ANNALES
POLONICI MATHEMATICI
XXXVI (1979)

On inequalities between solutions of first order partial differential-functional equations

by ZDZISŁAW KAMONT (Gdańsk)

Abstract. Suppose that the solutions u and v of the first order partial differential-functional equation

$$z_x(x,y) = f\big(x, y, z(x,y), Z(\alpha(x,y), \beta(x,y)), z_y(x,y)\big),$$

where

$$Z(\alpha(x,y), \beta(x,y)) = \big(z(\alpha_1(x,y), \beta_1(x,y)), \ldots, z(\alpha_m(x,y), \beta_m(x,y))\big),$$

are defined in a domain containing the set

$$D = \{(x,y):\ 0 \leqslant x-x_0 < a,\ \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\}$$

and satisfy the initial conditions

$$u(x,y) = \varphi(x,y), \quad v(x,y) = \psi(x,y) \quad \text{for } (x,y) \in E,$$

where φ and ψ are given initial functions and

$$E = \{(x,y):\ x \in [x_0-\tau_0, x_0],\ r_0(x) \leqslant y \leqslant s_0(x)\}.$$

Let

$$\tilde{E} = \{(x,y):\ x \in [x_0-\tau_0, x_0],\ g(x) \leqslant y \leqslant h(x)\},$$

where $r_0(x) \leqslant g(x) \leqslant h(x) \leqslant s_0(x)$ for $x \in [x_0-\tau_0, x_0]$.

Suppose that φ and ψ satisfy the conditions

$$\varphi(x,y) = \psi(x,y) \quad \text{for } (x,y) \in \tilde{E},$$
$$\varphi(x,y) < \psi(x,y) \quad \text{for } (x,y) \in E-\tilde{E}.$$

In this paper we investigate the mutual situation of the solutions u and v satisfying the above conditions in E.

Let

$$E_1 = \{(x,y):\ x \in [x_0-\tau_0, x_0],\ r_0(x) \leqslant y < g(x)\},$$
$$E_2 = \{(x,y):\ x \in [x_0-\tau_0, x_0],\ h(x) < y \leqslant s_0(x)\}.$$

We also give theorems concerning the mutual situation of the solutions u and v in the case when φ and ψ satisfy the conditions of the form

$$\varphi(x,y) < \psi(x,y) \quad \text{for } (x,y) \in E_1,$$
$$\varphi(x,y) = \psi(x,y) \quad \text{for } (x,y) \in \tilde{E},$$
$$\varphi(x,y) > \psi(x,y) \quad \text{for } (x,y) \in E_2.$$

The non-linear differential-functional inequalities of the form

$$u_x(x, y) < F(x, y, u(x, y), u(\cdot), u_y(x, y)),$$
$$v_x(x, y) > F(x, y, v(x, y), v(\cdot), v_y(x, y))$$

are treated in the paper.

Our theorems are generalizations of some results from [1], [4]–[7], [12].

Suppose that the solutions $u(x, Y)$ and $v(x, Y)$ of the first order partial differential equation

$$z_x = f(x, Y, z, z_Y),$$

where $Y = (y_1, \ldots, y_n), z_Y = (z_{y_1}, \ldots, z_{y_n})$, are generated by characteristics in a domain containing the set $\Delta = \{(x, Y)\colon\ x \in [x_0, x_0+a),\ |y_i - \overset{0}{y}_i| \leqslant b_i - M(x-x_0),\ i = 1, \ldots, n\}$, where $a > 0$, $b_i > 0$, $a < b_i/_M$. Suppose that these solutions satisfy the initial inequality

$$u(x_0, Y) < v(x_0, Y) \quad \text{for } (x_0, Y) \in A,$$

where $A = \{(x, Y)\colon\ x = x_0,\ |y_i - \overset{0}{y}_i| \leqslant b_i,\ i = 1, \ldots, n\}$. In [8], Theorem 59.2, p. 179 (see also [3], [9]) sufficient conditions are given for the functions $u(x, Y)$ and $v(x, Y)$ to satisfy the inequality

$$u(x, Y) < v(x, Y)$$

in Δ.

Let B and C denote n-dimensional domains such that $C \subset B \subset A$, $C \neq B$, $B \neq A$, and suppose that $u(x, Y)$ and $v(x, Y)$ satisfy the initial conditions

$$\begin{aligned} u(x_0, Y) &< v(x_0, Y) && \text{for } (x_0, Y) \in C,\\ u(x_0, Y) &= v(x_0, Y) && \text{for } (x_0, Y) \in B - C,\\ u(x_0, Y) &< v(x_0, Y) && \text{for } (x_0, Y) \in A - B.\end{aligned}$$

Papers [4]–[7] contain theorems concerning the mutual situation of the solutions $u(x, Y)$ and $v(x, Y)$ in Δ satisfying the above condition sin A.

Suppose that $u(x, Y)$ and $v(x, Y)$ satisfy the mixed initial inequalities

$$\begin{aligned} u(x_0, Y) &< v(x_0, Y) && \text{for } (x_0, Y) \in C,\\ u(x_0, Y) &= v(x_0, Y) && \text{for } (x_0, Y) \in B - C,\\ u(x_0, Y) &> v(x_0, Y) && \text{for } (x_0, Y) \in A - B.\end{aligned}$$

In [4]–[7] theorems concerning the mixed inequalities in Δ between solutions satisfying the above conditions in A have been established.

In this paper we shall consider similar problems for solutions of first order partial differential-functional equations

$$(1)\qquad z_x(x, y) = f\big(x, y, z(x, y), z\big(a_1(x, y), \beta_1(x, y)\big), \ldots \\ \ldots, z\big(a_m(x, y), \beta_m(x, y)\big), z_y(x, y)\big).$$

This paper is a continuation of paper [2], where theorems concerning mixed inequalities between solutions of an almost linear equation

$$z_x(x, y) + P(x, y) z_y(x, y) = R\big(x, y, z(x, y), z\big(x-\tau(x), y\big)\big)$$

have been established.

The paper is divided into three parts. The first part deals with inequalities between solutions of an almost linear equation

$$(2)\qquad z_x(x, y) + P(x, y) z_y(x, y) \\ = R\big(x, y, z(x, y), z\big(a_1(x, y), \beta_1(x, y)\big), \ldots, z\big(a_m(x, y), \beta_m(x, y)\big)\big).$$

The non-linear differential-functional inequalities are treated in the second part. The third part contains theorems on mixed inequalities between solutions of non-linear equation (1).

1. INEQUALITIES BETWEEN SOLUTIONS OF AN ALMOST LINEAR EQUATION

1.1. Notations, assumptions and lemma. For given functions $a_i(x, y)$, $\beta_i(x, y)$, $i = 1, \ldots, m$ we define

$$z\big(a(x, y), \beta(x, y)\big) = \big(z\big(a_1(x, y), \beta_1(x, y)\big), \ldots, z\big(a_m(x, y), \beta_m(x, y)\big)\big).$$

Equation (2) can be written briefly

$$(3)\qquad z_x(x, y) + P(x, y) z_y(x, y) = R\big(x, y, z(x, y), Z\big(a(x, y), \beta(x, y)\big)\big).$$

Let (see [13])

$$E = \{(x, y):\ x_0-\tau_0 \leqslant x \leqslant x_0,\ r_0(x) \leqslant y \leqslant s_0(x)\},\qquad \tau_0 > 0,$$
$$\tilde{E} = \{(x, y):\ x_0-\tau_0 \leqslant x \leqslant x_0,\ g(x) \leqslant y \leqslant h(x)\},$$
$$D = \{(x, y):\ 0 \leqslant x-x_0 < a,\ \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\},$$

where r_0, s_0, g, h and $\tilde{r}, \tilde{s}$ are functions defined in $[x_0-\tau_0, x_0]$ and $[x_0, x_0+a)$, respectively, and $r_0(x) \leqslant g(x) \leqslant h(x) \leqslant s_0(x)$ for $x \in [x_0-\tau_0, x_0]$, $\tilde{r}(x) < \tilde{s}(x)$ for $x \in [x_0, x_0+a)$, and $r_0(x_0) = \tilde{r}(x_0)$, $s_0(x_0) = \tilde{s}(x_0)$.

We introduce

ASSUMPTION H_1. *We assume that*

1° *The function R of the variables (x, y, z, U), $U = (u_1, \ldots, u_m)$, is continuous and satisfies the Lipschitz condition with respect to z and is strongly increasing with respect to each of the variables $u_1, \ldots, u_m$ separately in a domain Ω of the space (x, y, z, U).*

2° *The projection of Ω onto the plane (x, y) contains a domain Ω_0. The function P of the variables (x, y) is continuous in Ω_0 and satisfies the Lipschitz condition with respect to y in Ω_0.*

3° *The functions* r_0, s_0, g, h *are continuous in* $[x_0-\tau_0, x_0]$, $\tilde{r}, \tilde{s}$ *are of class* C^1 *in* $[x_0, x_0+a)$ *and*

$$\tilde{r}'(x) \geqslant P(x, \tilde{r}(x)), \quad \tilde{s}'(x) \leqslant P(x, \tilde{s}(x)) \quad \textit{for} \quad x \in [x_0, x_0+a).$$

Assume that $E \cup D \subset \Omega_0$.

4° *The initial functions* φ *and* ψ *of the variables* (x, y) *are continuous in* E. *The solutions* u *and* v *of equation* (3) *satisfy the initial conditions*

$$u(x, y) = \varphi(x, y), \quad v(x, y) = \psi(x, y) \quad \textit{for} \ (x, y) \in E.$$

These solutions are of class C^1 *in* D *and* $\big(x, y, u(x, y), U(\alpha(x, y), \beta(x, y))\big)$, $\big(x, y, v(x, y), V(\alpha(x, y), \beta(x, y))\big) \in \Omega$ *for* $(x, y) \in D$.

ASSUMPTION H_2. *Suppose that*

1° *The functions* a_i, β_i, $i = 1, \ldots, m$, *are continuous in* D *and* $a_i(x, y) \leqslant x$, $\big(a_i(x, y), \beta_i(x, y)\big) \in E \cup D$ *for* $(x, y) \in D$, $i = 1, \ldots, m$.

2° $\min\limits_i [\inf\limits_{(x,y)\in D} a_i(x, y)] = x_0-\tau_0$. *There exists a constant* $\delta > 0$ *such that* $a_i(x, y) \leqslant x-\delta$ *for* $(x, y) \in D$, $i = 1, \ldots, m$.

We define the sequence $a_1, \ldots, a_n$, where $x_0 < a_1 < a_2 < \ldots < a_n = x_0+a$, in the following way:

Put

$$I_1^* = \{x^*\colon x_0 < x^* \leqslant x_0+a, a_i(x, y) \leqslant x_0 \text{ for } x \in [x_0, x^*), \\ (x, y) \in D, \ i = 1, \ldots, m\}.$$

Denote by a_1 the least upper bound of I_1^*. I_1^* is non-void and $a_1 \geqslant x_0+\delta$.

Assuming that the numbers $a_1, \ldots, a_k$ have already been defined, we define a_{k+1} as follows. Let

$$I_{k+1}^* = \{x^*\colon a_k < x \leqslant x_0+a, a_i(x, y) \leqslant a_k \text{ for } x \in [a_k, x^*), (x, y) \in D, \\ i = 1, \ldots, m\}.$$

Denote by a_{k+1} the least upper bound of I_{k+1}^*. I_{k+1}^* is non-void and $a_{k+1} \geqslant a_k+\delta$.

There exists such an index n that $a_n = x_0+a$.

Let $I_1 = [x_0\ a_1)$, $I_k = [a_{k-1}, a_k)$ for $k = 2, 3, \ldots, n$ and

$$D_k = \{(x, y)\colon x \in I_k, \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\}, \quad k = 1, \ldots, n.$$

We adopt the following notation:

$$K = \big\{(x, y)\colon x = x_0, y \in (\tilde{r}(x_0), \tilde{s}(x_0))\big\},$$

$$(4) \quad \tilde{K}_i = \big\{(x, y)\colon (x, y) \in \bar{K} \cap \tilde{E}, \big(a_i(x, y), \beta_i(x, y)\big) \in \tilde{E}\big\}, \quad i = 1, \ldots, m,$$

$$\tilde{K} = \bigcap_{i=1}^m \tilde{K}_i, \quad \tilde{L} = \tilde{E} \cap (\bar{K} - \tilde{K}).$$

(We do not assume that $\tilde{K}$ is non-empty, see Theorem 1.2.)

Let $y = y(x)$ be a solution of the equation

$$\frac{dy}{dx} = P(x, y). \tag{5}$$

We denote by T the following set of integral curves of (5). A curve $y = y(x)$ is an element of T for x belonging to some interval $I \subset [x_0, a_1)$ if $y(x)$ is a solution of (5), $y(x_0) = y_0$ for some $(x_0, y_0) \in \bar{K}$, and $(x, y(x)) \in D_1$ for $x \in I$.

For a curve $y = y(x)$ belonging to T for $x \in I$ we define

$$u(x) = u(x, y(x)), \quad v(x) = v(x, y(x)) \tag{6}$$

and

$$\bar{R}(x, z) = R\Big(x, y(x), z, \Psi\big(a(x, y(x)), \beta(x, y(x))\big)\Big) \tag{7}$$

where

$$\Psi\big(a(x, y), \beta(x, y)\big) = \Big(\psi\big(a_1(x, y), \beta_1(x, y)\big), \ldots, \psi\big(a_m(x, y), \beta_m(x, y)\big)\Big).$$

We introduce the following definitions.

