\partial_v\bar{\partial}_v \log K(z,\bar{z}).$$

Consequently,

$$\partial_v\bar{\partial}_v \log K \geqslant |\partial_v f(\xi)|^2$$

and the corollary follows by appealing to (2.8) and (3.4).

COROLLARY 4. *Let* $M \in \mathcal{M}$. *Then*

(4.1) $$\delta_M \leqslant C_M \leqslant \tilde{C}_M \leqslant B_M, \quad \delta_M \leqslant \sqrt{2}\,\varrho_M.$$

Moreover, if in addition $M \in \mathscr{B}$, then all these pseudo-distances are in fact distances on M. Also, (M, ϱ_M) is complete whenever (M, δ_M) is complete. Furthermore,

(4.2) $$(M, \delta_M) \text{ complete} \Rightarrow (M, C_M) \text{ complete} \Rightarrow (M, \tilde{C}_M) \text{ complete} \Rightarrow (M, B_M) \text{ complete}.$$

Proof. From Corollary 3 we have $c_M \leqslant b_M$ and passing to the integrated metrices we obtain that $C_M \leqslant B_M$. This with (2.7), (2.9) and Corollary 2 implies (4.1). Now, δ_M is a distance on M if and only if $M \in \mathscr{B}$ and therefore the above pseudo-distances are indeed distances on M whenever $M \in \mathscr{B}$. If (M, δ_M) is complete, then since $\delta_M \leqslant \sqrt{2}\,\varrho_M$ we have that (M, ϱ_M) is complete also. Exactly as in Proposition 3, (4.2) follows from (4.1). This concludes the proof.

5. The Szegö kernel. As was mentioned earlier, some care is needed when $d\mu$ does not act on M but rather, say, on its Šilov boundary. This care is mostly needed with the arguments leading to the generalization of Pick's theorem (Theorem 3). The crucial relevant item here is, of course, Lemma 1 and the precise meaning of the kernels involved in its proof. This, however, does not constitute a serious problem for, the relevant arguments can be always modified in a natural and an obvious way. As an illustration we shall treat the case when M is a plane region $D \notin O_{AB}$. In this special case, it was shown in our previous work [3], [4] that the present Corollary 3 implies numerous relationships between the analytic capacity and the curvatures of certain conformal invariant metrics. The most important relationship is the one showing that the curvature of the analytic capacity metric is always $\leqslant -4$. Here, we shall extend these results to yet another generalization of Pick's theorem.

We assume first that D is a plane region whose boundary D consists of a finite number of rectifiable curves. In this case, there is no problem of introducing the Hardy–Szegö space $H_2(\partial D)$. This space stands for the Hilbert space of holomorphic functions in D with the scalar product

$$(f, g) = \int_{\partial D} f(z)\overline{g(z)}\,|dz|.$$

The integration is carried over the boundary values of the holomorphic functions f and g (this refers to an arbitrary non-tangential approach). The space $H_2(\partial D)$ admits a reproducing kernel $K(z, \bar{\xi})$ which is the classical Szegö kernel for D. Moreover, as is well known, $c_D(\xi) = 2\pi K(\xi, \bar{\xi})$ is precisely the analytic capacity and the Ahlfors function is given by $F(z: \xi) = K(z, \bar{\xi})/L(z, \xi)$. (Cf. [2], p. 118.) Here, $L(z, \xi)$ is the *adjoint* of the kernel $K(z, \bar{\xi})$. Under these circumstances the entire previous results hold true. Especially, if $f \in H(D: \Delta)$, then the above mentioned kernels $B(z, \bar{\xi})$, $R(z, \xi)$

and $S(z, \xi)$ are well defined, and, the statement of Lemma 1 leads to the following interesting identity:

$$B(z, \bar{\xi}) = \int_{\partial D} [R(t, z)\overline{R(t, \xi)} + S(t, z)\overline{S(t, \xi)}]\,|dt| \tag{5.1}$$

for any $z, \xi \in D$. Consequently, Theorem 3 and its corollaries are also valid.

We now pass to the more general case where D is merely assumed to be $D \notin O_{AB}$. Let $\{D_m\}$ be a canonical exhaustion of D such that each ∂D_m consists of a finite number of analytic curves. Here, D_m eventually contains each compact subset of D. In every D_m we have the Szegö kernel $K_m(z, \bar{\xi})$, the analytic capacity $c_m(z) \equiv c_{D_m}(z)$ and the Ahlfors function $F_m(z) = F_m(z\colon \xi)$. Clearly, $F'_m(\xi) = c_M(\xi) = 2\pi K_m(\xi, \bar{\xi})$ and $F_m(\xi) = 0$. Under these circumstances, $\{F_m(z)\}$ and $\{c_m(z)\}$ converge uniformly on compacta of D to $F(z)$ and $c_D(z)$. The same is true for the sequence $\{K_m(z, \bar{\xi})\}$ as was pointed out by Suita in [10]. Indeed, we clearly have

$$|K_m(z, \bar{\xi})|^2 \leqslant K_m(z, \bar{z})\,K_m(\xi, \bar{\xi})$$

which shows that $\{K_m(z, \bar{\xi})\}$ is uniformly bounded on compacta of D. Therefore, $\{K_m(z, \bar{\xi})\}$ forms a normal family of holomorphic functions in the two complex variables $(z, \bar{\xi})$ with $(z, \xi) \in D \times D$. If there were two limit functions $K(z, \bar{\xi})$ and $K^*(z, \bar{\xi})$, then the identity $c_D(z) = 2\pi K(z, \bar{z}) = 2\pi K^*(z, \bar{z})$ on the diagonal, implies that these two limits must coincide. Consequently, $\{K_m(z, \bar{\xi})\}$ converges uniformly on compacta of D to a function $K(z, \bar{\xi})$ which is holomorphic in $(z, \bar{\xi})$ with $(z, \xi) \in D \times D$.

The function $K(z, \bar{\xi})$ will be called the *generalized Szegö kernel* for $D \notin O_{AB}$. It is clearly an Hermitian positive definite kernel with $2\pi K(z, \bar{z}) = c_D(z)$ for every $z \in D$. When $f \in H(D\colon \Delta)$, the kernels $B_m(z, \bar{\xi})$, $R_m(z, \xi)$ and $S_m(z, \xi)$ are obviously well-defined and, of course, $\{B_m(z, \bar{\xi})\}$ converges uniformly on compacta of D to $B(z, \xi) = K(z, \bar{\xi})\,[1 - f(z)\overline{f(\xi)}]$. This, with (5.1) implies that

$$B(z, \xi) = \lim_{m\to\infty} \int_{\partial D_m} [R_m(t, z)\overline{R_m(t, \xi)} + S_m(t, z)\overline{S_m(t, \xi)}]\,|dt|$$

for any $z, \xi \in D$. Exactly as before, this leads to yet another generalization of Pick's theorem as follows:

THEOREM 5. *Let D be a plane region, $D \notin O_{AB}$, and let $K(z, \bar{\xi})$ be its generalized Szegö kernel. Assume that $f \in H(D\colon \Delta)$. Then*

$$\det\left[K(z_m, \bar{z}_n)\left(1 - f(z_m)\overline{f(z_n)}\right)\right]_{m,n=1}^{N} \geqslant 0$$

for any $z_1, \ldots, z_n \in D$.

The rest of the other extensions of the previous results follow in a similar way. Especially, one obtains several interesting results when $f \in H(D\colon \Delta)$ is replaced by the Ahlfors functions (see [3], [4] for more details).

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Reçu par la Rédaction le 26. 1. 1979