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A variational characterization of condenser capacities in C^n within a class of plurisubharmonic functions

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Abstract. A study of extermal properties of functional (**) introduced by S. S. Chern, H. I. Levine and L. Nirenberg [1] in the case of a bounded domain in C^n is given.

Introduction. Chern, Levine and Nirenberg [1] introduced a class of semi-norms on the homology groups of a complex manifold.

Namely, let M be a complex manifold of complex dimension n. Consider the family F of C^2-plurisubharmonic functions u on M satisfying the condition $0 \leqslant u \leqslant 1$. For any homology class γ of M with real coefficients we set, following [1],

$$(*) \quad N[\gamma] = \begin{cases} \sup\limits_{u \in F} \inf\limits_{T \in \gamma} |T(d^c u \wedge [dd^c u]^{k-1})|, & \text{if } \dim\gamma = 2k-1, \\ \sup\limits_{u \in F} \inf\limits_{T \in \gamma} |T(du \wedge d^c u \wedge [dd^c u]^{k-1})|, & \text{if } \dim\gamma = 2k, \end{cases}$$

where T runs over all currents of γ in the sense of de Rham. It can be easily verified that the mapping

$$N\colon \Gamma \to R^+,$$

Γ denoting the family of all homology classes on M, is a semi-norm on Γ. The main theorems given in [1] state that $N[\gamma]$ is always finite and non-increasing under a holomorphic mapping of M (in particular, $N[\gamma]$ is invariant under biholomorphic mappings).

In the case of complex manifolds endowed with an hermitian structure a class of semi-norms (capacities) has been introduced and investigated by Ławrynowicz [3].

Let D be a bounded domain in C^n whose boundary consists of two components which are $(2n-1)$-dimensional smooth differentiable manifolds. We denote by $\operatorname{Adm} D$ the class of C^2-smooth real-valued functions on $\bar{D}$ which are supposed to be plurisubharmonic on D and equal zero on one component of the boundary and one on the other. In our considerations we assume that $\operatorname{Adm} D \neq \emptyset$.

Let us define the functional ([1])

$$(**) \qquad \operatorname{Adm} D \ni u \to \int_D du \wedge d^c u \wedge [dd^c u]^{n-1} \in R.$$

We call the domain D a *condenser*. With any condenser D we associate the real number

$$\operatorname{Cap} D = \inf_{u \in \operatorname{Adm} D} \left(\int_D du \wedge d^c u \wedge [dd^c u]^{n-1} \right),$$

which is called its *capacity*.

This paper is devoted to a discussion of the extremal properties of (**). We find explicit formulae for the first and second variation of (**) within the class $\operatorname{Adm} D$, and prove that the second variation is non-negative. We also prove that functional (**) attains its minimum at $u_0 \in \operatorname{Adm} D$ if and only if

$$[dd^c u_0]^n = 0.$$

Finally, we prove that if the upper envelope u of $\operatorname{Adm} D$ belongs to $\operatorname{Adm} D$, then u minimizes functional (**).

To begin with we give three definitions.

DEFINITION 1. Let D_0 and D_1 be two bounded simply connected domains in C^n. The set $D = D_2 \setminus \bar{D}_0$ is called a *condenser* if

1° $\bar{D}_0 \subset D_1$,

2° $C_0 = \bar{D}_0 \setminus D_0$ and $C_1 = \bar{D}_1 \setminus D_1$ are $(2n-1)$-dimensional smooth differentiable manifolds.

3° there exists a real-valued C^2-smooth function u on $\bar{D}_1 \setminus D_0$ which fulfils the conditions: $0 \leqslant u \leqslant 1$, $u|_{C_0} = 0$, $u|_{C_1} = 1$ and

$$H(u, w)(z_0) = \sum_{i,k=1}^{n} \partial^2 u / \partial z^i \partial \bar{z}^k (z_0) w^i \bar{w}^k \geqslant 0$$

for $z_0 \in D$ and $w = (w^1, \ldots, w^n) \in C^n$.

Remark. Condition 3° implies that u is C^2-plurisubharmonic on D.

EXAMPLE. Let $r_1 > r_0 > 0$. Then the set

$$P(r_0, r_1) = \{z \in C^n \colon r_0 < |z|^2 < r_1\},$$

where $|z|^2 = \sum_{i=1}^{n} |z^i|^2$, is a condenser.

DEFINITION 2. By the class $\operatorname{Adm} D$ of functions admissible for a condenser D we mean the class of real-valued functions u which fulfil condition 3° of Definition 1.

Remark. The class $\operatorname{Adm} D$ is convex.

We will now be concerned with the functional

$$\text{Adm}\, D \ni u \mapsto J(u) = \int_D du \wedge d^c u \wedge [dd^c u]^{n-1} \in R, \tag{1}$$

where $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. This functional has been introduced by Chern, Levine and Nirenberg [1] in the general case of a complex manifold.