DEFINITION 1.1. The curve $y = y(x)$ belonging to T for $x \in I$ is said to *satisfy condition* T_1 in $[\bar{x}, \bar{a}) \subset I$ if

$$u(\bar{x}) < v(\bar{x}),$$

$$u'(x) \leqslant \bar{R}(x, u(x)), \quad v'(x) = \bar{R}(x, v(x)) \quad \text{for } x \in [\bar{x}, \bar{a}).$$

DEFINITON 1.2. The curve $y = y(x)$ (belonging to T for $x \in I$) is said to *satisfy condition* T_2 in $[\bar{x}, \bar{a}) \subset I$ if

$$u(\bar{x}) = v(\bar{x}),$$

$$u'(x) \leqslant \bar{R}(x, u(x)), \quad v'(x) = \bar{R}(x, v(x)) \quad \text{for } x \in [\bar{x}, \bar{a})$$

and for each $\varepsilon > 0$ there exists a point $x_\varepsilon \in (\bar{x}, \bar{x} + \varepsilon)$ such that

$$u'(x_\varepsilon) < \bar{R}(x_\varepsilon, u(x_\varepsilon)).$$

Now we prove

LEMMA 1.1. *Suppose that Assumptions* H_1 *and* H_2 *are satisfied. If a curve* $y = y(x)$ *belonging to* T *for* $x \in I$ *satisfies condition* T_1 *or condition* T_2 *in* $[\bar{x}, \bar{a}) \subset I$, *then*

$$u(x) < v(x) \quad \textit{for } x \in (\bar{x}, \bar{a}). \tag{8}$$

If we assume additionaly that $(\bar{a}, y(\bar{a})) \in D_1$, *then*

$$u(\bar{a}) < v(\bar{a}). \tag{9}$$

Proof. If the curve $y = y(x)$ satisfies condition T_1 in $[\bar{x}, \bar{a})$, then inequality (8) follows from Lemma 1 in [2].

Suppose that $y = y(x)$ satisfies condition T_2 in $[\bar{x}, \bar{a})$. From theorems on differential inequalities it follows that $u(x) \leqslant v(x)$ for $x \in [\bar{x}, \bar{a})$. Suppose that there exists a point $x^* \in (\bar{x}, \bar{a})$ such that

$$u(x^*) = v(x^*). \tag{10}$$

For each $\varepsilon > 0$ there exists a point $x \in (\bar{x}, \bar{x}+\varepsilon)$ such that

$$u'(x) < \bar{R}(x, u(x)) \tag{11}$$

and $u'(x)$, $\bar{R}(x, z)$ are continuous functions. Therefore there exists an interval $(x', x'') \subset (\bar{x}, x^*)$ such that inequality (11) is satisfied for $x \in (x', x'')$. Since $u(x') \leqslant v(x')$ and

$$v'(x) = \bar{R}(x, v(x)) \quad \text{for } x \in [x', x''],$$

it follows from Lemma 2 in [2] (see also [1]) that $u(x) < v(x)$ for $x \in (x', x'')$. The functions u and v satisfy the conditions

$$u(\tilde{x}) < v(\tilde{x}),$$
$$u'(x) \leqslant \bar{R}(x, u(x)), \quad v'(x) = \bar{R}(x, v(x)) \quad \text{for } x \in [\tilde{x}, \bar{a}).$$

It follows from Lemma 1 in [2] that $u(x) < v(x)$ for $x \in (\bar{x}, \bar{a})$. In particular, for $x = x^*$ we have $u(x^*) < v(x^*)$, what contradicts (10).

Inequality (8) is proved.

Suppose that $(\bar{a}, y(\bar{a})) \in D_1$. It follows from (8) that $u(\bar{a}) \leqslant v(\bar{a})$. If $u(\bar{a}) = v(\bar{a})$, then, by the conditions

$$u'(x) \leqslant \bar{R}(x, u(x)), \quad v'(x) = \bar{R}(x, v(x)) \quad \text{for } x \in (\bar{x}, \bar{a}]$$

and by theorems on differential inequalities, we get $u(x) \geqslant v(x)$ for $x \in (\bar{x}, \bar{a}]$, what contradicts (8). Therefore $u(\bar{a}) < v(\bar{a})$.

Thus the proof of Lemma 1.1 is complete.

1.2. Theorems on inequalities.

THEOREM 1.1. *If Assumptions* H_1 *and* H_2 *are satisfied and the initial functions satisfy the conditions*

$$\begin{aligned} &\varphi(x, y) \leqslant \psi(x, y) \quad &&\textit{for } (x, y) \in E, \\ &\varphi(x_0, y) < \psi(x_0, y) \quad &&\textit{for } (x_0, y) \in \bar{K}, \end{aligned} \tag{12}$$

then

$$u(x, y) < v(x, y) \tag{13}$$

for $(x, y) \in D$.

Proof. (i) At first we prove (13) for $(x, y) \in D_1$. Let $y = y(x)$ be a solution of (5) and $y(x_0) = \tilde{y}$, where $(x_0, \tilde{y}) \in \bar{K}$. Suppose that $(x, y(x)) \in D_1$ for $x \in \tilde{I}$. $\tilde{I} = [x_0, a_1)$ or there exists an $\tilde{a} \in (x_0, a_1)$ such that $\tilde{I} = [x_0, \tilde{a}]$ and $(\tilde{a}, y(\tilde{a})) \in \mathrm{Fr}(D_1)$. Let $u(x)$, $v(x)$, $\bar{R}(x, z)$ be the functions defined by (6), (7). It follows from (12) and from Assumptions H_1, H_2 that

$$u(x_0) < v(x_0),$$
$$u'(x) \leqslant \bar{R}(x, u(x)), \quad v'(x) = \bar{R}(x, v(x)), \quad x \in \tilde{I}.$$

By Lemma 1.1 we get

$$u(x, y(x)) = u(x) < v(x) = v(x, y(x)) \quad \text{for } x \in \tilde{I}.$$

To complete the proof of (13) for $(x, y) \in D_1$, it is sufficient to show that each point $\bar{P}(\bar{x}, \bar{y})$ belonging to D_1 can be joined by an integral curve $y = y(x)$ of equation (5) with some point $(x_0, \tilde{y}) \in \bar{K}$ and $(x, y(x)) \in D_1$ for $x \in [x_0, \bar{x}]$.

Suppose that this is not true and that there exists a point $(\bar{x}, \bar{y}) \in D_1$ and a curve $y = \bar{y}(x)$, where $\bar{y}(x)$ satisfies (5) for $x \in [x_0, \bar{x}]$, and such that $\bar{y}(\bar{x}) = \bar{y}$, $\tilde{s}(x') = \bar{y}(x')$, where $x_0 < x' < \bar{x}$, and

$$\bar{y}(x) > \tilde{s}(x) \tag{14}$$

for x belonging to some interval $(x' - \varepsilon', x')$, $\varepsilon' > 0$. (We proceed similarly in the case when the curve $y = \bar{y}(x)$ has a common point with the curve $y = \tilde{r}(x)$.)

As

$$\bar{y}'(x) = P(x, \bar{y}(x)), \quad \tilde{s}'(x) \leqslant P(x, \tilde{s}(x)) \quad \text{for } x \in (x' - \varepsilon', x'),$$

and $\tilde{s}(x') = \bar{y}(x')$, it follows (see [8], Chapter III; [11], Chapter II) that $\tilde{s}(x) \geqslant \bar{y}(x)$ for $x \in (x' - \varepsilon', x')$, what contradicts (14).

For $x_0 + a = a_1$ the proof of Theorem 1.1 is completed.

(ii) Assume that $a_1 < x_0 + a$. It is easy to prove that in this case $u(x, y) < v(x, y)$ for $(x, y) \in \bar{D}_1$. Consider the differential equation (3) in the set $E \cup D_1 \cup D_2$ and take $E \cup \bar{D}_1$ as the initial set and

$$\varphi_1(x, y) = \begin{cases} \varphi(x, y) & \text{for } (x, y) \in E, \\ u(x, y) & \text{for } (x, y) \in \bar{D}_1, \end{cases}$$

$$\psi_1(x, y) = \begin{cases} \psi(x, y) & \text{for } (x, y) \in E, \\ v(x, y) & \text{for } (x, y) \in \bar{D}_1, \end{cases}$$

as the initial functions. Then we see that

$$\varphi_1(x, y) \leqslant \psi_1(x, y) \quad \text{for } (x, y) \in E \cup \bar{D}_1,$$
$$\varphi_1(a_1, y) < \psi_1(a_1, y) \quad \text{for } y \in [\tilde{r}(a_1), \tilde{s}(a_1)].$$

Just as in (i) we can show that $u(x, y) < v(x, y)$ for $(x, y) \in D_2$.

In an analogous manner we show that $u(x, y) < v(x, y)$ for $(x, y) \in D_i$ for $i = 3, \ldots, n$.

Since $D = \bigcup_{i=1}^{n} D_i$, the proof of Theorem 1.1 is finished.

THEOREM 1.2. *If*

1° *Assumptions* H_1 *and* H_2 *are satisfied,*

2° *the initial functions fulfil the conditions*

$$\begin{aligned} \varphi(x, y) &= \psi(x, y) \quad \text{for } (x, y) \in \tilde{E}, \\ \varphi(x, y) &< \psi(x, y) \quad \text{for } (x, y) \in E - \tilde{E}, \end{aligned} \tag{15}$$

3° $\tilde{K}$ *is empty,*

then

$$u(x, y) < v(x, y) \tag{16}$$

for $(x, y) \in D - \bar{K}$.

Proof. (i) At first we prove that (16) is satisfied in $D_1 - \bar{K}$. Let

$$L_1 = \{(x, y): (x, y) \in \bar{K},\ u(x_0, y) < v(x_0, y)\},$$

$$L_2 = \{(x, y): (x, y) \in \bar{K},\ u(x_0, y) = v(x_0, y)\}.$$

(a) We prove that $u(x, y) < v(x, y)$ along solutions of equation (5) issuing from L_1 and situated in $D_1 - \bar{K}$.

Assume that $(x_0, y_1) \in L_1$ and that $y = y_1(x)$ is the solution of (5) satisfying the initial condition $y_1(x_0) = y_1$. Assume also that $(x, y_1(x)) \in D_1$ for $x \in \tilde{I}$, where $\tilde{I} = [x_0, a_1)$ or there exists an $\tilde{a} \in (x_0, a_1)$ such that $\tilde{I} = [x_0, \tilde{a}]$ and $(\tilde{a}, y_1(\tilde{a})) \in \mathrm{Fr}(D_1)$. Put $u_1(x) = u(x, y_1(x))$, $v_1(x) = v(x, y_1(x))$ for $x \in \tilde{I}$. It follows from (15) and from H_1 that

$$u_1(x_0) < v_1(x_0),$$

$$u_1'(x) \leqslant R_1(x, u_1(x)), \quad v_1'(x) = R_1(x, v_1(x)) \quad \text{for } x \in \tilde{I},$$

where $R_1(x, z) = R\big(x, y_1(x), z, \Psi\big(a(x, y_1(x)), \beta(x, y_1(x))\big)\big)$. Thus we see that the curve $y = y_1(x)$ satisfies condition T_1. From Lemma 1.1 we obtain $u(x, y_1(x)) = u_1(x) < v_1(x) = v(x, y_1(x))$ for $x \in \tilde{I}$.

(b) We prove that $u(x, y) < v(x, y)$ along solutions of (5) issuing from L_2 and situated in $D_1 - \bar{K}$.

Assume that $(x_0, y_2) \in L_2$ and denote by $y = y_2(x)$ the solution of (5) satisfying the condition $y_2(x_0) = y_2$. Suppose also that $(x, y_2(x)) \in D_1$ for $x \in [x_0, \tilde{\tilde{a}})$ and $(\tilde{\tilde{a}}, y_2(\tilde{\tilde{a}})) \in \mathrm{Fr}(D_1)$. We prove that the curve $y = y_2(x)$ satisfies condition T_2 in $[x_0, \tilde{\tilde{a}})$.

Since $\tilde{K}$ is empty, it follows that there exists an index j, $1 \leqslant j \leqslant m$, and an interval $[x_0, x')$, $x_0 < x' \leqslant \tilde{\tilde{a}}$, such that

$$\big(a_j(x, y_2(x)), \beta_j(x, y_2(x))\big) \in E - \tilde{E} \quad \text{for } x \in [x_0, x').$$

By assumption (15) we get

$$\varphi\big(a_j(x, y_2(x)), \beta_j(x, y_2(x))\big) < \psi\big(a_j(x, y_2(x)), \beta_j(x, y_2(x))\big) \tag{17}$$

for $x \in [x_0, x')$. From Assumptions H_1, H_2 and from (17) we obtain that the functions $u_2(x) = u(x, y_2(x))$, $v_2(x) = v(x, y_2(x))$ satisfy the conditions

$$u_2(x_0) \leqslant v_2(x_0),$$

$$u_2'(x) \leqslant R_2(x, u_2(x)), \qquad v_2'(x) = R_2(x, v_2(x)) \quad \text{for } x \in [x_0, \tilde{\tilde{a}}]$$

and

$$u_2'(x) < R_2(x, u_2(x)) \quad \text{for } x \in [x_0, x'),$$

where $R_2(x, z) = R\Big(x, y_2(x), z, \Psi\big(a(x, y_2(x)), \beta(x, y_2(x))\big)\Big)$.

