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KNEADING THEORY OF LORENZ MAPS*

LLUIS ALSEDÀ[1,2] AND JAUME LLIBRE[1]

[1] *Departament de Matemàtiques, Facultat de Ciències*

[2] *Departament d'Economía i Història Econòmica, Facultat de Ciències Econòmiques*
Universitat Autonòma de Barcelona
08193 *Bellaterra, Barcelona, Spain*

1. Introduction

In the study of the geometrical model of the Lorenz attractor, a class of one-dimensional maps plays an important role. We refer to such maps as the Lorenz maps (see [GH], [Sp] and [T]) although they are different from the one-dimensional maps presented by Lorenz (see [L]). We describe the use of the kneading theory to study the dynamics of Lorenz maps.

Let $I = [-1, 1]$. We say that a map $f\colon I \to I$ is *Lorenz* if

(L1) f has a single discontinuity at 0, $\lim_{x\to 0^+} f(x) = -1$ and $\lim_{x\to 0^-} f(x) = 1$,

(L2) f is odd on $I\setminus\{0\}$ (i.e. $f(-x) = -f(x)$ for all $x \in I\setminus\{0\}$),

(L3) $f(-1) < 0$ and $f(0) = 1$,

(L4) f is once continuously differentiable on $I\setminus\{0\}$, and $f'(x) > 1$ for all $x \in I\setminus\{0\}$.

Note that every Lorenz map is strictly monotone on the intervals $[-1, 0)$ and $(0, 1]$. Many of our results would also be true for maps satisfying (L1), (L2), (L3) and

(L5) f is strictly increasing on $[-1, 0)$ and $(0, 1]$,

instead of (L4); but the ideas of some proofs seem more transparent for Lorenz maps. It is also easy to see that the particular choice of the interval $[-1, 1]$ and fixing the discontinuity at $x = 0$ involves no loss of generality.

Let $J = [-1, 0]$. We say that a map $g\colon J \to J$ is *piecewise expanding unimodal* if

(U1) g is continuous,

(U2) there exists $c \in (-1, 0)$ such that $g(c) = 0$,

(U3) $g(0) = -1$,

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* This is a summary of [AL].

(U4) g is once continuously differentiable on $J \setminus \{c\}$, and $|g'(x)| > 1$ for all $x \in J \setminus \{c\}$.

We note that there is a close correspondence between the Lorenz maps and the piecewise expanding unimodal maps. In fact, if f is Lorenz then the map $g = -|f||_{[-1,0]}$ is piecewise expanding unimodal. From (L2) f and g are semiconjugate. That is, $h \circ f = g \circ h$, where h: $I \to J$ is given by $h(x) = -|x|$.

The kneading sequences and itineraries for the maps f and g are closely related. Thus, most of the results obtained by using the standard techniques of kneading theory for Lorenz maps, can also be obtained easily from the well-known results of the kneading theory for the piecewise expanding unimodal maps (see [DGP] and [CE]). The main advantage of applying kneading theory to Lorenz maps is that a Lorenz map is strictly increasing, and some basic tools work easier.

For unimodal maps the set of kneading sequences is characterized by the notion of maximality (see [CE]). The maximality is not sufficient to characterize the kneading sequences of piecewise expanding unimodal maps. Derrida, Gervois and Pomeau gave this characterization for the set of finite kneading sequences (see [DGP]). In this paper we show (see Theorem 8) that a similar characterization holds for the set of all kneading sequences of Lorenz maps (regardless if they are finite or not). Hence, as it was expected Derrida–Gervois–Pomeau's characterization extends to the whole set of kneading sequences.

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2. Itineraries

Let f be a Lorenz map. For a point $x \in I$ we define its *itinerary* (an infinite sequence of symbols I_n),

$$\underline{I}(x) = \underline{I}_f(x) = I_0 I_1 I_2 \ldots,$$

by $I_n = L$, C or R according as $f^n(x) > 0$, $f^n(x) = 0$ or $f^n(x) < 0$.

Notice that $\underline{I}(x)$ and $\underline{I}(f(x))$ are related in a very simple way. If $\underline{I}(x) = I_0 I_1 I_2 \ldots$ then $\underline{I}(f(x)) = I_1 I_2, \ldots$, because $f^n(f(x)) = f^{n+1}(x)$. This motivates the definition of the *shift* map S on sequences; defined by:

$$\text{if } \underline{A} = A_0 A_1 A_2 \ldots \text{ then } S(\underline{A}) = A_1 A_2 \ldots$$

Thus $\underline{I}(f(x)) = S(\underline{I}(x))$ and, by induction, $S(f^n(x)) = S^n(\underline{I}(x))$.

