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ON CERTAIN CLASSES OF DISTRIBUTIONS

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1. Introduction. A characteristic function $\alpha(u)$ is said to be *infinitely divisible* if $[\alpha(u)]^{1/n}$ is also a characteristic function for every positive integer n, provided that one selects the principal branch for the n-th root. Let $F(x)$ be an arbitrary distribution function and $\varphi(u)$ its characteristic function. Set

$$\gamma(u) = \exp\left\{-\lambda \int_0^u \int_0^y \varphi(x)\,dx\,dy\right\} \tag{1.1}$$

with $\lambda > 0$. Then $\gamma(u)$ is an infinitely divisible characteristic function and its distribution function has a finite variance ([3], Theorem 12.2.8). The characteristic function $\gamma(u)$ is said to *belong to the class* $\mathscr{K}$. If $F(x)$ has a unique mode at $x = 0$, then $\gamma(u)$ is said to *belong to the class* $\mathscr{L}$. Furthermore, $\gamma(u)$ is said to *belong to the class* $\mathscr{M}$ if the probability density of $F(x)$ is a convex function. The purpose of the present paper is to investigate the classes $\mathscr{K}$, $\mathscr{L}$, and $\mathscr{M}$. The connections of these classes with the classes L and U are also investigated.

2. The results. A characteristic function $\gamma(u)$ is infinitely divisible if and only if it has a unique representation of the form

$$\gamma(u) = \exp\left\{icu - \frac{\sigma^2}{2}u^2 + \int_{R\setminus\{0\}} \left[e^{iux} - 1 - \frac{iux}{1+x^2}\right] dM(x)\right\}, \tag{2.1}$$

where $M(x)$ is a non-decreasing function on $R\setminus\{0\}$ with

$$\int_{-\varepsilon}^{\varepsilon} x^2\,dM(x) < \infty \quad \text{for every } \varepsilon > 0$$

and the constants σ^2 and c satisfy the conditions $\sigma^2 \geqslant 0$, $c \in R$. The representation (2.1) is the Levy canonical representation of $\gamma(u)$, and $M(x)$ is its Levy spectral function. The Levy canonical representation is a generalization of the Kolmogorov canonical representation which is valid for the characteristic functions of infinitely divisible distributions with finite variance [3]. An infinitely divisible characteristic function $\gamma(u)$ is said to *belong to the class L* if the function $M(x)$ has left and right derivatives $M'(x)$ everywhere and if the function $xM'(x)$ is non-increasing on $R\backslash\{0\}$ (see [3]). If $M(x)$ is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$, then $\gamma(u)$ is said to *belong to the class U*. The representation

$$\gamma(u) = \exp\left\{\frac{1}{u}\int_0^u \log\theta(y)\,dy\right\}, \tag{2.2}$$

where $\theta(u)$ is an infinitely divisible characteristic function, is a necessary and sufficient condition for $\gamma(u)$ to belong to the class U (see [4]).

A distribution function $F(x)$ is said to be *unimodal* with mode at $x = 0$ if $F(x)$ is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. The characteristic function $\varphi(u)$ of a (0) unimodal distribution function is of the form

$$\varphi(u) = \int_0^1 \beta(ux)\,dx,$$

where $\beta(u)$ is a characteristic function ([3], Theorem 4.5.1). Below we use the transformation (1.1) to investigate the connection of the family of (0) unimodal distributions with the family of infinitely divisible distributions.

THEOREM 1. *The characteristic function $\gamma(u)$ of the class $\mathscr{K}$ belongs to the class $\mathscr{L}$ if and only if $\gamma(u) = \gamma^{1/r}(ru)\gamma_r(u)$, where $0<r<1$ and $\gamma_r(u)$ is a characteristic function of the class $\mathscr{K}$.*

Proof. First assume that $\gamma(u)$ belongs to the class $\mathscr{L}$ and let $f(x)$ be the probability density function of $F(x)$. The (0) unimodality of $F(x)$ implies that $f(x)$ is non-decreasing for $x<0$ and non-increasing for $x>0$. Hence $f(x) - f(x/r) > 0$ on $R\backslash\{0\}$ with $0<r<1$. The function F_r defined by

$$F_r(x) = [F(x) - rF(x/r)]/(1-r)$$

is a distribution function, and

$$\varphi_r(u) = [\varphi(u) - r\varphi(ru)]/(1-r)$$

is its characteristic function. Since

$$\gamma_r(u) = \gamma(u)/\gamma^{1/r}(ru) = \exp\left\{-\mu\int_0^u\int_0^y \varphi_r(x)\,dx\,dy\right\},$$

where $\mu = \lambda/(1-r)$, it follows that $\gamma_r(u)$ belongs to the class $\mathscr{K}$. The converse can be proved by reversing the argument.