Thus we see that the curve $y = y_2(x)$ satisfies condition T_2 in $[x_0, \tilde{\tilde{a}})$. It follows from Lemma 1.1 that $u_2(x) < v_2(x)$ for $x \in (x_0, \tilde{\tilde{a}})$ and $u_2(\tilde{\tilde{a}}) < v_2(\tilde{\tilde{a}})$ if $(\tilde{\tilde{a}}, y_2(\tilde{\tilde{a}})) \in D_1$.

(c) Every point $(\bar{x}, \bar{y}) \in D_1 - \bar{K}$ can be joined by means of an integral curve $y = y(x)$ of (5) with some point of the segment $\bar{K}$ and $(x, y(x)) \in D_1$ for $x \in [x_0, \bar{x}]$. (See proof of Theorem 1.1.)

As $\bar{K} = L_1 \cup L_2$, it follows from (a)–(c) that $u(x, y) < v(x, y)$ for $(x, y) \in D_1 - \bar{K}$.

(ii) Assume that (16) holds in the sets $D_1 - \bar{K}, D_2, \ldots, D_k$. We shall show that the same inequality is satisfied also in D_{k+1}.

If $a_k < x_0 + a$, then it is easy to prove that $u(x, y) < v(x, y)$ for $(x, y) \in (D_1 - \bar{K}) \cup D_2 \cup \ldots \cup D_{k-1} \cup \bar{D}_k$. Consider equation (3) in $E \cup D_1 \cup \ldots \cup D_{k+1}$ and take $E \cup D_1 \cup \ldots \cup \bar{D}_k$ as the initial set and

$$\varphi_k(x, y) = \begin{cases} \varphi(x, y) & \text{for } (x, y) \in E, \\ u(x, y) & \text{for } (x, y) \in D_1 \cup \ldots \cup D_{k-1} \cup \bar{D}_k, \end{cases}$$

$$\psi_k(x, y) = \begin{cases} \psi(x, y) & \text{for } (x, y) \in E, \\ v(x, y) & \text{for } (x, y) \in D_1 \cup \ldots \cup D_{k-1} \cup \bar{D}_k, \end{cases}$$

as the initial functions. Then we have

$$\varphi_k(x, y) \leqslant \psi_k(x, y) \quad \text{for } (x, y) \in E \cup D_1 \cup \ldots \cup D_{k-1} \cup \bar{D}_k,$$

$$\varphi_k(a_k, y) < \psi_k(a_k, y) \quad \text{for } y \in [\tilde{r}(a_k), \tilde{s}(a_k)].$$

Thus we infer from Theorem 1.1 that $u(x, y) < v(x, y)$ for $(x, y) \in D_{k+1}$. As $D - \bar{K} = D_1 - \bar{K} \cup \bigcup_{k=2}^{n} D_k$, the proof of Theorem 1.2 is finished.

We now consider the mutual situation of the solutions $u(x, y)$ and $v(x, y)$ of (3) in D in the case when the initial functions satisfy (15) and $\tilde{K}$ is non-empty.

Assume that $y = \tilde{y}(x)$ is a solution of (5) and $\tilde{y}(x_0) = \tilde{y}$, where $(x_0, \tilde{y}) \in \tilde{K}$. Then $(a_i(x_0, \tilde{y}), \beta_i(x_0, \tilde{y})) \in \tilde{E}$ for $i = 1, \ldots, m$. Let I' be the largest interval contained in $[x_0, a_1)$ such that

$$(18) \qquad \big(a_i(x, \tilde{y}(x)), \beta_i(x, \tilde{y}(x))\big) \in \tilde{E} \quad \text{for } x \in I', \ i = 1, \ldots, m.$$

($I' = [x_0, a_1)$ or there exists an $a' < a_1$ such that $I' = [x_0, a']$. If $I' = [x_0, a']$, then condition (18) holds in $[x_0, a']$ and for every $\varepsilon > 0$ there exists an $x_\varepsilon \in (a', a'+\varepsilon)$ and an index j, $1 \leqslant j \leqslant m$, such that $\big(a_j(x_\varepsilon, \tilde{y}(x_\varepsilon)), \beta_j(x_\varepsilon, \tilde{y}(x_\varepsilon))\big) \in E - \tilde{E}$.)

We shall denote by $\tilde{C}$ the curve $y = \tilde{y}(x)$ for $x \in I'$. Let Δ_1 denote the plane set formed by all curves $\tilde{C}$ issuing from $\tilde{K}$.

Now we have

THEOREM 1.3. *If*

1° *Assumptions* H_1 *and* H_2 *are satisfied,*

2° *the initial functions satisfy* (15),

3° $\tilde{K}$ *is non-empty,*

then

$$(19) \qquad u(x, y) = v(x, y) \quad \text{for } (x, y) \in \Delta_1,$$

$$(20) \qquad u(x, y) < v(x, y) \quad \text{for } (x, y) \in D_1 - \Delta_1 - \tilde{L}.$$

Proof. (I) We start with proving (19). Let $y = \tilde{y}(x)$ be a solution of (5) and $\tilde{y}(x_0) = \tilde{y}$, where $(x_0, \tilde{y}) \in \tilde{K}$. Suppose that the curve $y = \tilde{y}(x)$ is situated in Δ_1 for $x \in I'$, $I' \subset [x_0, a_1)$. (I' is the interval of the form $I' = [x_0, a']$, where $a' < a_1$ and $\big(a', y(a')\big) \in \mathrm{Fr}(\Delta_1)$ or $I' = [x_0, a_1)$.) Thus we obtain

$$(21) \qquad \big(a_i(x, \tilde{y}(x)), \beta_i(x, \tilde{y}(x))\big) \in \tilde{E} \quad \text{for } x \in I', \ i = 1, \ldots, m.$$

It is easy to verify that the functions $u(x) = u\big(x, \tilde{y}(x)\big)$, $v(x) = v\big(x, \tilde{y}(x)\big)$, $x \in I'$, satisfy respectively the differential equations

$$\frac{dz}{dx} = \bar{R}_1(x, z), \qquad \frac{dz}{dx} = \bar{R}_2(x, z),$$

where $\bar{R}_1(x, z) = R\Big(x, \tilde{y}(x), z, \ \Phi\big(a(x, \tilde{y}(x)), \beta(x, \tilde{y}(x))\big)\Big)$, $\bar{R}_2(x, z) = R\Big(x, \tilde{y}(x), z, \Psi\big(a(x, \tilde{y}(x)), \beta(x, \tilde{y}(x))\big)\Big)$. It follows from (21) that $\bar{R}_1(x, z) = \bar{R}_2(x, z)$ for $x \in I'$. Since $u(x_0) = v(x_0)$, it follows from condition 1° of Assumption H_1 that $u(x) = v(x)$ for $x \in I'$. The solutions $u(x, y)$ and $v(x, y)$ of (3) are therefore equal along any arbitrary curve $\tilde{C}$ issuing from $\tilde{K}$. The proof of statement (19) is completed.

(II) We now prove (20).

(a) Assume that $y = y(x)$ is a solution of (5), $\big(x_0, y(x_0)\big) \in \tilde{K}$ and that $\big(x, y(x)\big) \in \Delta_1$ for $x \in [x_0, a']$ and $\big(x, y(x)\big) \in D_1 - \Delta_1$ for $x \in (a', a'')$, where $\big(a'', y(a'')\big) \in \mathrm{Fr}(D_1)$. We shall prove that $u(x, y) < v(x, y)$ along, the curve $y = y(x)$ for $x \in (a', a'')$ and that $u(a'', y(a'')) < v(a'', y(a''))$ if $(a'', y(a'')) \in D_1$.

Since $(x, y(x)) \in D_1 - \Delta_1$ for $x \in (a', a'')$, it follows that in an arbitrary right-hand neighbourhood of the point a' there exists a number x such that $(a_k(x, y(x)), \beta_k(x, y(x))) \in E - \tilde{E}$ for some k, where $1 \leqslant k \leqslant m$. It follows from (15) and from H_1 that the functions $u(x) = u(x, y(x))$ and $v(x) = v(x, y(x))$ satisfy the conditions

$$u(a') = v(a'),$$

$$\frac{du(x)}{dx} \leqslant \bar{R}(x, u(x)), \quad \frac{dv(x)}{dx} = \bar{R}(x, v(x)), \quad x \in [a', a'')$$

and for every $\varepsilon > 0$ there exists an $x_\varepsilon \in (a', a'+\varepsilon)$ such that

$$\frac{du(x_\varepsilon)}{dx} < \bar{R}(x_\varepsilon, u(x_\varepsilon)),$$

where $\bar{R}(x, z)$ is defined by (7). Thus we see that the curve $y = y(x)$ satisfies condition T_2. It follows from Lemma 1.1 that $u(x, y(x)) < v(x, y(x))$ for $x \in (a', a'')$ and that $u(a'', y(a'')) < v(a'', y(a''))$ if $(a'', y(a'')) \in D_1$.

We shall now prove that inequality (20) holds along solutions of (5) issuing from $\bar{K} - \tilde{K}$. Let

$$\tilde{L}_1 = \{(x, y)\colon (x, y) \in \bar{K} - \tilde{K},\ u(x_0, y) < v(x_0, y)\},$$

$$\tilde{L}_2 = \{(x, y)\colon (x, y) \in \bar{K} - \tilde{K},\ u(x_0, y) = v(x_0, y)\}.$$

(b) We shall show that $u(x, y) < v(x, y)$ along the curve $y = y_1(x)$, where $y_1(x)$ is a solution of (5) and $(x_0, y_1(x_0)) \in \tilde{L}_1$.

Suppose that $(x, y_1(x)) \in D_1 - \Delta_1$ for $x \in [x_0, a')$ and $(a', y_1(a')) \in \mathrm{Fr}(D_1)$. The functions $u_1(x) = u(x, y_1(x))$, $v_1(x) = v(x, y_1(x))$ satisfy the conditions

$$u_1(x_0) < v_1(x_0),$$

$$u_1'(x) \leqslant R_1(x, u_1(x)), \quad v_1'(x) = R_1(x, v_1(x)), \quad x \in [x_0, a'),$$

where $R_1(x, z) = R\Big(x, y_1(x), z, \Psi\big(a(x, y_1(x)), \beta(x, y_1(x))\big)\Big)$. It follows from Lemma 1.1 that $u(x, y_1(x)) < v(x, y_1(x))$ for $x \in [x_0, a')$ and also that $u(a', y_1(a')) < v(a', y_1(a'))$ if $(a', y_1(a')) \in D_1$.

(c) Suppose that $y = y_2(x)$ is a solution of (5) and $(x_0, y_2(x_0)) \in \tilde{L}_2$. Assume that $(x, y_2(x)) \in D_1 - \Delta_1$ for $x \in [x_0, a'')$ and $(a'', y_2(a'')) \in \mathrm{Fr}(D_1)$. Since $(x_0, y_2(x_0)) \in \tilde{L}_2$, there exists an interval $[x_0, x')$, $x_0 < x' \leqslant a''$ such that for each $x \in [x_0, x')$ there exists an index k such that

$$\varphi\big(a_k(x, y_2(x)), \beta_k(x, y_2(x))\big) < \psi\big(a_k(x, y_2(x)), \beta_k(x, y_2(x))\big).$$

The functions $u_2(x) = u(x, y_2(x))$, $v_2(x) = v(x, y_2(x))$ satisfy the conditions

$$u_2(x_0) = v_2(x_0),$$
$$u_2'(x) \leqslant R_2(x, u_2(x)), \quad v_2'(x) = R_2(x, v_2(x)), \quad x \in [x_0, a'')$$

and

$$u_2'(x) < R_2(x, u_2(x)) \quad \text{for } x \in (x_0, x'),$$

where $R_2(x, z) = R\Big(x, y_2(x), z, \Psi\big(a(x, y_2(x)), \beta(x, y_2(x))\big)\Big)$. Thus we see that the curve $y = y_2(x)$ satisfies condition T_2 in $[x_0, a'')$. It follows from Lemma 1.1 that $u(x, y_2(x)) < v(x, y_2(x))$ for $x \in (x_0, a'')$ and $u(a'', y_2(a'')) < v(a'', y_2(a''))$ if $(a'', y_2(a'')) \in D_1$.

(d) Each point $\bar{P}(\bar{x}, \bar{y})$, $\bar{x} > x_0$, belonging to $D_1 - \Delta_1$ can be connected by an integral curve $y = y(x)$ of equation (5) with some point $(x_0, y(x_0)) \in \bar{K}$ and $(x, y(x)) \in D_1$ for $x \in [x_0, \bar{x}]$. The proof of this property of the set D_1 is given in the proof of Theorem 1.1.

It follows from (a)–(d) that $u(x, y) < v(x, y)$ for $(x, y) \in D_1 - \Delta_1 - \tilde{L}$.

The proof of Theorem 1.3 is completed.

Remark 1.1. If the assumptions of Theorem 1.3 are satisfied and $a_1 < a + x_0$, then it is easy to prove that $u(x, y) = v(x, y)$ for $(x, y) \in \bar{\Delta}_1$ and $u(x, y) < v(x, y)$ for $(x, y) \in \bar{D}_1 - \bar{\Delta}_1 - \tilde{L}$.

Theorem 1.3 concerns the mixed inequalities between solutions of equation (3) in that part of D, where $x \in [x_0, a_1)$. In the sequel we consider the mutual situation of solutions of (3) in the entire D.