Defining the order $L < C < R$ on the symbols, one can extend it to an order on the sequences as follows: if $\underline{A}$ and $\underline{B}$ are two different sequences, there is a least integer k for which $A_k \neq B_k$. Then we say $\underline{A} < \underline{B}$ if and only if

$A_k < B_k$. The key point of this definition is the following: $x < y$ if and only if $\underline{I}(x) < \underline{I}(y)$ (see the next theorem).

Since $<$ is a complete ordering in the set of all itineraries, we consider in this set the ordering topology.

PROPOSITION 1. *Let f be a Lorenz map. Then we have*:

(1) *The map $x \to \underline{I}_f(x)$ is an order preserving homeomorphism from the interval I to the set of all itineraries of f.*

(2) *The set of all itineraries of f containing C is dense in the set of all itineraries of f.*

(3) *The set of all itineraries of f which do not contain C is dense in the set of all itineraries of f.*

If $\underline{A} = A_0 A_1 \ldots$ is a sequence, then we define the sequence $\underline{A}' = A'_0 A'_1 \ldots$ where $L' = R$, $C' = C$ and $R' = L$. Hence, since f is odd we have $\underline{I}(-x) = \underline{I}(x)'$ for all $x \in I \setminus \{0\}$.

Defining the *kneading sequence* $K(f)$ of f to be the itinerary $\underline{I}_f(1)$, the above theorem implies that $K(f)$ is the largest itinerary that can occur for f. Also $K(f)' = \underline{I}(-1)$ is the smallest one that occurs on I. Given an itinerary $\underline{A} = \underline{I}(x)$, $S^n(\underline{A}) = \underline{I}(f^n(x))$ is also an itinerary. Thus, $\underline{A}$ must satisfy

$$K(f)' \leqslant S^n(\underline{A}) \leqslant K(f) \quad \text{for all integer } n \geqslant 0.$$

The next theorem almost gives the converse statement. Note that if $\underline{A} = \underline{I}(x)$ contains a C, then its future is determined by the orbit of 0, and so it must have the form $\underline{A} = \underline{B}CK(f)$, for some finite sequence $\underline{B}$ containing no C. We call an arbitrary infinite sequence $\underline{A}$ of L's, C's and R's *admissible* for f if it contains no C's or has the above form.

We define $\bar{K}(f)$ as follows:

$$\bar{K}(f) = \begin{cases} K(f) & \text{if } K(f) \text{ does not contain } C, \\ (\underline{B}L)^\infty & \text{if } K(f) = (\underline{B}C)^\infty = \underline{B}C\underline{B}C\ldots \end{cases}$$

We shall say that an admissible sequence $\underline{A}$ is *dominated* by f if

$$K(f)' \leqslant S^n(\underline{A}) \leqslant \bar{K}(f) \quad \text{for all integer } n \geqslant 0.$$

PROPOSITION 2. *The set of all itineraries of a Lorenz map f is formed by all dominated sequences by f plus the kneading sequence.*

We shall say that an itinerary $\underline{B}$ is *periodic of period p* if there exists a finite sequence $\underline{A} = A_0 A_1 \ldots A_{p-1}$ of symbols L's, C's and R's such that $\underline{B} = \underline{A}^\infty$ and $S^k(\underline{B}) \neq \underline{B}$ for $k = 1, 2, \ldots, p-1$. The next proposition tells us the relation between periodic orbits and periodic itineraries.

PROPOSITION 3. *Let f be a Lorenz map. Then $\underline{I}(x)$ is a periodic itinerary of period p if and only if x is a periodic point of period p.*

3. Topological equivalence

Two Lorenz maps f and g are said to be *topologically conjugate*, $f \approx g$, if there exists a homeomorphism h: $I \to I$ such that h preserves orientation, $h(0) = 0$ and $hf = gh$. If $f \approx g$, then $\underline{I}_f(x) = \underline{I}_g(h(x))$. In particular $\underline{I}_f(1) = \underline{I}_g(1)$. Thus, we see that topologically conjugate maps have the same kneading sequence. The converse is also true. The following theorem is due to Guckenheimer and Williams (see [GW]).