Below we extend a modified form of Theorem 2 of [4] to characterize the class $\mathscr{M}$.

THEOREM 2. *The characteristic function $\gamma(u)$ of the class $\mathscr{K}$ belongs to the* *class $\mathscr{M}$ if and only if $\gamma(u) = \gamma(ru)\gamma_r(u)$, where $0 < r < 1$ and $\gamma_r(u)$ is a characteristic function of the class $\mathscr{L}$.*

Proof. First assume that $\gamma(u)$ belongs to the class $\mathscr{M}$. The convexity of $f(x)$ implies that $f'(x)$ is non-decreasing on $R\setminus\{0\}$, where $f'(x)$ denotes the right derivative of $f(x)$. Hence $f'(x) - f'(x/r) > 0$ for $x < 0$ and $f'(x) - f'(x/r) < 0$ for $x > 0$. Furthermore, the function F_r defined by

$$F_r(x) = [F(x) - r^2 F(x/r)]/(1-r^2)$$

is a (0) unimodal distribution function, and

$$\varphi_r(u) = [\varphi(u) - r^2 \varphi(ru)]/(1-r^2)$$

is its characteristic function. Since

$$\gamma_r(u) = \gamma(u)/\gamma(ru) = \exp\{-\mu \int_0^u \int_0^y \varphi_r(x)\,dx\,dy\},$$

where $\mu = \lambda/(1-r^2)$, it follows that $\gamma_r(u)$ belongs to the class $\mathscr{L}$.

Conversely, assume that $\gamma_r(u) = \gamma(u)/\gamma(ru)$ belongs to the class $\mathscr{L}$. The (0) unimodality of the distribution function $F_r(x)$ implies that its density $f_r(x) = [f(x) - rf(x/r)]/(1-r^2)$ is non-decreasing for $x < 0$ and non-increasing for $x > 0$. We shall show that $f(x)$ is convex on $R\setminus\{0\}$. Following O'Connor we get

$$f(x_1) - f(x_2) \geqslant rf(x_1/r) - rf(x_2/r) \quad \text{with } 0 < x_1 < x_2.$$

For $r = x_1/x_2$ we have

$$f(x_1) - f(x_2) \geqslant rf(x_2) - rf\left(x_2 - \frac{x_1 - x_2}{r}\right)$$

or, equivalently,

$$\frac{1}{1+r} f(x_1) + \frac{r}{1+r} f\left(x_2 - \frac{x_1 - x_2}{r}\right) \geqslant f(x_2).$$

Hence $f(x)$ is convex for $x > 0$. In a similar way we can prove that $f(x)$ is convex for $x < 0$. Hence $\gamma(u)$ belongs to the class $\mathscr{M}$.

For any $\gamma(u)$ belonging to the class $\mathscr{K}$ the characteristic function ψ defined by

$$\psi(u) = \exp\left\{\frac{1}{u}\int_0^u \log\gamma(y)\,dy\right\} = \exp\left\{-\frac{\lambda}{u}\int_0^u \int_0^y \int_0^w \varphi(x)\,dx\,dw\,dy\right\}$$

belongs to the class U. By Lemma 2 of [1] and the correspondence between the Kolmogorov and Levy canonical representations, the Levy spectral function of $\psi(u)$ is given by

$$M(x) = \begin{cases} -\lambda \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} \frac{dF(w)}{w^3} dy, & x < 0, \\ -\lambda \int\limits_{x}^{+\infty} \int\limits_{y}^{+\infty} \frac{dF(w)}{w^3} dy, & x > 0. \end{cases}$$

Letting $\gamma(u)$ belong to the class $\mathscr{L}$ we infer that $M_r(x) = M(x) - rM(x/r)$ is a Levy spectral function which is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. Hence $\psi(u)$ satisfies the functional equation $\psi(u) = \psi^r(ru)\psi_r(u)$, where $\psi_r(u)$ is a member of the class U.