We adopt the following

ASSUMPTION H_3. *Suppose that there exists a finite sequence of intervals* $\tilde{I}_0, \tilde{I}_1, \ldots, \tilde{I}_n$, *where* $\tilde{I}_0 = [x_0 - \tau_0, x_0]$, $\tilde{I}_1 = [x_0, \tilde{a}_1]$, $\tilde{a}_1 = a_1$, $\tilde{I}_k = [\tilde{a}_{k-1}, \tilde{a}_k]$ *for* $k = 2, 3, \ldots, n-1$, $\tilde{I}_n = [\tilde{a}_{n-1}, x_0 + a)$, *satisfying the following condition: for each* $k \in \{1, \ldots, n\}$ *there exists an* $l \in \{0, 1, 2, \ldots, k-1\}$ *such that if* $(x, y) \in D$, $x \in \tilde{I}_k$, *then* $a_i(x, y) \in \tilde{I}_l$ *for* $i = 1, \ldots, m$.

Suppose that Assumptions $\mathrm{H}_1 - \mathrm{H}_3$ hold, $\tilde{K}$ is non-empty and that the initial functions satisfy (15). Put

$$\tilde{D}_k = \{(x, y)\colon x \in \tilde{I}_k, \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\}, \quad k = 1, \ldots, n.$$

We shall define a sequence of sets $\tilde{\Delta}^{(1)}, \ldots, \tilde{\Delta}^{(n)}$ in the following way:

Consider the differential equation (3) in $E \cup \tilde{D}_1$. It follows from Theorem 1.3 (see also Remark 1.1) that there exists a set $\tilde{\Delta}^{(1)}$ such that $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{\Delta}^{(1)}$ and $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_1 - \tilde{\Delta}^{(1)} - \tilde{L}$. The set $\tilde{\Delta}^{(1)}$ has the form

$$\tilde{\Delta}^{(1)} = \{(x, y)\colon x_0 \leqslant x \leqslant c_1, \; g_1(x) \leqslant y \leqslant h_1(x)\},$$

where $c_1 \leqslant \tilde{a}_1$ and g_1, h_1 are continuous functions.

Now suppose that the sets $\tilde{\Delta}^{(1)}, \ldots, \tilde{\Delta}^{(k)}$ have already been constructed. We define $\tilde{\Delta}^{(k+1)}$ as follows. Consider equation (3) in the set $E \cup \tilde{D}_1 \cup \ldots \cup \tilde{D}_{k+1}$ and take $E \cup \tilde{D}_1 \cup \ldots \cup \tilde{D}_k$ as the initial set and

$$\varphi_k(x, y) = \begin{cases} \varphi(x, y) & \text{for } (x, y) \in E, \\ u(x, y) & \text{for } (x, y) \in \tilde{D}_1 \cup \ldots \cup \tilde{D}_k, \end{cases}$$

$$\psi_k(x, y) = \begin{cases} \psi(x, y) & \text{for } (x, y) \in E, \\ v(x, y) & \text{for } (x, y) \in \tilde{D}_1 \cup \ldots \cup \tilde{D}_k, \end{cases}$$

as the initial functions. It follows from Assumptions H_2, H_3 that there exists an $l \in \{0, 1, \ldots, k\}$ such that if $(x, y) \in \tilde{D}_{k+1}$, then $\big(a_i(x, y), \beta_i(x, y)\big) \in \tilde{D}_l$ for $i = 1, \ldots, m$. The set $\tilde{\Delta}^{(l)} \subset \tilde{D}_l$ has the form

$$\tilde{\Delta}^{(l)} = \{(x, y) \colon x \in \tilde{\tilde{I}}_l, g_l(x) \leqslant y \leqslant h_l(x)\},$$

where $\tilde{\tilde{I}}_l \subset \tilde{I}_l$ and g_l, h_l are continuous functions. For $l = 0$ we define $\tilde{\Delta}^{(0)} = \tilde{E}$.

(i) If $\varphi_k(\tilde{a}_k, y) < \psi_k(\tilde{a}_k, y)$ for $y \in [\tilde{r}(\tilde{a}_k), \tilde{s}(\tilde{a}_k)]$, then we obtain by Theorem 1.1 that $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_{k+1}$. In this case $\tilde{\Delta}^{(k+1)}$ is empty.

(ii) Let

$$K^{(k)} = \big\{(x, y) \colon x = \tilde{a}_k, y \in \big(\tilde{r}(\tilde{a}_k), \tilde{s}(\tilde{a}_k)\big)\big\},$$

$$\tilde{K}_i^{(k)} = \big\{(x, y) \colon (x, y) \in \overline{K^{(k)}} \cap \tilde{\Delta}^{(k)}, \big(a_i(x, y), \beta_i(x, y)\big) \in \tilde{\Delta}^{(l)}\big\},$$

$$\tilde{K}^{(k)} = \bigcap_{i=1}^{m} \tilde{K}_i^k, \qquad \tilde{L}^{(k)} = \tilde{\Delta}^{(k)} \cap (\overline{K^{(k)}} - \tilde{K}^{(k)}).$$

If $\tilde{K}^{(k)}$ is empty, then we obtain easily from Theorem 1.2 that $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_k - \overline{K^{(k)}}$. In this case $\tilde{\Delta}^{(k+1)}$ is empty.

(iii) If $\tilde{K}^{(k)}$ is non-empty, then Theorem 1.3 implies (see also Remark 1.1) the existence of the set $\tilde{\Delta}^{(k+1)}$ formed by integral curves of equation (5) such that

$$u(x, y) = v(x, y) \quad \text{for } (x, y) \in \tilde{\Delta}^{(k+1)},$$
$$u(x, y) < v(x, y) \quad \text{for } (x, y) \in \tilde{D}_{k+1} - \tilde{\Delta}^{(k+1)} - \tilde{L}^{(k)}.$$

$\tilde{\Delta}^{(k+1)}$ has the form

$$\tilde{\Delta}^{(k+1)} = \{(x, y) \colon x \in \tilde{\tilde{I}}_{k+1}, g_{k+1}(x) \leqslant y \leqslant h_{k+1}(x)\},$$

where $\tilde{\tilde{I}}_{k+1} \subset \tilde{I}_{k+1}$, g_{k+1} and h_{k+1} are continuous functions.

We have therefore the following

THEOREM 1.4. *If*

1° *Assumptions* H_1–H_3 *are satisfied,*

2° $\tilde{K}$ *is non-empty,*

3° *the initial functions satisfy* (15),

then

$$u(x,y) = v(x,y) \quad \text{for } (x,y) \in \tilde{D},$$
$$u(x,y) < v(x,y) \quad \text{for } (x,y) \in D - \tilde{D} - \tilde{L},$$

where $\tilde{D} = \bigcup_{i=1}^{n} \tilde{\Delta}^{(i)}$.

2. DIFFERENTIAL-FUNCTIONAL INEQUALITIES

Denote by $\tilde{\Phi}$ the class of continuous functions defined in $E \cup D$. Elements of $\tilde{\Phi}$ will be denoted by $z(\cdot)$, $u(\cdot)$, $v(\cdot)$ and the like. Let

$$\tilde{H}_x = \{(s,t): (s,t) \in E \cup D, s \leqslant x\}.$$

We introduce

ASSUMPTION H_4. *Suppose that*

1° *F is a real function defined in the set $\tilde{\Omega} \times \tilde{\Phi} \times Q$, where $\tilde{\Omega}$ is a domain contained in the three-dimensional Euclidean space. The projection of $\tilde{\Omega}$ onto the plane (x,y) contains the set $E \cup D$. $\tilde{\Omega}$ satisfies the following condition: if $(x,y,z) \in \tilde{\Omega}$ and $\bar{z} > z$, then $(x,y,\bar{z}) \in \tilde{\Omega}$. Q is an interval of real numbers.*

2° *F is continuous in $\tilde{\Omega} \times \tilde{\Phi} \times Q$ and satisfies the following Volterra condition: if $(x,y,z) \in \tilde{\Omega}$, $q \in Q$, $z_1(\cdot)$, $z_2(\cdot) \in \tilde{\Phi}$, $z_1(s,t) = z_2(s,t)$ for $(s,t) \in H_x$, then $F(x,y,z,z_1(\cdot),q) = F(x,y,z,z_2(\cdot),q)$. F is non-decreasing with respect to the fourth variable.*

3° *The functions r_0 and s_0 are continuous in $[x_0 - \tau_0, x_0]$, the functions $\tilde{r}$ and $\tilde{s}$ are of class C^1 in $[x_0, x_0 + a)$ and*

$$F(x,y,z,\bar{u}(\cdot),q_1) - F(x,y,z,\bar{u}(\cdot),q_2) \geqslant -\tilde{r}'(x)(q_1 - q_2),$$
$$(22) \qquad F(x,y,z,\bar{u}(\cdot),q_1) - F(x,y,z,\bar{u}(\cdot),q_2) \leqslant -\tilde{s}'(x)(q_1 - q_2)$$

for $(x,y,z) \in \tilde{\Omega}$, $\bar{u}(\cdot) \in \tilde{\Phi}$, $q_1, q_2 \in Q$, $q_1 \geqslant q_2$.

4° *The functions u and v are continuous in $E \cup D$, have first order derivatives in D and, moreover, they have Stolz differentials in* $\mathrm{Fr}(D) - \tilde{K}$.

5° *If $(x,y) \in D$, then $(x,y,u(x,y))$, $(x,y,v(x,y)) \in \tilde{\Omega}$ and $u_y(x,y)$, $v_y(x,y) \in Q$.*

Remark 2.1. If F satisfies the Lipschitz condition

$$|F(x,y,z,\bar{u}(\cdot),q_1) - F(x,y,z,\bar{u}(\cdot),q_2)| \leqslant M|q_1 - q_2|$$

for $(x, y, z, \bar{u}(\cdot)) \in \tilde{\Omega}\times\tilde{\Phi}$, $q_1, q_2 \in Q$ *and*

$$\tilde{r}(x) = y_0 - b + M(x-x_0), \qquad \tilde{s}(x) = y_0 + b - M(x-x_0), \qquad x \in [x_0, x_0+a),$$

where $b > 0$, $y_0 - b = r_0(x_0)$, $y_0 + b = s_0(x_0)$, $a < b/M$, then condition (22) is satisfied.

We have

THEOREM 2.1. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *the functions u and v fulfil the initial inequalities*

$$(23)\qquad \begin{aligned} u(x, y) &\leqslant v(x, y) && \text{for } (x, y) \in E,\\ u(x_0, y) &< v(x_0, y) && \text{for } (x_0, y) \in \bar{K} \end{aligned}$$

and the differential inequalities

$$(24)\qquad u_x(x, y) < F\big(x, y, u(x, y), u(\cdot), u_y(x, y)\big), \qquad (x, y) \in D,$$

$$(25)\qquad v_x(x, y) \geqslant F\big(x, y, v(x, y), v(\cdot), v_y(x, y)\big), \qquad (x, y) \in D.$$

Under these assumptions,

$$(26)\qquad u(x, y) < v(x, y)$$

for $(x, y) \in D$.

Proof. (The proof of this theorem is patterned on that given in [8]. If assertion (26) is false, then the set

$$Z = \{x\colon\ x \in [x_0, x_0+a),\ u(x, y) \geqslant v(x, y) \text{ for some } y \in [\tilde{r}(x), \tilde{s}(x)]\}$$

is non-empty. Defining $x^* = \inf Z$, it is clear from (23) that $x^* > x_0$ and

$$(27)\qquad u(x^*, y^*) = v(x^*, y^*)$$

for some $y^* \in [\tilde{r}(x^*), \tilde{s}(x^*)]$.

If $(x^*, y^*) \in \mathrm{Int}(D)$, then $u_y(x^*, y^*) = v_y(x^*, y^*)$ and

$$(28)\qquad u_x(x^*, y^*) - v_x(x^*, y^*) \geqslant 0.$$

Since $u(s, t) \leqslant v(s, t)$ for $(s, t) \in \tilde{H}_x$, it follows from condition 2° of Assumption H_4 and from (24), (25), (27) that

$$\begin{aligned} &u_x(x^*, y^*) - v_x(x^*, y^*)\\ &< F\big(x^*, y^*, u(x^*, y^*), u(\cdot), u_y(x^*, y^*)\big) - F\big(x^*, y^*, v(x^*, y^*), v(\cdot), v_y(x^*, y^*)\big)\\ &\leqslant F\big(x^*, y^*, u(x^*, y^*), v(\cdot), u_y(x^*, y^*)\big) - F\big(x^*, y^*, u(x^*, y^*),\\ &\qquad\qquad v(\cdot), v_y(x^*, y^*)\big) \leqslant 0, \end{aligned}$$

which contradicts (28).

Suppose that $(x^*, y^*) \in \mathrm{Fr}(D)$ and assume that for example $y^* = \tilde{r}(x^*)$. Then $u_y(x^*, y^*) \leqslant v_y(x^*, y^*)$ and the function $\bar{u}(x) = u(x, \tilde{r}(x)) - v(x, \tilde{r}(x))$,

$x \in [x_0, x^*]$ attains maximum at $x = x^*$. Therefore

(29) $$u_x(x^*, y^*) - v_x(x^*, y^*) + \tilde{r}'(x^*)[u_y(x^*, y^*) - v_y(x^*, y^*)] \geqslant 0.$$

Since $u(s, t) \leqslant v(s, t)$ for $(s, t) \in \tilde{H}_{x^*}$, then we obtain by the monotonicity condition and by (22), (24), (25), (27)

$$\begin{aligned}
&u_x(x^*, y^*) - v_x(x^*, y^*)\\
&< F(x^*, y^*, u(x^*, y^*), u(\cdot), u_y(x^*, y^*)) - F(x^*, y^*, v(x^*, y^*), v(\cdot), v_y(x^*, y^*))\\
&\leqslant [F(x^*, y^*, u(x^*, y^*), u(\cdot), u_y(x^*, y^*)) - F(x^*, y^*, u(x^*, y^*), v(\cdot), u_y(x^*, y^*))] +\\
&+ [F(x^*, y^*, u(x^*, y^*), v(\cdot), u_y(x^*, y^*)) - F(x^*, y^*, v(x^*, y^*), v(\cdot), u_y(x^*, y^*))]\\
&\leqslant -r'(x^*)[u_y(x^*, y^*) - v_y(x^*, y^*)],
\end{aligned}$$

which contradicts (29).