THEOREM 4. *Let f and g be Lorenz maps. Then f and g are topologically conjugate if and only if $K(f) = K(g)$.*

Following the ideas of Misiurewicz and Szlenk (see [MS1] and [MS2]; see also [MT]) we shall define the growth number of a Lorenz map. We shall say that a map f from I to itself is *piecewise-increasing* if I can be subdivided into finitely many subintervals $I_i = [c_i, c_{i+1}]$ for $i = 0, 1, \ldots, k-1$ in such a way that $c_0 = -1 < c_1 < c_2 \ldots < c_k = 1$, and the restriction of f to each interval (c_i, c_{i+1}) is strictly increasing. Such a maximal interval will be called a *lap* of the function f, and the number $l = l(f)$ of distinct laps will be called the *lap number* of f. Note that if f is a Lorenz map, then f^n is piecewise-increasing for all integer $n \geqslant 0$ and the endpoints of the laps are the discountinuity points of f^n together with the endpoints of I.

If f is a Lorenz map, then there exists the limit $l(f^n)^{1/n}$ as $n \to \infty$. This limit, $s(f)$, will be called the *growth number* of f, and $s(f) \in [1, 2]$.

For each $s \in (1, 2]$ we define the Lorenz map f_s where $f_s(x)$ is either $sx+1$ if $x \in [-1, 0]$ or $sx-1$ if $x \in (0, 1]$.

PROPOSITION 5. *Let f be a Lorenz map and $s = s(f)$ be its growth number. Then f and f_s are topologically conjugate and $s = s(f_s)$.*

In a similar way to the continuous maps of the interval we define the *topological entropy* of a Lorenz map f, $h(f)$, as the logarithm of its growth number. So, $h(f) \in [0, \log 2]$.

PROPOSITION 6. *Let f and g be Lorenz maps.*

(1) $h(f) = h(g)$ *if and only if* $K(f) = K(g)$.

(2) $h(f) < h(g)$ *if and only if* $K(f) < K(g)$.

Let K be the set of all kneading sequences. Since $<$ is a complete ordering on K, we consider the ordering topology in K. If $\underline{A}$ is kneading, then $\underline{A} = K(f)$ for some Lorenz map f. We define the *topological entropy* of $\underline{A}$, $h(\underline{A})$, as the topological entropy of f. From Theorem 4 and Proposition 5, $h(\underline{A})$ is well defined.

COROLLARY 7. *The topological entropy map $h : K \to (0, \log 2]$ is a homeomorphism.*

4. Kneading sequences

In this section a sequence $\underline{A}$ formed of symbols L, C, R will be called *admissible* if either $\underline{A}$ is an infinite sequence of L's and R's, or $\underline{A}$ is a finite (or empty) sequence of L's and R's, followed by C. We denote by $|\underline{A}|$ the cardinality of $\underline{A}$. In the set of all admissible sequences we can define the ordering $\leqslant$ as above.

Let f be a Lorenz map and let $\underline{I}(x) = I_0 I_1 I_2 \dots$ be the itinerary of $x \in J$ by f. We define the *stopped itinerary* of $x \in I$ as

$$\underline{It}(x) = \underline{It}_f(x) = \begin{cases} \underline{I}(x) & \text{if } I_n \neq C \text{ for all } n \geqslant 0, \\ I_0 I_1 \dots I_n & \text{if } I_n = C \text{ and } I_j \neq C \text{ for } j = 0, 1, \dots, n-1. \end{cases}$$

Note that every stopped itinerary is an admissible sequence.

An admissible sequence $\underline{A}$ is called *maximal* if $S^k(\underline{A}) \leqslant \underline{A}$ and $S^k(\underline{A}') \leqslant \underline{A}$ for all $k < |\underline{A}|$.

In what follows the *kneading sequence* $K(f)$ of a Lorenz map f will be the stopped itinerary $\underline{It}_f(1)$. Note that since $K(f)$ is the largest stopped itinerary that occurs for f, it is in particular larger than $\underline{It}(f^n(1))$ and $\underline{It}(f^n(-1))$ for all integer $n \geqslant 0$. Thus $S^n(K(f)) \leqslant K(f)$ and $S^n(K(f)') \leqslant K(f)$ for all integer $n \geqslant 0$ for which S^n is defined. Therefore, every kneading sequence is maximal.

To study the properties of the maximal sequences we shall use the $*$-*product* between sequences.