A distribution function $F(x)$ with a well-defined (except possibly at $x = 0$) density $f(x)$ is said to be *α-unimodal* (with mode at $x = 0$) if $f(x)/\alpha|x|^{\alpha-1}$ is non-decreasing for $x < 0$ and non-increasing for $x > 0$. In terms of characteristic functions, $F(x)$ is α-unimodal if and only if its characteristic function $\varphi(u)$ is of the form

$$\varphi(u) = \alpha \int\limits_0^1 \beta(ux) x^{\alpha-1} dx,$$

where $\beta(u)$ is a characteristic function [5]. From the fact the Levy spectral function of $\gamma(u)$ is given by

$$M(x) = \lambda \int\limits_{-\infty}^{x} \frac{dF(y)}{y^2}, \ x < 0, \qquad M(x) = -\lambda \int\limits_{x}^{+\infty} \frac{dF(y)}{y^2}, \ x > 0,$$

it follows that $\gamma(u)$ belongs to the class U if and only if $F(x)$ is α-unimodal with $\alpha = 3$. Equivalently, using the representation (2.2) for the members of the class U we infer that $\gamma(u)$ belongs to the class U if and only if

$$\theta(u) = \exp\{-\lambda u \int\limits_0^u \varphi(y) dy - \lambda \int\limits_0^u \int\limits_0^y \varphi(x) dx dy\}$$

is an infinitely divisible characteristic function. Similarly, $\gamma(u)$ belongs to the class L if and only if $F(x)$ is α-unimodal with $\alpha = 2$.

Any $\gamma(u)$ member of the class $\mathscr{L}$ can be decomposed into two characteristic functions of the class L. Indeed, the (0) unimodality of $F(x)$ implies that its characteristic function

$$\varphi(u) = \int\limits_0^1 \beta(ux) dx$$

can be written in the form

$$\varphi(u) = \tfrac{1}{2}\varphi_1(u) + \tfrac{1}{2}\varphi_2(u),$$

where

$$\varphi_1(u) = 2\int_0^1 \varphi(ux)\,x\,dx \quad \text{and} \quad \varphi_2(u) = 2\int_0^1 \beta(ux)\,x\,dx$$

are characteristic functions of α-unimodal distributions ($\alpha = 2$). Letting

$$\gamma_i(u) = \exp\{-\mu \int_0^u \int_0^y \varphi_i(x)\,dx\,dy\} \quad \text{with } \mu = \lambda/2,$$

we obtain $\gamma(u) = \gamma_1(u)\gamma_2(u)$. In a similar way we can prove that any $\gamma(u)$ belonging to the class $\mathscr{M}$ can be decomposed into two characteristic functions of the class U.

In Theorem 3, members of the classes $\mathscr{L}$ and $\mathscr{M}$ are represented as limits of sequences of characteristic functions belonging to the class $\mathscr{K}$.

THEOREM 3. *Let $\gamma_1(u)$ and $\gamma_2(u)$ be characteristic functions of the classes $\mathscr{L}$ and $\mathscr{M}$, respectively. Then*

(i)
$$\gamma_1(u) = \lim_{n\to\infty} \prod_{\varkappa=1}^{n} \left[\delta_1\left(\frac{\varkappa u}{n}\right)\right]^{n/\varkappa^2},$$

where $\delta_1(u)$ is a member of the class $\mathscr{K}$;

(ii)
$$\gamma_2(u) = \lim_{n\to\infty} \prod_{\varkappa=1}^{n} \left[\delta_2\left(\frac{\varkappa u}{n}\right)\right]^{(n-\varkappa)/\varkappa^2},$$

where $\delta_2(u)$ is a member of the class $\mathscr{K}$.

Proof. (i) The characteristic function $\gamma_1(u)$ is of the form (1.1) with

$$\varphi(u) = \int_0^1 \beta(ux)\,dx.$$

Using the partition $\{0, 1/n, 2/n, \ldots, n/n\}$ of the interval $[0, 1]$ and the fact that

$$\delta_1(u) = \exp\{-\lambda \int_0^u \int_0^y \beta(x)\,dx\,dy\}$$

belongs to the class $\mathscr{K}$, we infer that part (i) of the theorem is valid.