Hence Z is empty, and the statement (26) follows.

Remark 2.2. If $\tau_0 = 0$ and F does not contain the functional argument, then from Theorem 2.1 we obtain the well-known theorem on strong first order partial differential inequalities (see [3], [8]). Theorem 2.1 is a generalization of some results on first order partial differential inequalities with lagging argument which are given in [12].

Remark 2.3. In Theorem 2.1 we can assume that in (24) the weak inequality holds and that in (25) the strong inequality is satisfied. In Theorem 2.1 we can assume instead of (24), (25) that

$$\begin{aligned}
u_x(x, y) &\leqslant F(x, y, u(x, y), u(\cdot), u_y(x, y)), \quad (x, y) \in D,\\
v_x(x, y) &\geqslant F(x, y, v(x, y), v(\cdot), v_y(x, y)), \quad (x, y) \in D,
\end{aligned}$$

where for each $(x, y) \in D$ at least one of the above inequalities must be strong.

THEOREM 2.2. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *there exists a constant* $N > 0$ *such that*

(30) $$|F(x, y, z_1, \bar{u}(\cdot), q) - F(x, y, z_2, \bar{u}(\cdot), q)| \leqslant N|z_1 - z_2|$$

for $(x, y, z_i, \bar{u}(\cdot), q) \in \tilde{\Omega} \times \tilde{\Phi} \times Q$, $i = 1, 2$,

3° *the initial inequalities*

(31) $$\begin{aligned}
u(x, y) &\leqslant v(x, y) \quad \textit{for} \quad (x, y) \in E,\\
u(x_0, y) &< v(x_0, y) \quad \textit{for} \quad (x_0, y) \in \bar{K}
\end{aligned}$$

hold and the differential inequalities

(32) $$\begin{aligned}
u_x(x, y) &\leqslant F(x, y, u(x, y), u(\cdot), u_y(x, y)), \quad (x, y) \in D,\\
v_x(x, y) &\geqslant F(x, y, v(x, y), v(\cdot), v_y(x, y)), \quad (x, y) \in D
\end{aligned}$$

are satisfied.

Under these assumptions we have

$$(33) \qquad u(x,y) < v(x,y)$$

for $(x,y) \in D$.

Proof. Let $\varepsilon_0 = \min_{(x,y)\in\bar{K}} [v(x,y)-u(x,y)]$. It follows from (31) that $\varepsilon_0 > 0$. Let $\varepsilon \in (0,\varepsilon_0)$. There exists a constant $\delta > 0$ such that

$$(34) \qquad (\varepsilon+\delta/N)e^{-N(x-x_0)} - \delta/N > 0$$

for $x \in [x_0, x_0+a)$. Denote by $\tilde{z}$ a continuous function defined in E such that

$$(35) \qquad u(x,y) \leqslant \tilde{z}(x,y) \leqslant v(x,y) \quad \text{for } (x,y)\in E$$

and

$$(36) \qquad \tilde{z}(x_0,y) = u(x_0,y)+\varepsilon, \quad y \in [\tilde{r}(x_0), \tilde{s}(x_0)].$$

Let

$$(37) \quad \tilde{u}(x,y) = \begin{cases} \tilde{z}(x,y) & \text{for } (x,y)\in E, \\ u(x,y)+(\varepsilon+\delta/N)e^{-N(x-x_0)} - \delta/N & \text{for } (x,y)\in D. \end{cases}$$

We shall prove that $\tilde{u}(x,y) < v(x,y)$ for $(x,y) \in D$.

The function $\tilde{u}$ is continuous in $E\cup D$, has first derivatives in D and possesses a Stolz differential in $\mathrm{Fr}(D)-\bar{K}$. Moreover, $\tilde{u}(x,y) \leqslant v(x,y)$ for $(x,y)\in E$ and $\tilde{u}(x_0,y) < v(x_0,y)$ for $(x_0,y)\in\bar{K}$. It follows from (30), (32), (34), (35), (37) and by the monotonicity of F with respect to the functional argument that

$$\begin{aligned}
&\tilde{u}_x(x,y) = u_x(x,y) - N(\varepsilon+\delta/N)\, e^{-N(x-x_0)} \\
&\leqslant F\big(x,y,\tilde{u}(x,y),\tilde{u}(\cdot),\tilde{u}_y(x,y)\big) + \big[F\big(x,y,u(x,y),u(\cdot),u_y(x,y)\big) - \\
&\quad - F\big(x,y,\tilde{u}(x,y),u(\cdot),u_y(x,y)\big)\big] + \big[F\big(x,y,\tilde{u}(x,y),u(\cdot),u_y(x,y)\big) - \\
&\quad - F\big(x,y,\tilde{u}(x,y),\tilde{u}(\cdot),u_y(x,y)\big)\big] - N(\varepsilon+\delta/N)e^{-N(x-x_0)} \\
&\leqslant F\big(x,y,\tilde{u}(x,y),\tilde{u}(\cdot),\tilde{u}_y(x,y)\big) + N|u(x,y)-\tilde{u}(x,y)| - N(\varepsilon+\delta/N)e^{-N(x-x_0)} \\
&= F\big(x,y,\tilde{u}(x,y),\tilde{u}(\cdot),\tilde{u}_y(x,y)\big) - \delta.
\end{aligned}$$

Since $\delta > 0$, we have

$$(38) \qquad \tilde{u}_x(x,y) < F\big(x,y,\tilde{u}(x,y),\tilde{u}(\cdot),\tilde{u}_y(x,y)\big), \quad (x,y)\in D.$$

It follows from (32), (35), (37), (38) and from Theorem 2.1 that $\tilde{u}(x,y) < v(x,y)$ for $(x,y)\in D$. From this inequality and from (34), (37) we obtain assertion (33). This completes the proof of Theorem 2.2.

THEOREM 2.3. *Suppose that*

1° *Assumption* H_4 *is satisfied,* 2° *there exists* $N > 0$ *such that*

$$(39)\qquad \left|F\big(x, y, z_1, u_1(\cdot), q\big) - F\big(x, y, z_2, u_2(\cdot), q\big)\right| \leqslant N\Big[|z_1 - z_2| + \sup_{(s,t)\in\tilde{H}_x} |u_1(s, t) - u_2(s, t)|\Big]$$

for $\big(x, y, z_i, u_i(\cdot), q\big) \in \tilde{\Omega} \times \tilde{\Phi} \times Q$, $i = 1, 2$,

3° *the initial inequality*

$$(40)\qquad u(x, y) \leqslant v(x, y), \quad (x, y) \in E$$

and the differential inequalities

$$(41)\qquad \begin{aligned} u_x(x, y) &\leqslant F\big(x, y, u(x, y), u(\cdot), u_y(x, y)\big), \quad (x, y) \in D,\\ v_x(x, y) &\geqslant F\big(x, y, v(x, y), v(\cdot), v_y(x, y)\big), \quad (x, y) \in D \end{aligned}$$

hold.

Under these assumptions we have

$$(42)\qquad u(x, y) \leqslant v(x, y) \quad \text{for } (x, y) \in D.$$

Proof. Let

$$(43)\qquad \tilde{v}(x, y) = v(x, y) + \frac{\varepsilon}{N}\big[(1+N)e^{2N(x-x_0+\tau_0)} - \tfrac{1}{2}\big], \quad \varepsilon > 0,\ (x, y) \in E \cup D.$$

We shall prove that

$$(44)\qquad u(x, y) < \tilde{v}(x, y) \quad \text{for } (x, y) \in D.$$

Since

$$\begin{aligned}
\tilde{v}_x(x, y) &= v_x(x, y) + 2\varepsilon(1+N)e^{2N(x-x_0+\tau_0)}\\
&\geqslant F\big(x, y, v(x, y), v(\cdot), v_y(x, y)\big) + 2\varepsilon(1+N)e^{2N(x-x_0+\tau_0)}\\
&= F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big) + 2\varepsilon(1+N)e^{2N(x-x_0+\tau_0)} +\\
&\quad + \big[F\big(x, y, v(x, y), v(\cdot), v_y(x, y)\big) - F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big)\big]\\
&\geqslant F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big) - N\big[|v(x, y) - \tilde{v}(x, y)| +\\
&\quad + \sup_{(s,t)\in\tilde{H}_x} |v(s, t) - \tilde{v}(s, t)|\big] + 2\varepsilon(1+N)e^{2N(x-x_0+\tau_0)}\\
&= F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big) - N\Big[\frac{\varepsilon}{N}\big((1+N)e^{2N(x-x_0+\tau_0)} - \tfrac{1}{2}\big) +\\
&\quad + \frac{\varepsilon}{N}\big((1+N)e^{2N(x-x_0+\tau_0)} - \tfrac{1}{2}\big)\Big] + 2\varepsilon(1+N)e^{2N(x-x_0+\tau_0)}\\
&= F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big) + \varepsilon,
\end{aligned}$$

we get the strong differential inequality

$$(45)\qquad \tilde{v}_x(x, y) > F\big(x, y, \tilde{v}(x, y), \tilde{v}(\cdot), \tilde{v}_y(x, y)\big).$$

From (41), (43), (45) and from the initial inequality

$$u(x, y) < \tilde{v}(x, y), \qquad (x, y) \in E,$$

we obtain, in virtue of Theorem 2.1, that

$$u(x, y) < v(x, y) + \frac{\varepsilon}{N}\left[(1+N)e^{2N(x-x_0+\tau_0)} - \tfrac{1}{2}\right], \qquad (x, y) \in D.$$

From the above inequality we obtain in the limit (letting ε tend to 0) inequality (42). Theorem 2.3 is proved.

Remark 2.4. Suppose that Assumption H_4 is satisfied and that u and v are solutions of the equation

$$z_x(x, y) = F\big(x, y, z(x, y), z(\cdot), z_y(x, y)\big), \qquad (x, y) \in D.$$

Then

(a) if $u(x, y) < v(x, y)$ for $(x, y) \in E$ and the Lipschitz condition (30) is satisfied, then $u(x, y) < v(x, y)$ on D (see Theorem 2.2),

(b) if $u(x, y) \leqslant v(x, y)$ for $(x, y) \in E$ and the Lipschitz condition (39) is satisfied, then $u(x, y) \leqslant v(x, y)$ on D (see Theorem 2.3),

(c) if $u(x, y) \leqslant v(x, y)$ for $(x, y) \in E$, condition (30) holds and condition (39) is not satisfied, then it can be $v(x, y) < u(x, y)$ for $(x, y) \in D - \bar{K}$.

For example, the functions

$$u(x, y) = \begin{cases} 0 & \text{for } (x, y) \in [-1, 0] \times (-\infty, +\infty), \\ x^2 & \text{for } (x, y) \in (0, 1) \times (-\infty, +\infty), \end{cases}$$

$$v(x, y) = 0 \qquad \text{for } (x, y) \times [-1, 1) \times (-\infty, +\infty),$$

satisfy the equation

$$z_x(x, y) = \big[z\big(-x, \beta_1(x, y)\big)\big]^{1/2} + 4\big[z\big(\tfrac{1}{2}x, \beta_2(x, y)\big)\big]^{1/2}$$

and $u(x, y) \leqslant v(x, y)$ for $(x, y) \in [-1, 0] \times (-\infty, +\infty)$; however, $u(x, y) > v(x, y)$ for $(x, y) \in (0, 1) \times (-\infty, +\infty)$.

THEOREM 2.4. *Suppose that*

1° *Assumption* H_4 *and the Lipschitz condition* (39) *are satisfied,*

2° $\qquad u(x, y) \leqslant v(x, y) \quad$ *for* $(x, y) \in E$

and

$$u_x(x, y) < F\big(x, y, u(x, y), u(\cdot), u_y(x, y)\big), \quad (x, y) \in D, \tag{46}$$

$$v_x(x, y) \geqslant F\big(x, y, v(x, y), v(\cdot), v_y(x, y)\big), \quad (x, y) \in D. \tag{47}$$

Under these assumptions we have

$$u(x, y) < v(x, y) \tag{48}$$

for $(x, y) \in D - \bar{K}$.