Remark. Functional (1) is non-negative [2].

DEFINITION 3. The real number

$$\text{Cap}\, D = \inf_{u \in \text{Adm}\, D} J(u)$$

is called the *capacity of D*.

Let u and $\tilde{u}$ be arbitrary functions of $\text{Adm}\, D$. For any $t_0 \in [0, 1]$ the function $u(t_0) = u + t_0(\tilde{u} - u)$ belongs to $\text{Adm}\, D$ (see remark after Definition 2). Let us consider the function

$$F(u, h, \cdot)\colon [0, 1] \ni t \mapsto F(u, h, t) = J(u(t)) \in R, \tag{2}$$

where $h = \tilde{u} - u$. Function (2) is a polynomial of degree $n+1$ with respect to t.

Let

$$\delta J(u)(h, t) = \begin{cases} \lim\limits_{s \to 0^+} [F(u, h, s) - F(u, h, 0)]/s & \text{for } t = 0, \\ \lim\limits_{s \to 0} [F(u, h, t+s) - F(u, h, t)]/s & \text{for } t \in (0, 1), \\ \lim\limits_{s \to 0^-} [(u, h, 1+s) - F(u, h, 1)]/s & \text{for } t = 1. \end{cases} \tag{3}$$

DEFINITION 4. We call the limit given by (3) the *variation of J* at the point $u(t) = u + t(\tilde{u} - u)$ in the direction of $h = \tilde{u} - u$.

Remark. From the above definition it follows that with any $u \in \text{Adm}\, D$ we may associate the set $H_u = \{h = \tilde{u} - u\colon \tilde{u} \in \text{Adm}\, D\}$ of all directions of variation at u. It can be easily seen that H_u is convex.

LEMMA 1. *The variation of functional* (3) *at a point* $u(t_0) = u + t_0(\tilde{u} - u)$, $t_0 \in [0, 1]$, *in the direction of* $h = \tilde{u} - u$ *has the form*

$$\delta J(u)(h, t_0) = -(n+1) \int_D h [dd^c u(t_0)]^n.$$

Proof. Let $W(s) = F(u, h, t_0 + s)$ for $s \in S = \{p \in R\colon (t_0 + p) \in [0, 1]\}$. From the definition of F it follows that W is a polynomial of degree $n+1$ with respect to $s \in S$. By Definition 4, $\delta J(u)(h, t_0)$ is equal to the coefficient of W standing at the first power of s. Hence we obtain

(4) $$\delta J(u)(h, t_0) = \int_D dh \wedge d^c u(t_0) \wedge [dd^c u(t_0)]^{n-1} + \int_D du(t_0) \wedge d^c h \wedge [dd^c u(t_0)]^{n-1} + (n-1) \int_D du(t_0) \wedge d^c u(t_0) \wedge dd^c h \wedge [dd^c u(t_0)]^{n-2}.$$

For the proof we need the following identity:

(4′) $$dh \wedge d^c u(t_0) \wedge [dd^c u(t_0)]^{n-1} = du(t_0) \wedge d^c h \wedge [dd^c u(t_0)]^{n-1}.$$

In order to prove (4′) we express the left-hand side of it in terms of ∂ and $\bar{\partial}$:

(5) $$\begin{aligned} i(\partial+\bar{\partial}) h \wedge (\bar{\partial}-\partial) u(t_0) \wedge [2i\partial\bar{\partial} u(t_0)]^{n-1} &= 2^{n-1} i^n \{[\partial h \wedge \bar{\partial} u(t_0) - \bar{\partial} h \wedge \partial u(t_0) + \bar{\partial} h \wedge \bar{\partial} u(t_0) - \\ &\quad - \partial h \wedge \partial u(t_0)] \wedge [\partial\bar{\partial} u(t_0)]^{n-1}\} \\ &= 2^{n-1} i^n \{[\partial h \wedge \bar{\partial} u(t_0) - \bar{\partial} h \wedge \partial u(t_0)] \wedge [\partial\bar{\partial} u(t_0)]^{n-1}\}. \end{aligned}$$

The latter equality in (5) is obtained from the identities

$$\partial h \wedge \partial u(t_0) \wedge [\partial\bar{\partial} u(t_0)]^{n-1} = 0 \quad \text{and} \quad \bar{\partial} h \wedge \bar{\partial} u(t_0) \wedge [\partial\bar{\partial} u(t_0)]^{n-1} = 0.$$

Analogously, regarding the right-hand side of (4′) we have

(6) $$\begin{aligned} i(\partial+\bar{\partial}) u(t_0) \wedge (\bar{\partial}-\partial) h \wedge [2i\partial\bar{\partial} u(t_0)]^{n-1} &= 2^{n-1} i^n \{[\partial u(t_0) \wedge \bar{\partial} h - \bar{\partial} u(t_0) \wedge \partial h + \bar{\partial} u(t_0) \wedge \bar{\partial} h - \\ &\quad - \partial u(t_0) \wedge \partial h] \wedge [\partial\bar{\partial} u(t_0)]^{n-1}\} \\ &= 2^{n-1} i^n \{[\partial h \wedge \bar{\partial} u(t_0) - \bar{\partial} u \wedge \partial u(t_0)] \wedge [\partial\bar{\partial} u(t_0)]^{n-1}\}. \end{aligned}$$

From (5) and (6) we arrive at (4′).

