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The range of vector valued holomorphic mappings

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Abstract. Let E be an arbitrary separable complex Banach space with open unit ball B. A holomorphic mapping $f\colon \Delta \to E$ ($\Delta = \{z \in C\colon |z| < 1\}$) is constructed such that $f(\Delta)$ is densely contained in B.

Introduction. The purpose of this note is to provide an affirmative answer to the following question, posed by D. Patil at the Conference on Infinite Dimensional Holomorphy, University of Kentucky, 1973:

(*) Let E be an arbitrary separable complex Banach space with open unit ball B. Is there a holomorphic mapping $f\colon \Delta \to E$ ($\Delta = \{z \in C\colon |z| < 1\}$) such that $f(\Delta)$ is contained and dense in B?

This problem has also been studied by J. Globevnik, who has independently obtained a solution, by totally different means [5]. In fact, when E is finite dimensional, Globevnik has proved that there is a holomorphic function $f\colon \Delta \to E$ such that $\overline{f(\Delta)} = \bar{B}$ and such that f is continuous on $\bar{\Delta}$ [4]. This follows from his extension to vector values of the Rudin–Carleson theorem (see for example [7]), while his solution in the general case follows from his generalization of results of Heard and Wells [6] to vector values. In addition, partial and related results on interpolation into Banach spaces were obtained in [1].

Our approach here is entirely different from that of Globevnik. Making use of cluster set properties of the Blaschke product, we construct the solution f to (*) as a composition of three holomorphic functions, $f = f_3 \circ f_2 \circ f_1$, each of which we now sketch. f_1 will be the solution to (*) in the case $E = c_0$. Denoting the open unit balls of c_0 by B_0 and of l_2 by B_2, f_2 will be a holomorphic mapping from B_0 to B_2 with dense range. Finally, f_3 will be a holomorphic mapping of B_2 into a dense subset of B.

Construction. A frequent tool in our construction will be a Blaschke product $b(z)$ which has the entire unit circle $\partial\Delta$ as the cluster set for its zeroes. Such a function has the following important property: for any point $e^{i\theta}$ on $\partial\Delta$ and for any deleted neighborhood $U \subset \Delta$ of $e^{i\theta}$, $b(U)$

is dense in Δ [3], p. 95. Without loss of generality, we will assume that $|b(z)| \leqslant |z|$, for all $z \in \Delta$.

Step 1. Let $A \subset \partial\Delta$ be a Cantor set of Lebesgue measure 0. By [8], p. 165, there is a continuous surjection $h\colon A \to [0,1]^N$ (N the natural numbers), $h(a) = (h_1(a), \ldots, h_n(a), \ldots)$. By a theorem of Rudin [7], p. 81, for each $j = 1, 2, \ldots$, there is a function $g_j \in C(\bar{\Delta}) \cap H(\Delta)$, extending h_j on A, and having maximum modulus 1. Define $f_1\colon \Delta \to B_0$ as follows:

$$f_1(z) = \big(b(z)g_1(z), (b \circ b(z))^2 g_2(z), \ldots, (\underbrace{b \circ b \circ \ldots \circ b}_{n \text{ iterations}}(z))^n g_n(z), \ldots\big).$$

It is easy to see that f_1 does in fact map into B_0; that it is analytic follows by an application of the result in [2]. Finally, we indicate why $f_1(\Delta)$ is dense in B_0. Let $(w_1, \ldots, w_n, \ldots) \in B_0$ and let $\varepsilon > 0$. For some $M \in N$, $|w_n| < \varepsilon$ for all $n > M$. Let $Z_0 \in A$ such that $g_j(Z_0) = |w_j|$ for all $j \in N$. For $k = 1, 2, \ldots, M$, let $\theta_k = \arg w_k$, and choose $Z_k \in \partial\Delta$ such that $Z_k^k = e^{i\theta_k}$ ($k = 1, \ldots, M$). Now, choose z_M in a suitably small (deleted) neighbourhood U_M of Z_M, $U_M \subset \Delta$, such that for all $z \in V_M$, some neighbourhood of z_M in U_M, $|b(z)| < \varepsilon$. Note that then $|b \circ b \circ \ldots \circ b(z)| < \varepsilon$ for any iteration of b, if $z \in V_M$. Let U_{M-1} be a small deleted neighbourhood of Z_{M-1}, $U_{M-1} \subset \Delta$, and choose z_{M-1} and a neighbourhood $V_{M-1} \subset U_{M-1}$ of z_{M-1} such that for all $z \in V_{M-1}$, $b(z) \in V_M$. Continuing in this manner, we choose a small deleted neighbourhood U_0 of Z_0, $U_0 \subset \Delta$, such that $\big|g_j(z) - |w_j|\big| < \varepsilon$ for $j = 1, \ldots, M$ and $z \in U_0$. There is some point $z_0 \in U_0$ such that $b(z_0) \in V_1$. It is easy to verify that if each of the neighbourhoods U_j and V_j ($j = 1, \ldots$ $\ldots, M$) is chosen "small enough", then $\|f_1(z_0) - (w_1, \ldots, w_n, \ldots)\| < 2\varepsilon$. This completes Step 1.