Proof. It follows from Theorem 2.3 that $u(x,y) \leqslant v(x,y)$ for $(x,y) \in D-\bar{K}$. Let

$$\tilde{Z} = \{(x,y)\colon (x,y) \in D-\bar{K},\ u(x,y) = v(x,y)\}.$$

If $(\bar{x},\bar{y}) \in \tilde{Z} \cap \mathrm{Int}(D)$, then $u_y(\bar{x},\bar{y}) = v_y(\bar{x},\bar{y})$ and

$$u_x(\bar{x},\bar{y}) - v_x(\bar{x},\bar{y}) \geqslant 0. \tag{49}$$

Since $u(s,t) \leqslant v(s,t)$ for $(s,t) \in H_{\bar{x}}$, it follows from the monotonicity condition and from (46), (47) that $u_x(\bar{x},\bar{y}) - v_x(\bar{x},\bar{y}) < 0$, which is incompatible with (49). Assume that $(\bar{x},\bar{y}) \in \tilde{Z} \cap \mathrm{Fr}(D)$ and suppose that $\bar{y} = \tilde{r}(\bar{x})$. Then (see the proof of Theorem 2.1)

$$u_x(\bar{x},\bar{y}) - v_x(\bar{x},\bar{y}) + \tilde{r}'(\bar{x})[u_y(\bar{x},\bar{y}) - v_y(\bar{x},\bar{y})] \geqslant 0 \tag{50}$$

and

$$u_y(\bar{x},\bar{y}) \leqslant v_y(\bar{x},\bar{y}). \tag{51}$$

It follows from (22), (46), (47), (51) and from the monotonicity condition that

$$u_x(\bar{x},\bar{y}) - v_x(\bar{x},\bar{y}) < -r'(\bar{x})[u_y(\bar{x},\bar{y}) - v_y(\bar{x},\bar{y})],$$

which contradicts (50). Hence $\tilde{Z}$ is empty, and the statement (48) follows.

Remark 2.5. In Theorem 2.4 we can assume that in (46) the weak inequality holds and that in (47) the strong inequality is satisfied. We can also assume that instead of (46), (47) the differential inequalities of the form (41) are satisfied, where for each $(x,y) \in D-\bar{K}$ at least one of inequalities (41) must be strong.

3. MIXED INEQUALITIES BETWEEN SOLUTIONS OF A NON-LINEAR EQUATION

In this part of the paper we investigate the mutual situation of two solutions of the non-linear equation

$$z_x(x,y) = f\big(x, y, z(x,y), Z\big(a(x,y), \beta(x,y)\big), z_y(x,y)\big), \tag{52}$$

where $Z\big(a(x,y), \beta(x,y)\big) = \big(z\big(a_1(x,y), \beta_1(x,y)\big), \ldots, z\big(a_m(x,y), \beta_m(x,y)\big)\big)$.

3.1. Assumptions and lemma. We introduce

ASSUMPTION H_5. *Suppose that*

1° *The function f the variables (x,y,z,U,q), $U = (u_1, \ldots, u_m)$ and its first partial derivatives with respect to $y, z, u_1, \ldots, u_m, q$ are continuous for $(x,y,z) \in \tilde{\Omega}$, $q \in Q$ and for arbitrary U. $\tilde{\Omega}$ is a domain such that $(x,y,\bar{z}) \in \tilde{\Omega}$ if $\bar{z} \geqslant z$ and $(x,y,z) \in \tilde{\Omega}$. The projection of $\tilde{\Omega}$ onto the plane (x,y) contains $E \cup D$. Q is an interval real numbers.*

2° *f is strongly increasing with respect to each of the variables $u_1, \ldots, u_m$*

separately. The derivatives f_y, f_z, f_{u_i} $(i = 1, \ldots, m)$, f_q *satisfy the Lipschitz condition with respect to* (y, z, U, q).

3° *The function* r_0 *and* s_0 *are continuous in* $[x_0 - \tau_0, x_0]$, *where* $\tau_0 > 0$, $\tilde{r}$ *and* $\tilde{s}$ *are of class* C^1 *in* $[x_0, x_0 + a)$ *and*

$$\begin{aligned} \tilde{r}'(x) &\geqslant -f_q\big(x, \tilde{r}(x), z, U, q\big), \\ \tilde{s}'(x) &\leqslant -f_q\big(x, \tilde{s}(x), z, U, q\big) \end{aligned} \tag{53}$$

for $x \in [x_0, x_0 + a)$, $\big(x, \tilde{r}(x), z\big), \big(x, \tilde{s}(x), z\big) \in \tilde{\Omega}$, $q \in Q$, U *arbitrary.*

4° *The initial functions* φ *and* ψ *of the variables* (x, y) *and their first partial derivatives* φ_y, ψ_y *are continuous in* E. φ_y *and* ψ_y *satisfy the Lipschitz condition with respect to* y *in* E.

5° u *and* v *are solutions of* (52) *satisfying the initial conditions*

$$u(x, y) = \varphi(x, y), \quad v(x, y) = \psi(x, y), \quad (x, y) \in E. \tag{54}$$

u *and* v *are of class* C^1 *in* D, u_y *and* v_y *satisfy the Lipschitz condition with respect to* y *in* D. *If* $(x, y) \in D$, *then* $\big(x, y, u(x, y)\big), \big(x, y, v(x, y)\big) \in \tilde{\Omega}$, $u_y(x, y), v_y(x, y) \in Q$.

6° g *and* h *are continuous functions in* $[x_0 - \tau_0, x_0]$ *and* $r_0(x) \leqslant g(x) \leqslant h(x) \leqslant s_0(x)$ *for* $x \in [x_0 - \tau_0, x_0]$,

$$\tilde{E} = \{(x, y) \colon x \in [x_0 - \tau_0, x_0], g(x) \leqslant y \leqslant h(x)\}.$$

ASSUMPTION H_6. *Suppose that*

1° *The functions* α_i, β_i, $i = 1, \ldots, m$, *satisfy Assumption* H_2.

2° *There exist continuous derivatives* $\partial\alpha_i/\partial y$, $\partial\beta_i/\partial y$ *and they satisfy the Lipschitz condition with respect to* y *in* D.

Let us consider equation (52) in $E \cup D_1$ and suppose that Assumptions $\mathrm{H}_5, \mathrm{H}_6$ are satisfied and that the initial functions fulfil the inequality $\varphi(x, y) \leqslant \psi(x, y)$ for $(x, y) \in E$. Then we have by Theorem 2.3 that $u(x, y) \leqslant v(x, y)$ in D_1. The functions u and v satisfy in D_1 following first order partial differential equations

$$z_x = \tilde{f}(x, y, z, z_y) \tag{55}$$

and

$$z_x = \tilde{\tilde{f}}(x, y, z, z_y), \tag{56}$$

respectively, where

$$\begin{aligned} \tilde{f}(x, y, z, q) &= f\big(x, y, z, \Phi\big(\alpha(x, y), \beta(x, y)\big), q\big), \\ \tilde{\tilde{f}}(x, y, z, q) &= f\big(x, y, z, \Psi\big(\alpha(x, y), \beta(x, y)\big), q\big). \end{aligned} \tag{57}$$

It follows from Assumptions H_5 and H that u and v are generated in D_1

by characteristics of equations (55) and (56), respectively (see [10]). Let $(\bar{x}, \bar{y})$, $\bar{x} > x_0$, be an arbitrary point of D_1. Denote by $y = \bar{y}(x)$ a solution of the equation

$$\frac{dy}{dx} = -\tilde{f}_q(x, y, u(x, y), u_y(x, y)) \tag{58}$$

satisfying the initial condition $\bar{y}(\bar{x}) = \bar{y}$. It follows from (53), (57), (58) and from theorems on differential inequalities that the function $y = \bar{y}(x)$ is defined for $x \in [x_0, \bar{x}]$ and $(x, \bar{y}(x)) \in D_1$ for $x \in [x_0, \bar{x}]$.

We introduce the following definitions:

DEFINITION 3.1. A point $(\bar{x}, \bar{y}) \in D_1$ is said to *satisfy condition* W_1 if there exists a $\tilde{x} \in [x_0, \bar{x})$ such that $u(\tilde{x}, \bar{y}(\tilde{x})) < v(\tilde{x}, \bar{y}(\tilde{x}))$.

DEFINITION 3.2. A point $(\bar{x}, \bar{y}) \in D_1$ is said to *satisfy condition* W_2 if there exists a $\tilde{x} \in [x_0, \bar{x})$ and an index j, $1 \leqslant j \leqslant m$, such that

$$\varphi\big(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))\big) < \varphi\big(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))\big).$$

Now we prove

LEMMA 3.1. *Suppose that Assumptions* H_5, H_6 *are satisfied and that* $\varphi(x, y) \leqslant \psi(x, y)$ *for* $(x, y) \in E$. *If a point* $(\bar{x}, \bar{y}) \in D_1$, $\bar{x} > x_0$, *satisfies condition* W_1 *or condition* W_2, *then*

$$u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y}). \tag{59}$$

Proof. I. At first we prove (59) in the case when $(\bar{x}, \bar{y}) \in \operatorname{Int} D_1$.

(a) Suppose that $(\bar{x}, \bar{y})$ satisfies W_1, i.e., there exists a $\tilde{x} \in [x_0, \bar{x})$ such that $u(\tilde{x}, \bar{y}(\tilde{x})) < v(\tilde{x}, \bar{y}(\tilde{x}))$, and $y = \bar{y}(x)$ is a solution of (58) such that $\bar{y}(\bar{x}) = \bar{y}$. Let $\delta > 0$ be such a small constant that $(\bar{x}+\delta, \bar{y}(\bar{x}+\delta)) \in \operatorname{Int}(D_1)$. There exists an $\varepsilon_1 > 0$ such that any solution $y = y(x)$ of (58) issuing from the segment

$$\{(x, y)\colon\ x = \tilde{x},\ \ y \in [\bar{y}(\tilde{x}) - \varepsilon_1, \bar{y}(\tilde{x}) + \varepsilon_1]\}$$

is defined for $x \in [\tilde{x}, \bar{x}+\delta)$ and $(x, y(x)) \in D_1$ for $x \in [\tilde{x}, \bar{x}+\delta)$. There exists an $\varepsilon_2 > 0$ such that $u(x, y) < v(x, y)$ in the segment

$$\{(x, y)\colon\ x = \tilde{x}, y \in [\bar{y}(\tilde{x}) - \varepsilon_2, \bar{y}(\tilde{x}) + \varepsilon_2]\}.$$

Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (58) satisfying the initial conditions $y_1(\tilde{x}) = \bar{y}(\tilde{x}) - \varepsilon$, $y_2(\tilde{x}) = \bar{y}(\tilde{x}) + \varepsilon$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Let

$$T_1(\tilde{x}, \varepsilon) = \{(x, y)\colon\ \tilde{x} \leqslant x < \bar{x}+\delta, y_1(x) \leqslant y \leqslant y_2(x)\}.$$

Since $u(\tilde{x}, y) < v(\tilde{x}, y)$ for $y \in [y_1(\tilde{x}), y_2(\tilde{x})]$ and

$$\begin{aligned} u_x(x, y) &= \tilde{f}(x, y, u(x, y), u_y(x, y)), \\ v_x(x, y) &\geqslant \tilde{f}(x, y, v(x, y), v_y(x, y)) \end{aligned} \tag{60}$$

for $(x, y) \in T_1(\tilde{x}, \varepsilon)$ it follows from Theorem 2.2 (see also [1]) that $u(x, y)$

$< v(x, y)$ for $(x, y) \in T_1(\tilde{x}, \varepsilon)$. In particular, for $(\bar{x}, \bar{y}) \in T_1(\tilde{x}, \varepsilon)$ we have $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$.

(b) Suppose that $(\bar{x}, \bar{y})$ satisfies W_2, i.e., there exists a $\tilde{x} \in [x_0, \bar{x})$ and an index j, $1 \leqslant j \leqslant m$, such that $\varphi\big(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))\big) < \psi\big(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))\big)$. Let $\delta > 0$ be such a constant that $\big(\bar{x}+\delta, \bar{y}(\bar{x}+\delta)\big) \in \operatorname{Int}(D_1)$. It follows from the continuity of the functions a_j, β_j, φ, ψ that there exists a neighbourhood $U(\tilde{x})$ of the point $(\tilde{x}, \bar{y}(\tilde{x}))$ such that $\varphi\big(a_j(x, y), \beta_j(x, y)\big) < \psi\big(a_j(x, y), \beta_j(x, y)\big)$ for $(x, y) \in U(\tilde{x})$. Let

$$\tilde{T}_1 = \{(x, y):\ x' \leqslant x < x'+\varepsilon',\ \varepsilon' > 0,\ \tilde{y}_1(x) \leqslant y \leqslant \tilde{y}_2(x)\}$$

be a set such that $\tilde{T}_1 \subset U(\tilde{x})$; $y = \tilde{y}_1(x)$, $y = \tilde{y}_2(x)$ are solutions of (58). By (55), (57) and by the monotonicity of f with respect to u_j we infer that u and v satisfy in T_1 the following conditions:

$$u(x', y) \leqslant v(x', y), \qquad y \in [\tilde{y}_1(x'), \tilde{y}_2(x')],$$

$$u_x(x, y) = \tilde{f}\big(x, y, u(x, y), u_y(x, y)\big),$$

$$v_x(x, y) > \tilde{f}\big(x, y, v(x, y), v_y(x, y)\big), \qquad (x, y) \in \tilde{T}_1.$$

Then we obtain, in virtue of Theorem 2.4 (see Remark 2.3, see also [1]), that $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{T}_1$ and $x > x'$. For a fixed $t_0 \in (x', x'+\varepsilon')$ we choose $\varepsilon_1 > 0$ such that any solution $y = y(x)$ of (58) issuing from the segment

$$\{(x, y):\ x = t_0,\ \bar{y}(t_0) - \varepsilon_1 \leqslant y \leqslant \bar{y}(t_0) + \varepsilon_1\}$$

is defined for $x \in [t_0, \bar{x}+\delta)$ and $\big(x, y(x)\big) \in D_1$ for $x \in [t_0, \bar{x}+\delta)$. Let $\varepsilon_2 > 0$ be such a constant that the segment

$$\{(x, y):\ x = t_0,\ \bar{y}(t_0) - \varepsilon_2 \leqslant y \leqslant \bar{y}(t_0) + \varepsilon_2\}$$

is contained in $\tilde{T}_1$. Denote by $T_2(t_0, \varepsilon)$ the set

$$T_2(t_0, \varepsilon) = \{(x, y):\ t_0 \leqslant x < \bar{x}+\delta,\ y_1(x) \leqslant y \leqslant y_2(x)\},$$

where $y = y_1(x)$ and $y = y_2(x)$ are solutions of (58) such that $y_1(t_0) = \bar{y}(t_0) - \varepsilon$, $y_2(t_0) = \bar{y}(t_0) + \varepsilon$ and $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Since $u(t_0, y) < v(t_0, y)$ for $y \in [y_1(t_0), y_2(t_0)]$ and (60) holds for $(x, y) \in T_2(t_0, \varepsilon)$, it follows from Theorem 2.2 (see also [1]) that $u(x, y) < v(x, y)$ for $(x, y) \in T_2(t_0, \varepsilon)$. Because $(\bar{x}, \bar{y}) \in T_2(t_0, \varepsilon)$, we have $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$.