In view of (4′) the variation given by (4) takes the form

(7) $$\begin{aligned} \delta J(u)(h, t_0) &= 2 \int_D dh \wedge d^c u(t_0) \wedge [dd^c u(t_0)]^{n-1} + \\ &\quad + (n-1) \int_D du(t_0) \wedge d^c u(t_0) \wedge dd^c h \wedge [dd^c u(t_0)]^{n-2} \\ &= 2 \int_{\partial D} h d^c u(t_0) \wedge [dd^c u(t_0)]^{n-1} - 2 \int_D h [dd^c u(t_0)]^n + \\ &\quad + (n-1) \int_{\partial D} du(t_0) \wedge d^c u(t_0) \wedge d^c h \wedge [dd^c u(t_0)]^{n-2} - \\ &\quad - (n-1) \int_D h [dd^c u(t_0)]^n + (n-1) \int_{\partial D} h d^c u(t_0) \wedge [dd^c u(t_0)]^{n-1} \\ &= -(n+1) \int_D h [dd^c u(t_0)]^n. \end{aligned}$$

In order to obtain the latter equality we have to apply Weierstrass' approximation theorem [4] and then Stokes' theorem using the boundary conditions

$$h|\partial D = 0 \quad \text{and} \quad d(u(t_0)|\partial D) = 0.$$

This proves the lemma.

By Lemma 1 the function

$$\delta J(u)(h, \cdot): [0, 1] \to R \tag{8}$$

is a polynomial of degree n with respect to $t \in [0, 1]$.

DEFINITION 5. By the second variation of the functional (1) we mean the first derivative of function (8); we denote it by $\delta^2 J(u)(t, h)$ for $t \in [0, 1]$.

Remark. At the points $t = 0$ and $t = 1$ the derivative of (8) is understood as the corresponding one-sided derivative.

LEMMA 2. *For every pair of functions u, $\tilde{u}$ in* Adm D *and any $t_0 \in [0, 1]$ the following inequality holds*:

$$\delta^2 J(u)(h, t_0) \geqslant 0, \quad h = \tilde{u} - u.$$

Proof. Let u and $\tilde{u}$ be arbitrary functions in Adm D and $t_0 \in [0, 1]$. We notice that $\delta J(u)(h, t_0 + s) = \delta J(u(t_0))(h, s)$, $h = \tilde{u} - u$. By Definition 5 and by the above remark we have

$$\delta^2 J(u)(h, t_0) = -\tfrac{1}{2} n(n+1) \int_D h dd^c h \wedge [dd^c u(t_0)]^{n-1}, \quad h = \tilde{u} - u. \tag{9}$$

Taking into account the boundary condition

$$h|\partial D = 0$$

and the identity

$$\int_{\partial D} h dd^c h \wedge [dd^c u(t_0)]^{n-1}$$
$$= \int_{\partial D} h d^c h \wedge [dd^c u(t_0)]^{n-1} - \int_D dh \wedge d^c h \wedge [dd^c u(t_0)]^{n-1},$$

we see that (9) becomes

$$\delta^2 J(u)(h, t_0) = \tfrac{1}{2} n(n+1) \int_D dh \wedge d^c h \wedge [d^c u(t_0)]^{n-1} - \tag{10}$$
$$- \tfrac{1}{2} n(n+1) \int_{\partial D} h d^c h \wedge [dd^c u(t_0)]^{n-1}$$
$$= \tfrac{1}{2} n(n+1) \int_D dh \wedge d^c h \wedge [dd^c u(t_0)]^{n-1}.$$

Since the form $dh \wedge d^c h \wedge [dd^c u(t_0)]^{n-1}$ is non-negative [2], our result follows.

THEOREM 1. *If D is a condenser and $u \in \operatorname{Adm} D$, then the following conditions are equivalent:*

(a) $[dd^c(u \mid D)]^n = 0$,

(b) $\delta J(u)(h, 0) \geqslant 0$ *for every* $h \in H_u$,

(c) $J(u) = \operatorname{Cap} D$.