(ii) Set $\delta_2(u) = \exp\{-\lambda u^2 \varphi(u)\}$. Then $\delta_2(u)$ belongs to the class $\mathscr{K}$ if and only if $\varphi(u)$ belongs to a distribution function $F(x)$ having a convex density ([3], Theorem 12.2.8, and [6], Theorem 1). Since any characteristic function $\gamma(u)$ of the class $\mathscr{K}$ can be written in the form

$$\gamma(u) = \exp\{-\lambda u^2 \int_0^1 \varphi(ux)(1-x)\,dx\}, \tag{2.3}$$

from (i) it follows that part (ii) of the theorem is also valid.

Below we extend the representation (2.3) to introduce an interesting sequence of subclasses of the class $\mathscr{K}$.

THEOREM 4. *Let $\mathscr{K}_p$ $(p = 2, 3, \ldots)$ be the class of all functions defined by*

$$\psi(u) = \exp\{-\lambda(p+1)\int_0^1 \varphi(ux)(1-x)^p\,dx\},$$

where $\varphi(u)$ is a characteristic function. Then $\mathscr{K}_{p+1} \subset \mathscr{K}_p \subset \mathscr{K}$.

Proof. For any infinitely divisible characteristic function $\gamma(u)$ the transformation defined by

$$\psi_q(u) = \exp\left\{\frac{q}{u^q}\int_0^u \log\gamma(y)\,y^{q-1}\,dy\right\}, \qquad q > 0,$$

yields an infinitely divisible characteristic function [2]. Applying this transformation to $\gamma(u)$ in (1.1) successively for $q = 1, 2, \ldots, p-1$, we get the infinitely divisible characteristic function

$$\psi_p(u) = \exp\left\{-\lambda\frac{p-1}{u^{p-1}}\int_0^u\left[\frac{p-2}{u_{p-2}^{p-2}}\int_0^{u_{p-2}} \cdots \frac{1}{u^1}\int_0^{u_1}\int_0^y\int_0^w \varphi(x)\,dx\,dw\,du_1 \ldots\right]u_{p-2}^{p-2}\,du_{p-2}\right\}$$

$$= \exp\left\{-\lambda\frac{(p-1)!}{u^{p-1}}\int_0^u\int_0^{u_{p-2}} \cdots \int_0^{u_1}\int_0^y\int_0^w \varphi(x)\,dx\,dw\,du_1 \ldots du_{p-2}\right\}.$$

Using the Cauchy iterated integral we obtain

$$\psi_p(u) = \exp\left\{-\frac{\lambda}{p}\frac{1}{u^{p-1}}\int_0^u \varphi(x)(u-x)^p\,dx\right\}$$

$$= \exp\left\{-\frac{\lambda}{p}u^2\int_0^1 \varphi(ux)(1-x)^p\,dx\right\}.$$

Since

$$\psi_p(u) = \exp\{-\lambda\int_0^1\int_0^{u(1-x^{1/(p-1)})}\int_0^y \varphi(w)\,dw\,dy\,dx\}$$

$$= \lim_{n\to\infty}\prod_{x=1}^{n}\exp\left\{-\frac{\lambda}{n}\int_0^{u[1-(x/n)^{1/(p-1)}]}\int_0^y \varphi(w)\,dw\,dy\right\}$$

and the class $\mathscr{K}$ is closed under multiplication, passing to the limit and raising to a positive power we infer that

$$\psi(u) = \exp\{-\lambda(p+1)\int_0^1 \varphi(ux)(1-x)^p\,dx\}$$

belongs to the class $\mathscr{K}$. If X denotes the random variable corresponding to

$\varphi(u)$ and $Y_{q,p}$ denotes a beta random variable independent of X with parameters q, $p > 0$, then

$$(p+1)\int_0^1 \varphi(ux)(1-x)^p\,dx$$

is the characteristic function of the random variable $X \cdot Y_{1,p+1}$. From the fact

$$Y_{1,p+1} \stackrel{d}{=} Y_{1,p} \cdot Y_{p,1}$$

it follows that $\mathscr{K}_{p+1} \subset \mathscr{K}_p \subset \mathscr{K}$.

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