II. Consider the case when $(\bar{x}, \bar{y}) \in \operatorname{Fr}(D_1) \cap D_1$ and $\bar{x} > x_0$, i.e., $\bar{y} = \tilde{r}(\bar{x})$ or $\bar{y} = \tilde{s}(\bar{x})$. Suppose that $\bar{y} = \tilde{r}(\bar{x})$. Let $\delta > 0$ be such a small constant that $\big(\bar{x}+\delta, \tilde{r}(\bar{x}+\delta)\big) \in D_1$.

(a) Assume that $(\bar{x}, \bar{y})$ satisfies W_1, i.e., $u\big(\tilde{x}, \bar{y}(\tilde{x})\big) < v\big(\tilde{x}, \bar{y}(\tilde{x})\big)$ for some $\tilde{x} \in [x_0, \bar{x})$. Then there exists an $\varepsilon > 0$ such that

(i) $u(\tilde{x}, y) < v(\tilde{x}, y)$ for $y \in [\bar{y}(\tilde{x}), \bar{y}(\tilde{x}) + \varepsilon]$,

(ii) the solution $y = y_2(x)$ of (58) satisfying the condition $y_2(\tilde{x}) = \bar{y}(\tilde{x})+\varepsilon$ is defined for $x \in [\tilde{x}, \bar{x}+\delta)$ and $(x, y_2(x)) \in D_1$ for $x \in [\tilde{x}, \bar{x}+\delta)$.

Let $T_3(\tilde{x}, \varepsilon) = \{(x, y)\colon \tilde{x} \leqslant x < \bar{x}+\delta,\ y_1(x) \leqslant y \leqslant y_2(x)\}$, where

$$y_1(x) = \begin{cases} \bar{y}(x) & \text{for } x \in [\tilde{x}, \bar{x}], \\ \tilde{r}(x) & \text{for } x \in (\bar{x}, \bar{x}+\delta). \end{cases}$$

It is easy to see that u and v satisfy in $T_3(\tilde{x}, \varepsilon)$ the differential inequalities (60). Since $u(\tilde{x}, y) < v(\tilde{x}, y)$ for $y \in [y_1(\tilde{x}), y_2(\tilde{x})]$, it follows that $u(x, y) < v(x, y)$ in $T_3(\tilde{x}, \varepsilon)$. Because $(\bar{x}, \bar{y}) \in T_3(\tilde{x}, \varepsilon)$, we have (59).

(b) If $(\bar{x}, \bar{y})$ satisfies W_2, then the proof of (59) runs quite similarly to the proof of this inequality in cases I(b) and II(a).

The proof of statement (59) is completed.

3.2. Theorems on inequalities.

THEOREM 3.1. *Suppose that*

1° *Assumptions* H_5 *and* H_6 *are satisfied,*

2° *the initial functions satisfy the conditions*

$$\begin{aligned} \varphi(x, y) &= \psi(x, y) \quad \text{for } (x, y) \in \tilde{E}, \\ \varphi(x, y) &< \psi(x, y) \quad \text{for } (x, y) \in E-\tilde{E}, \end{aligned} \tag{61}$$

3° *the set* $\tilde{K}$ (see (4)) *is empty.*

Under these assumptions we have

$$u(x, y) < v(x, y) \tag{62}$$

for $(x, y) \in D-\bar{K}$.

Proof. At first we prove (62) for $(x, y) \in D_1-\bar{K}$. It follows from Theorem 2.3 that $u(x, y) \leqslant v(x, y)$. Suppose that there exists a point $(\bar{x}, \bar{y}) \in D_1-\bar{K}$ such that

$$u(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y}). \tag{63}$$

Denote by $y = \bar{y}(x)$ a solution of equation (58) satisfying the initial condition $\bar{y}(\bar{x}) = \bar{y}$. It follows from (53), (57), (58) that the function $y = \bar{y}(x)$ is defined for $x \in [x_0, \bar{x}]$ and $(x, \bar{y}(x)) \in D_1$ for $x \in [x_0, \bar{x}]$.

If $u(x_0, \bar{y}(x_0)) < v(x_0, \bar{y}(x_0))$, then the point $(\bar{x}, \bar{y})$ satisfies condition W_1. If $u(x_0, \bar{y}(x_0)) = v(x_0, \bar{y}(x_0))$, then by condition 3° and by (4) we obtain that $(\bar{x}, \bar{y})$ satisfies W_2. In both cases we conclude, in virtue of Lemma 3.1, that $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$, which gives a contradiction with (63). Therefore we have (62) in $D_1-\bar{K}$.

Consider equation (52) in the set

$$D(t) = \{(x, y): t \leqslant x < x_0 + a,\ \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\},$$

where $t \in (x_0, a_1)$, and take $E \cup \overline{D - D(t)}$ as the initial set and

$$\varphi^{(t)}(x, y) = \begin{cases} \varphi(x, y) & \text{for } (x, y) \in E, \\ u(x, y) & \text{for } (x, y) \in \overline{D - D(t)}, \end{cases}$$

$$\psi^{(t)}(x, y) = \begin{cases} \psi(x, y) & \text{for } (x, y) \in E, \\ v(x, y) & \text{for } (x, y) \in \overline{D - D(t)}, \end{cases}$$

as the initial functions. Then we have

$$\varphi^{(t)}(x, y) \leqslant \psi^{(t)}(x, y) \quad \text{for } (x, y) \in E \cup \overline{D - D(t)},$$

$$\varphi^{(t)}(t, y) < \psi^{(t)}(t, y) \quad \text{for } y \in [\tilde{r}(t), \tilde{s}(t)].$$

Thus we infer from Theorem 2.2 that $u(x, y) < v(x, y)$ for $(x, y) \in D(t)$.

The proof of Theorem 3.1 is completed.

Now we consider the mutual situation of solutions of (52) in the case when $\tilde{K}$ is non-empty.

Let $y = \tilde{y}(x)$, $z = \tilde{z}(x)$, $q = \tilde{q}(x)$ be a characteristic of equation (55) situated on the integral u of this equation (u satisfies (55) for $(x, y) \in D_1$). Assume that $(x_0, \tilde{y}(x_0)) \in \tilde{K}$. Then $\big(a_i(x_0, \tilde{y}(x_0)), \beta_i(x_0, \tilde{y}(x_0))\big) \in \tilde{E}$ for $i = 1, \ldots, m$. Let I' be the biggest interval contained in $[x_0, a_1)$ such that $\big(a_i(x, \tilde{y}(x)), \beta_i(x, \tilde{y}(x))\big) \in \tilde{E}$ for $x \in I'$, $i = 1, \ldots, m$. ($I' = [x_0, a_1)$ or there exists an $a' < a_1$ such that $I' = [x_0, a']$.) We shall denote by $\tilde{C}$ the curve $y = \tilde{y}(x)$ for $x \in I'$. Let G_1 be the plane set formed by all curves $\tilde{C}$ issuing from $\tilde{K}$.

Now we have

THEOREM 3.2. *Suppose that*

1° *Assumptions* H_5 *and* H_6 *are satisfied,*

2° *the initial functions satisfy* (61) *and* $\tilde{K}$ *is non-empty.*

Under these assumptions we have

$$u(x, y) = v(x, y) \quad \textit{for } (x, y) \in G_1, \tag{64}$$

$$u(x, y) < v(x, y) \quad \textit{for } (x, y) \in D_1 - G_1 - \tilde{L}. \tag{65}$$

Proof. At first we prove (64). Let $y = \tilde{y}_1(x)$, $z = \tilde{u}(x)$, $q = \tilde{q}_1(x)$ be a solution of the characteristic system corresponding to equation (55)

$$\frac{dy}{dx} = -\tilde{f}_q(x, y, z, q), \qquad \frac{dz}{dx} = \tilde{f}(x, y, z, q) - q\tilde{f}_q(x, y, z, q),$$

$$\frac{dq}{dx} = \tilde{f}_y(x, y, z, q) + q\tilde{f}_z(x, y, z, q).$$

Assume that this characteristic is situated on the integral u. Denote by $y = \tilde{y}_2(x)$, $z = \tilde{v}(x)$, $q = \tilde{q}_2(x)$ a solution of an analogous system corresponding to equation (56) and suppose that this characteristic lies on v. Assume that $(x_0, \tilde{y}_1(x_0)) = (x_0, \tilde{y}_2(x_0)) \in \tilde{K}$ and $(x, \tilde{y}_1(x)) \in G_1$ for $x \in I'$, $I' \subset [x_0, a_1)$. From (57) and (61) it follows that the characteristic systems corresponding to equations (55) and (56) are identical for $(x, y) \in \tilde{E}$. Solutions of these systems are uniquely determined by initial data, therefore we obtain from (61) that $\tilde{y}_1(x) = \tilde{y}_2(x)$, $\tilde{u}(x) = \tilde{v}(x)$, $\tilde{q}_1(x) = \tilde{q}_2(x)$ for $x \in I'$. Since $\tilde{u}(x) = u(x, \tilde{y}_1(x))$, $\tilde{v}(x) = v(x, \tilde{y}_2(x))$ for $x \in I'$, it follows that the solutions u and v are equal along an arbitrary curve $\tilde{C}$ issuing from $\tilde{K}$. The proof of statement (64) is completed.

Now we prove (65). From Theorem 2.3 we obtain that $u(x, y) \leqslant v(x, y)$ in D_1. Suppose that there exists a point $(\bar{x}, \bar{y}) \in D_1 - G_1$, $\bar{x} > x_0$, such that

$$u(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y}). \tag{66}$$

Denote by $y = \bar{y}(x)$ the solution of (58) satisfying the condition $\bar{y}(\bar{x}) = \bar{y}$. This solution is defined for $x \in [x_0, \bar{x}]$ and $(x, \bar{y}(x)) \in D_1$ for $x \in [x_0, \bar{x}]$. Now, there are three cases to be distinguished.

(a) Suppose that $(x_0, \bar{y}(x_0)) = L_1$, where

$$L_1 = \{(x, y)\colon (x, y) \in \bar{K},\ u(x, y) < v(x, y)\}.$$

Then we see that the point $(\bar{x}, \bar{y})$ satisfies condition W_1 and we infer from Lemma 3.1 that $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$, which gives a contradiction with (66).

(b) Suppose that $(x_0, \bar{y}(x_0)) \in L_2$, where

$$L_2 = \{(x, y)\colon (x, y) \in \bar{K}, u(x, y) = v(x, y), (x, y) \notin \tilde{K}\}.$$

Then in an arbitrary neighbourhood of the point x_0 there exists a $\tilde{x}$, $\tilde{x} > x_0$, such that for a certain index j, $1 \leqslant j \leqslant m$, $(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))) \in E - \tilde{E}$. It follows from (61) and from the monotonicity of f with respect to u_j that $(\bar{x}, \bar{y})$ satisfies condition W_2. By Lemma 3.1 we obtain that $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$, which contradicts (66).

(c) Assume that $(x_0, \bar{y}(x_0)) \in \tilde{K}$. Suppose that $(x, \bar{y}(x)) \in G_1$ for $x \in [x_0, \bar{t}]$ and $(x, \bar{y}(x)) \in D_1 - G_1$ for $x \in (\bar{t}, \bar{x}]$. Then in an arbitrary neighbourhood of the point $\bar{t}$ there exists a $\tilde{x}$, $\tilde{x} > \bar{t}$, such that for a certain j, $1 \leqslant j \leqslant m$, $(a_j(\tilde{x}, \bar{y}(\tilde{x})), \beta_j(\tilde{x}, \bar{y}(\tilde{x}))) \in E - \tilde{E}$. Now, from the last condition we infer (see (b)) that $(\bar{x}, \bar{y})$ satisfies W_2 and consequently $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$, which contradicts (66). Theorem 3.2 is proved.

Remark 3.1. If $a_1 < a + x_0$ and the assumptions of Theorem 3.2 are satisfied, then it is easy to prove that $u(x, y) = v(x, y)$ for $(x, y) \in \bar{G}_1$ and $u(x, y) < v(x, y)$ for $(x, y) \in \bar{D}_1 - \bar{G}_1 - \bar{L}$.

If we accept additional assumptions concerning the functions g, h and a_i, β_i, $i = 1, \ldots, m$, we shall to state the set G_1 in a simply way.