Proof. Let $\tilde{u}$ be an arbitrary function in $\operatorname{Adm} D$ and $F = F(u, h, t)$ be the function given by (2). By Lemma 1 we have

$$F(0) = \delta J(u)(h, 0) = 0, \quad h = \tilde{u} - u \tag{11}$$

which proves that (a) implies (b).

Now we show that (b) is equivalent to (c). To see this, it is enough to remark that

$$F''(t) = \delta^2 J(u)(h, t) \quad \text{for} \quad t \in [0, 1] \text{ and } h = \tilde{u} - u. \tag{12}$$

From (b), (12) and Lemma 2 it follows that F is a non-decreasing function. Hence

$$F(0) = F(u, h, 0) = J(u) \leqslant F(1) = F(u, h, 1) = J(\tilde{u}).$$

Since $\tilde{u} \in \operatorname{Adm} D$ is arbitrary, the above implication is proved. The converse implication is a simple consequence of the mean value theorem applied to F.

Now we shall prove that (b) implies (a). We shall prove this assuming the contrary and arriving at a contradiction.

Suppose that there exists a point $z_0 \in D$ such that

$$f(z_0) = \det [u_{|i\bar{k}}(z_0)]_{1 \leqslant i,k \leqslant n} > 0.$$

Since D is a domain and f is a continuous function, we can find a polydisc $\Delta_0(z_0, r)$ such that $\bar{\Delta}_0 \subset D$ and

$$f \mid \bar{\Delta}_0 > 0.$$

Let

$$M_j = \det [u_{|i\bar{k}}]_{1 \leqslant i,k \leqslant j} \quad \text{for } j = 1, 2, \ldots, n.$$

Since $u \in \operatorname{Adm} D$, the functions M_j, $j = 1, \ldots, n$, are continuous and non-negative in D. The matrix $[u_{|i\bar{k}} \mid \bar{\Delta}_0]_{1 \leqslant i,k \leqslant n}$ is hermitian and positive definite. Therefore $M_j \mid \bar{\Delta}_0 > 0$ for $j = 1, 2, \ldots, n$. Let us denote by $\varphi \geqslant 0$ an arbitrary function of class $C^\infty(\bar{D})$ such that $\varphi \mid \Delta_1 > 0$ and $\Delta_1 \subset \operatorname{supp} \varphi \subset \Delta_0$. Consider the functions $\tilde{M}_j$: $D \times [-t_0; t_0] \to R$ for $j = 1, 2, \ldots, n$ and $t_0 \in R^+$, given by the formulae

$$\tilde{M}_j(z, t) = \det [(u + t\varphi)_{|i\bar{k}}(z)]_{1 \leqslant i,k \leqslant j} \quad \text{for } z \in D,$$

$t \in [-t_0; t_0]$ and $j = 1, 2, \ldots, n$. The functions $\tilde{M}_j$ are uniformly continuous on $\bar{\Delta}_0 \times [-t_0; t_0]$. Hence for

$$\eta_j = \frac{1}{2} \min_{z \in \bar{\Delta}_0} \tilde{M}_j(z, 0) > 0$$

we can find $\sigma_j > 0$ such that for every $z_0 \in \bar{\Delta}_0$ and $t \in [-t_0; t_0]$, $|t| < \sigma_j$, we have

$$|\tilde{M}_j(z, t) - \tilde{M}_j(z, 0)| < \eta_j.$$

Therefore we obtain

$$\tilde{M}_j(z, t) > 0 \quad \text{for } z \in \bar{\Delta}_0 \text{ and } |t| < \sigma_j.$$

Let

$$\sigma = \min\{\min_j \tfrac{1}{2}\sigma_j, (n+1) \int_D \varphi [dd^c u]^n\}.$$

Obviously, $\sigma > 0$, $\tilde{M}_j(z, \sigma) > 0$ for $z \in \bar{\Delta}_0$ and $j = 1, 2, \ldots, n$. This means that the matrix $[(u+\sigma\varphi)_{|i\bar{k}}(z)]_{1 \leqslant i,k \leqslant n}$ has all its characteristic roots positive for every $z \in \bar{\Delta}_0$. Since φ has its support in Δ_0, it follows that $\tilde{u} = u + \sigma\varphi \in \operatorname{Adm} D$. Thus the first variation of (1) at the point u in the direction of $h = \sigma\varphi$ has the form

$$\delta J(u)(h, 0) = -(n+1) \int_D \sigma\varphi [dd^c u]^n \leqslant -\sigma^2.$$

The last inequality contradicts our assumption and so the theorem follows.

As a consequence of Theorem 1 we have

THEOREM 2. *If the upper envelope of* Adm D *belongs to* Adm D, *then it minimizes functional* (1).

Proof. If u is the upper envelope of Adm D, then condition (b) in Theorem 1 must be satisfied. This implies our statement.

References

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