Step 2. We define a holomorphic function f_2 from B_0 to B_2 with dense range as follows. For $(z_1, z_2, \ldots) \in B_0$, first define $g\colon B_0 \to l_2$ as follows:

$$g(z_1, z_2, \ldots) = \big(g_1(z_1), g_2(z_1, z_2), \ldots, g_n(z_1, z_2, \ldots, z_n), \ldots\big),$$

where

$$g_1(z_1) = \frac{1+z_1^2}{2}, \qquad g_2(z_1, z_2) = \frac{1-z_1^2}{2}\,\frac{1+z_2^2}{2},$$

and in general

$$g_n(z_1, z_2, \ldots, z_n) = \frac{1-z_1^2}{2}\,\frac{1-z_2^2}{2} \cdots \frac{1-z_{n-1}^2}{2}\,\frac{1+z_n^2}{2}.$$

(The motivation for this definition comes from the observation that

$$\left|\frac{1+e^{2i\theta}}{2}\right| = |\cos\theta| \quad \text{and} \quad \left|\frac{1-e^{2i\theta}}{2}\right| = |\sin\theta|.)$$

One shows that g is in fact analytic into l_2 in a similar manner to that described in Step 1. Furthermore, an elementary calculation shows that $\|g(z_1, \dots, z_n, 0, \dots, 0, \dots)\| < 1$ for all points of the form $(z_1, \dots, z_n, 0, \dots, 0, \dots) \in B_0$, so that by a density argument, we see that $g(B_0) \subset B_2$. In order to get the required function, we must alter g, as follows.

Define f_2: $B_0 \to B_2$ by

$$f_2(z_0, z_1, \dots, z_n, \dots) = \big(b(z_0)g_1(z_1), (b \circ b(z_0))^2 g_2(z_1, z_2), \dots \\ \dots, (b \circ b \circ \dots \circ b(z_0))^n g_n(z_1, \dots, z_n), \dots\big).$$

There is no difficulty in showing that f_2 is analytic, with range contained in B_2, and so we will only sketch the proof that $f_2(B_0)$ is dense in B_2. Fix any point $e^{i\theta_0} \in \partial\Delta$. Let $(w_1, \dots, w_n, \dots) \in B_2$ and let $\varepsilon > 0$. For some $M \in N$, $\sum\limits_{j>M} |w_j|^2 < \varepsilon$ and $\sum\limits_{j>M} 2^{-j} < \varepsilon$. We note that $\sum |w_j|^2 < 1$, so that it is possible to choose $Z_j \in \partial\Delta$, $j = 1, \dots, M$, satisfying the following conditions:

$$|g_1(Z_1)| = \left|\frac{1+Z_1^2}{2}\right| = |w_1|,$$

$$|g_2(Z_1, Z_2)| = \left|\frac{1-Z_1^2}{2} \cdot \frac{1+Z_2^2}{2}\right| = |w_2|,$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$|g_M(Z_1, \dots, Z_M)| = \left|\frac{1-Z_1^2}{2}\,\frac{1-Z_2^2}{2} \cdots \frac{1-Z_{M-1}^2}{2}\,\frac{1+Z_M^2}{2}\right| = |w_M|.$$

Setting $w_j = g_j(Z_1, \dots, Z_j)e^{i\theta_j}$, $j = 1, \dots, M$, let $\gamma_j \in [0, 2\pi]$ such that $(e^{i\gamma_j})^j = e^{i\theta_j}$. In some suitably small deleted neighbourhood $U_M \subset \Delta$ of $e^{i\gamma_M}$, choose a neighbourhood V_M such that $|b(z)| < \frac{1}{2}$ for all $z \in V_M$. Choose a small deleted neighbourhood $U_{M-1} \subset \Delta$ of $e^{i\gamma_{M-1}}$ and a neighbourhood $V_{M-1} \subset U_{M-1}$ such that $b(V_{M-1}) \subset V_M$. Continuing in this manner, we ultimately choose a suitably small deleted neighbourhood $U_0 \subset \Delta$ of $e^{i\gamma_0}$ and a neighbourhood $V_0 \subset U_0$ such that $b(V_0) \subset V_1$. Choose any point $z_0 \in V_0$ and points $z_1, \dots, z_M$ sufficiently close to $Z_1, \dots, Z_M$, respectively. Then,

$$\|f(z_0, z_1, \dots, z_M, 0, \dots, 0, \dots) - (w_1, \dots, w_M, \dots)\| < 3\varepsilon,$$

provided our neighbourhoods U_j and V_j are "small enough" and the points z_j are "close enough" to the points Z_j, $j = 1, \dots, M$. This completes Step 2.

Step 3. We define here a holomorphic mapping f_3 from B_2 to B as follows (cf. [1]). Let $\{x_n\}$ be a countable dense subset of B. If $\{e_n\}$ is an orthonormal basis for l_2, define $f_3(\sum a_n e_n) = \sum a_n^2 x_n$. f_3 is clearly holomorphic; that $f(B_2)$ is dense follows from the observation that $f_3(e_n) = x_n$.

Therefore, if we let $f = f_3 \circ f_2 \circ f_1$, it is routine to check that $f\colon \Delta \to B$ holomorphically with dense range.

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