Let $\underline{A}$ be a finite nonempty sequence of L's and R's, and let $\underline{B}$ be a sequence of L's, C's and R's finite or not. We define $\underline{A} * \underline{B}$ as follows:

(P1) If $\underline{B}$ is infinite, then

$$\underline{A} * \underline{B} = \underline{A} B_0 \underline{A}_0 B_1 \underline{A}_1 B_2 \underline{A}_2 \dots$$

(P2) If $\underline{B} = B_0 B_1 \dots B_n$, then

$$\underline{A} * \underline{B} = \underline{A} B_0 \underline{A}_0 B_1 \underline{A}_1 \dots B_{n-1} \underline{A}_{n-1} B_n,$$

where $\underline{A}_i = \underline{A}$ if $B_i \in \{L, C\}$, or $\underline{A}_i = \underline{A}'$ if $B_i = R$. We define $\emptyset * \underline{B} = \underline{B}$.

We shall denote

$$R^{*0} * \underline{A} = \underline{A}, \quad R^{*1} * \underline{A} = R * A \quad \text{and} \quad R^{*k} * \underline{A} = R * (R^{*(k-1)} * \underline{A}) \quad \text{for } k > 1.$$

Let M be the set of all maximal sequences plus the sequence L^∞. We shall say that a maximal sequence $\underline{A}$ is *irreducible* if there exists a positive integer m such that $\underline{A} > R^{*m} * R^\infty$, and $\underline{A} \neq \underline{B} * \underline{D}$ for all maximal sequence $\underline{B}C > R^{*n} * R^\infty$ for some integer $n > 0$, and for all $\underline{D} \in M \setminus \{C\}$. If $\underline{A}$ is maximal and it is not irreducible then we shall say that $\underline{A}$ is *reducible*. Note that the notion of reducible sequence is equivalent to Derrida–Gervois–

Pomeau's characterization of "forgotten sequences" for piecewise expanding unimodal maps (see [DGP]).

The next theorem characterizes the set of kneading sequences for the Lorenz maps.

THEOREM 8. *The following conditions are equivalent*

(1) $\underline{A}$ *is kneading.*

(2) $\underline{A} = K(f_s)$ *for some* $s \in (1, 2]$.

(3) $\underline{A}$ *is irreducible.*

To clarify the notions of reducible and irreducible sequences we shall use the idea of a box in the set of all maximal sequences.

Let $\underline{A}C$ be a maximal sequence with $|\underline{A}| \geqslant 2$. We define the *box* of $\underline{A}C$ as

$$\langle \underline{A} \rangle = \{\underline{A} * \underline{B}\colon \ \underline{B} \in M\}.$$

The next proposition shows the "geometry" of a box.

PROPOSITION 9. *Let* $\underline{A}C$ *be a maximal sequence with* $|\underline{A}| \geqslant 2$. *Then* $\langle \underline{A} \rangle$ *is the set of all maximal sequences* $\underline{B}$ *such that*

$$\underline{A} * L^\infty = (\underline{A}L)^\infty \leqslant \underline{B} \leqslant \underline{A} * R^\infty = \underline{A}(R\underline{A}')^\infty.$$

Let N be the reunion of the sets $\langle \underline{A} \rangle \setminus \{\underline{A}C\}$ for all the irreducible sequences $\underline{A}C$.

PROPOSITION 10. *The set of all reducible sequences is the union of* N *with the set of all maximal sequences* $\underline{B}$ *such that* $\underline{B} \leqslant R^{*m} * R^\infty$ *for all* $m \geqslant 0$.

Note that Proposition 10 gives a new characterization of the irreducible sequences.

5. Periods

Let S_n be the set of all integers m such that $m = n$ or m stands to the right of n in the following ordering

$$1, 3, 5, 7, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \ldots, 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \ldots, \ldots, 2^3, 2^2, 2.$$

Note that this ordering is essentially Sarkovskii's one. The difference is that in Sarkovskii's ordering 1 is to the right instead of to the left. The following proposition gives the structure of the set of periods $P(f)$ for a Lorenz map f. In its proof it plays an important role the characterization of the kneading sequences.

PROPOSITION 11. (1) *If* f *is a Lorenz map with* $s(f) > 1$, *then there exist integers* $i \geqslant 0$ *and* $k \geqslant 1$ *odd, such that* $P(f) = S_n$ *with* $n = 2^i k$.

(2) *If we have integers $i \geqslant 0$ and $k \geqslant 1$ odd, then there is a Lorenz map f with $s(f) > 1$, such that $P(f) = S_n$ with $n = 2^i k$.*

(3) *Let f a Lorenz map. Then $f(1) = 1$ if and only if $P(f)$ is the set of all positive integers.*

(4) *Let f be a Lorenz map with $s(f) > 1$. Then $\{2, 2^2, 2^3, \ldots\} \subset P(f)$.*

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