Assume that

$$(67) \qquad \alpha_i(x, y) \equiv x - \tau_i(x), \quad \beta_i(x, y) \equiv y, \quad (x, y) \in D,\ i = 1, \ldots, m.$$

Let

$$(68) \qquad \begin{aligned} \tilde{h}_i(x) &= h(x - \tau_i(x)), \quad x \in [x_0, a_1),\ i = 1, \ldots, m, \quad \tilde{h}(x) = \min_i \tilde{h}_i(x), \\ \tilde{g}_i(x) &= g(x - \tau_i(x)), \quad x \in [x_0, a_1),\ i = 1, \ldots, m, \quad \tilde{g}(x) = \max_i \tilde{g}_i(x). \end{aligned}$$

Now the set $\tilde{K}$ is the segment $\tilde{K} = \{(x, y)\colon x = x_0,\ y \in [c, d]\}$, where $c = \max[g(x_0), \tilde{g}(x_0)]$, $d = \min[h(x_0), \tilde{h}(x_0)]$. (If $c > d$, then $\tilde{K}$ is empty.)

EXAMPLES (see [2]) 1. If

$$(69) \qquad \begin{aligned} D_-\tilde{h}(x) &\geqslant -f_q(x, \tilde{h}(x), z, U, q), \\ D_-\tilde{g}(x) &\leqslant -f_q(x, \tilde{g}(x), z, U, q) \end{aligned}$$

($D_-F(x)$ denotes the left-hand lower Dini derivative of F at the point x), then G_1 is the set formed by integral curves of (58) issuing from the segment $\tilde{K}$ for $x \in [x_0, a_1)$.

2. If

$$(70) \qquad D_-\tilde{g}(x) \geqslant -f_q(x, \tilde{g}(x), z, U, q), \quad D_-\tilde{h}(x) \leqslant -f_q(x, \tilde{h}(x), z, U, q)$$

and $\max[g(x_0), \tilde{g}(x_0)] = \tilde{g}(x_0)$, $\min[h(x_0), \tilde{h}(x_0)] = \tilde{h}(x_0)$, then

$$G_1 = \{(x, y)\colon x \in [x_0, a_1),\ \tilde{g}(x) \leqslant y \leqslant \tilde{h}(x)\}.$$

3. Assume that inequalities (70) are satisfied and $\max[g(x_0), \tilde{g}(x_0)] = g(x_0)$, $\min[h(x_0), \tilde{h}(x_0)] = h(x_0)$. Denote by $y_1(x)$ and $y_2(x)$ solutions of (58) satisfying the initial conditions $y_1(x_0) = g(x_0)$, $y_2(x_0) = h(x_0)$. Let

$$J_1 = \{x \in [x_0, a_1)\colon y_1(x) \geqslant \tilde{g}(x)\},$$
$$J_2 = \{x \in [x_0, a_1)\colon y_2(x) \leqslant \tilde{h}(x)\}$$

and

$$\tilde{y}_1(x) = \begin{cases} y_1(x) & \text{for } x \in J_1, \\ \tilde{g}(x) & \text{for } x \in [x_0, a_1) - J_1, \end{cases}$$

$$\tilde{y}_2(x) = \begin{cases} y_2(x) & \text{for } x \in J_2, \\ \tilde{h}(x) & \text{for } x \in [x_0, a_1) - J_2. \end{cases}$$

Under these assumptions

$$G_1 = \{(x, y)\colon x_0 \leqslant x < a_1,\ \tilde{y}_1(x) \leqslant y \leqslant \tilde{y}_2(x)\}.$$

4. Assume that inequalities (70) are satisfied and that

$$(71) \qquad \max[\tilde{g}(x_0), g(x_0)] = \tilde{g}(x_0), \quad \min[\tilde{h}(x_0), h(x_0)] = h(x_0).$$

Denote by $y = y_2(x)$ the solution of equation (58) satisfying the initial condition $y_2(x_0) = h(x_0)$ and assume that $y_2(x) > \tilde{g}(x)$ for $x \in [x_0, a_1)$.

Under these assumptions

$$G_1 = \{(x, y)\colon x_0 \leqslant x < a_1,\ \tilde{g}(x) \leqslant y \leqslant \tilde{\tilde{h}}(x)\},$$

where $\tilde{\tilde{h}}(x) = \min[y_2(x), \tilde{h}(x)]$.

5. Assume that conditions (70) and (71) are satisfied. Denote by $y = y_2(x)$ the solution of (58) satisfying the condition $y_2(x_0) = h(x_0)$. Assume that $y_2(x) > \tilde{g}(x)$ for $x \in [x_0, \bar{x})$, $x_0 < \bar{x} < a_1$, and $y_2(\bar{x}) = \tilde{g}(\bar{x})$.

Under these assumptions

$$G_1 = \{(x, y)\colon x_0 \leqslant x \leqslant \bar{x},\ \tilde{g}(x) \leqslant y \leqslant \tilde{\tilde{h}}(x)\},$$

where $\tilde{\tilde{h}}(x) = \min[y_2(x), \tilde{h}(x)]$.

6. Analogously to the case in Examples 4 and 5 the set G_1 can be determined in the case when inequalities (70) and the conditions

$$\max[\tilde{g}(x_0), g(x_0)] = g(x_0),$$
$$\min[\tilde{h}(x_0), h(x_0)] = \tilde{h}(x_0)$$

are satisfied.

The proofs of the construction of the set G_1 in Examples 1–6 is based on the fact that each point $(\bar{x}, \bar{y}) \in G_1$ can be joined by an integral curve $y = y(x)$ of (58) with some point $(x_0, \tilde{y}) \in \tilde{K}$ and $(x, y(x)) \in G_1$ for $x \in [x_0, \bar{x}]$.

Theorem 3.2 concerns the mixed inequalities between solutions of equation (52) in that part of D, where $x \in [x_0, a_1)$. In the sequel we consider the mutual situation of solutions of (52) in the entire D.

Suppose that Assumptions H_5 and H_6 are satisfied, $\tilde{K}$ is non-empty and that the functions a_i $(i = 1, \ldots, m)$ fulfil Assumption H_3. Let

$$\tilde{D}_k = \{(x, y)\colon x \in \tilde{I}_k, \tilde{r}(x) \leqslant y \leqslant \tilde{s}(x)\}, \qquad k = 1, \ldots, n.$$

($\tilde{I}_k$ are defined in H_3.) We shall now define a sequence of sets $\tilde{G}^{(1)}, \ldots, \tilde{G}^{(n)}$ in the following way.

Assume that conditions (61) hold and consider equation (52) in $E \cup \tilde{D}_1$. It follows from Theorem 3.2 (see also Remark 3.1) that there exists a set

$$\tilde{G}^{(1)} = \{(x, y)\colon x_0 \leqslant x \leqslant c_1, g_1(x) \leqslant y \leqslant h_1(x)\},$$

where $c_1 \leqslant \tilde{a}_1, g_1$ and h_1 are continuous functions, such that $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{G}^{(1)}$ and $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_1 - \tilde{G}^{(1)} - \tilde{L}$.

Suppose that the sets $\tilde{G}^{(1)}, \ldots, \tilde{G}^{(k)}$ have already been constructed. We define $\tilde{G}^{(k+1)}$ as follows. Consider equation (52) in $E \cup \tilde{D}_1 \cup \ldots \cup \tilde{D}_{k+1}$ and take $E \cup \tilde{D}_1 \cup \ldots \cup \tilde{D}_k$ as the initial set and

$$\varphi_k(x, y) = \begin{cases} \varphi(x, y) & \text{for } (x, y) \in E, \\ u(x, y) & \text{for } (x, y) \in \tilde{D}_1 \cup \ldots \cup \tilde{D}_k, \end{cases}$$

$$\psi_k(x, y) = \begin{cases} \psi(x, y) & \text{for } (x, y) \in E, \\ v(x, y) & \text{for } (x, y) \in \tilde{D}_1 \cup \ldots \cup \tilde{D}_k, \end{cases}$$

as the initial functions. It follows from Assumption H_3 that there exists an $l \in \{0, 1, \ldots, k\}$ such that $(a_i(x, y), \beta_i(x, y)) \in \tilde{D}_l$, $i = 1, \ldots, m$, if $(x, y) \in \tilde{D}_{k+1}$. The set $\tilde{G}^{(l)} \subset \tilde{D}_l$ has the form

$$\tilde{G}^{(l)} = \{(x, y):\ x \in \tilde{\tilde{I}}_l,\ g_l(x) \leqslant y \leqslant h_l(x)\},$$

where $\tilde{\tilde{I}}_l \subset \tilde{I}_l$ and g_l, h_l are continuous functions (for $l = 0$ we define $\tilde{G}^{(0)} = \tilde{E}$).

(i) If $\varphi_k(\tilde{a}_k, y) < \psi_k(\tilde{a}_k, y)$ for $y \in [\tilde{r}(\tilde{a}_k), \tilde{s}(\tilde{a}_k)]$, then we obtain by Theorem 2.2 that $u(x, y) < v(x, y)$ in $\tilde{D}_{k+1}$. In this case $\tilde{G}^{(k+1)}$ is the empty set.

(ii) Let

$$K^{(k)} = \{(x, y):\ x = \tilde{a}_k,\ y \in (\tilde{r}(\tilde{a}_k), \tilde{s}(\tilde{a}_k))\},$$

$$\tilde{K}_i^{(k)} = \{(x, y):\ (x, y) \in \overline{K^{(k)}} \cap \tilde{G}^{(k)},\ (a_i(x, y), \beta_i(x, y)) \in \tilde{G}^{(l)}\},$$

$$\tilde{K}^{(k)} = \bigcap_{i=1}^{m} \tilde{K}_i^{(k)}, \qquad \tilde{L}^{(k)} = \tilde{G}^{(k)} \cap (\overline{K^{(k)}} - \tilde{K}^{(k)}).$$

If $\tilde{K}^{(k)}$ is empty, then we obtain by Theorem 3.1 that $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_{k+1}$ and for $x > \tilde{a}_k$. In this case $\tilde{G}^{(k+1)}$ is the empty set.

(iii) If $\tilde{K}^{(k)}$ is non-empty, then Theorem 3.2 implies (see also Remark 3.1) the existence of a set $\tilde{G}^{(k+1)}$ such that $u(x, y) = v(x, y)$ for $(x, y) \in \tilde{G}^{(k+1)}$ and $u(x, y) < v(x, y)$ for $(x, y) \in \tilde{D}_{k+1} - \tilde{G}^{(k+1)} - \tilde{L}^{(k)}$.

These considerations imply

THEOREM 3.3. *If*

1° *Assumptions* H_3, H_5, H_6 *are satisfied,*

2° K *is non-empty and the initial functions satisfy* (61),

then

$$u(x, y) = v(x, y) \quad \textit{for } (x, y) \in \tilde{G},$$
$$u(x, y) < v(x, y) \quad \textit{for } (x, y) \in D - \tilde{G} - \tilde{L},$$

where $\tilde{G} = \bigcup_{i=1}^{n} \tilde{G}^{(i)}$.

Last of all we consider the mixed inequalities between solutions of equation (52) in the case when the initial functions satisfy the conditions

$$\begin{aligned} &\varphi(x, y) < \psi(x, y) && \text{for } (x, y) \in E_1, \\ &\varphi(x, y) = \psi(x, y) && \text{for } (x, y) \in \tilde{E}, \\ &\varphi(x, y) > \psi(x, y) && \text{for } (x, y) \in E_2, \end{aligned} \tag{72}$$

where

$$E_1 = \{(x, y):\ x \in [x_0 - \tau_0, x_0],\ r_0(x) \leqslant y < g(x)\},$$
$$E_2 = \{(x, y):\ x \in [x_0 - \tau_0, x_0],\ h(x) < y \leqslant s_0(x)\}.$$

THEOREM 3.4. *Suppose that*

1° *Assumption* H_5 *is satisfied and the functions* a_i, β_i $(i = 1, \ldots, m)$ *defined by formulas* (67) *fulfil Assumption* H_2.

2° *The functions* $\tilde{g}$ *and* $\tilde{h}$ *defined by* (68) *satisfy the differential inequalities* (69).

3° *The initial functions fulfil* (72).

4° $c = \max[g(x_0), \tilde{g}(x_0)] \leqslant \min[h(x_0), \tilde{h}(x_0)] = d$, *i.e.*, $\tilde{K}$ *in non-empty.*

5° $L^{(1)} = \{(x, y) \in \bar{K}:\ u(x, y) = v(x, y), y < c\}$, $L^{(2)} = \{(x, y) \in \bar{K}:\ u(x, y) = v(x, y), y > d\}$.

Under these assumptions we have

$$\begin{aligned} u(x, y) &< v(x, y) \quad \textit{for } (x, y) \in G_1^* - L^{(1)}, \\ u(x, y) &= v(x, y) \quad \textit{for } (x, y) \in G^*, \\ u(x, y) &> v(x, y) \quad \textit{for } (x, y) \in G_2^* - L^{(2)}, \end{aligned}$$

where G^* *is the set formed by integral curves of* (58) *issuing from* $\tilde{K}$, G_1^* *and* G_2^* *are the sets formed by integral curves of* (58) *issuing from the segments*

$$\{(x, y):\ x = x_0, y \in [\tilde{r}(x_0), c)\}, \qquad \{(x, y):\ x = x_0, y \in (d, \tilde{s}(x_0)]\},$$

respectively. Furthermore $D_1 = G^* \cup G_1^* \cup G_2^*$.

The proof of this theorem runs in a similar manner to the proof of Lemma 3.1 and Theorem 3.2.

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Reçu par la Rédaction le 17. 11. 1975