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ON THE EXTENSIONS OF UNIFORMLY CONTINUOUS MAPPINGS

BY

NGUYEN TO NHU (WARSZAWA)

In this note we consider the problem of extensions of uniformly continuous mappings in uniform spaces and in metric spaces. This problem has been investigated by several authors (see [3], [4], and [1]).

(E) Let A, X and Y be metric spaces (respectively, uniform spaces) such that A is a closed subset of X and let $f: A \to Y$ be a uniformly continuous mapping. Under what conditions can f be extended to a uniformly continuous mapping F from the whole space X into Y?

In Section 1 we consider problem (E) in uniform spaces. Using some corollaries to Katětov's theorem we prove that if Y is an injective locally convex space, then every bounded uniformly continuous mapping from a closed subset A of a uniform space X into Y can be extended to the whole space X. Since R^1 is injective, this generalizes Katětov's theorem.

In Section 2 we consider problem (E) in metric spaces. It is shown that a metric space $Y \in \mathrm{AEU}(\mathfrak{M})$ if and only if $Y \in \mathrm{ARU}(\mathfrak{M})$ and $\mathrm{diam}(Y) < \infty$. We note that, in the sense of Isbell [3] and [4], AEU-spaces are the same as ARU-spaces.

In [2] Borsuk proved that if X is the union of two closed subsets X_1 and X_2 such that X_1, X_2 and $X_1 \cap X_2$ are $\mathrm{AR}(\mathfrak{M})$-spaces, then so is X. An example in Section 3 shows that the corresponding proposition for $\mathrm{ARU}(\mathfrak{M})$-spaces is generally false, but under some additional assumptions the proposition holds true.

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1. Some corollaries to Katětov's theorem. First we recall the following theorem of Katětov [5]:

1.1. THEOREM. *Let R^1 denote the real line and let A be a subset of a uniform space X. Then every bounded uniformly continuous mapping f from*

A into R^1 admits a bounded uniformly continuous extension F from the whole space X into R^1. Moreover, the extension F of f satisfies the condition

$$\sup_{x\in X}\{|F(x)|\} = \sup_{x\in A}\{|f(x)|\}.$$

Now we prove some immediate consequences from Theorem 1.1 which will be used in the sequel.

Let D be any set. By $m(D)$ we denote the space of all bounded real functions on D with the supremum norm. The following corollary is an immediate consequence of Katětov's theorem:

1.2. COROLLARY. *For every uniform space X and for every bounded uniformly continuous mapping f from a subset A of X into $m(D)$, there exists a uniformly continuous mapping $F\colon X \to m(D)$ such that $F|A = f$ and*

$$\sup_{x\in X}\{\|F(x)\|\} = \sup_{x\in A}\{\|f(x)\|\}.$$

1.3. COROLLARY (Isbell [3]). *Let A be a subset of a uniform space X and let ϱ be a bounded uniformly continuous pseudometric on A. Then there exists a bounded uniformly continuous pseudometric ϱ^* on X such that $\varrho^*(x, y) = \varrho(x, y)$ for every $x, y \in A$ and*

$$\sup_{x,y\in X}\{\varrho^*(x, y)\} = \sup_{x,y\in A}\{\varrho(x, y)\}.$$

Proof. Let $g\colon A \to m(A)$ be a mapping defined by $(g(x))y = \varrho(x, y)$ for $x, y \in A$. Then g is bounded uniformly continuous. Thus, by Corollary 1.2, there exists a uniformly continuous mapping $G\colon X \to m(A)$ such that $G|A = g$ and

$$\sup_{x\in X}\{\|G(x)\|\} = \sup_{x\in A}\{\|g(x)\|\}.$$

Setting $\varrho^*(x, y) = \|G(x)-G(y)\|$, we get a uniformly continuous pseudometric on X having the required properties. This completes the proof.

1.4. Remark. *Suppose that X is a metric space with a metric d and let ϱ be a bounded uniformly continuous pseudometric on a closed subset A of X. Then there is a bounded uniformly continuous pseudometric $\bar{\varrho}$ on X extending ϱ such that A is also closed with respect to the pseudometric $\bar{\varrho}$.*

Indeed, putting

$$\bar{\varrho}(x, y) = \max\{\varrho^*(x, y), |d(x, A)-d(y, A)|\},$$

we get the uniformly continuous pseudometric $\bar{\varrho}$ having the required properties.

1.5. Definition. A locally convex space Y is said to be *injective* if whenever A and X are locally convex spaces such that A is a closed subspace of X and $T\colon A \to Y$ is a continuous linear operator, then there exists a continuous linear operator $T'\colon X \to Y$ which extends T.

By the Hahn-Banach theorem, R^1 is injective. The following theorem is a generalization of Katětov's theorem.

1.6. THEOREM. *Let A be a closed subset of a uniform space X and let Y be an injective locally convex space. Then every bounded uniformly continuous mapping f from A into Y can be extended to a uniformly continuous mapping $\tilde{F}$ from the whole space X into Y.*

Proof. Let $\{d_a\}_{a\in I}$ be a family of pseudometrics inducing the uniformity of X and let $\{p_\gamma\}_{\gamma\in J}$ be a family of pseudonorms inducing the topology of Y.

For every $\gamma \in J$, put

$$U_\gamma = \{y \in Y\colon\ p_\gamma(y) \leqslant 1\}$$

and let U_γ^0 be the polar of U_γ. Define a mapping $H_\gamma\colon Y \to m(U_\gamma^0)$ by $(H_\gamma y)\varphi = \varphi(y)$ for every $y \in Y$ and $\varphi \in U_\gamma^0$. It can easily be seen that H_γ is a continuous linear operator from Y into $m(U_\gamma^0)$ such that

$$\|H_\gamma(y)\| = p_\gamma(y) \tag{1}$$

for every $y \in Y$ and $\gamma \in J$.

Let $f\colon A \to Y$ be a bounded uniformly continuous mapping from a closed subset A of a uniform space X into Y. Then for every $\gamma \in J$ the mapping $g_\gamma = H_\gamma \circ f$ is also a bounded uniformly continuous mapping from A into $m(U_\gamma^0)$. Thus, by Corollary 1.2, for every $\gamma \in J$ there exists a bounded uniformly continuous mapping $G_\gamma\colon X \to m(U_\gamma^0)$ such that $G_\gamma | A = g_\gamma$ and

$$\sup_{x\in X}\{\|G_\gamma(x)\|\} = \sup_{x\in A}\{\|g_\gamma(x)\|\}.$$

Let $L(A)$ and $L(X)$ denote the linear spaces spanned formally by elements of A and X, respectively. Then $L(A) \subset L(X)$.

Given

$$x = \sum_{i=1}^{n} \lambda_i x_i \in L(X), \qquad x_i \in X \text{ and } \lambda_i \in R^1 \text{ for } 1 \leqslant i \leqslant n,$$

put

$$q_\gamma(x) = \max\Big\{\Big\|\sum_{i=1}^{n} \lambda_i G_\gamma(x_i)\Big\|,\ \sup_{\varphi\in\Phi}\Big|\sum_{i=1}^{n} \lambda_i \varphi(x_i)\Big|\Big\}, \tag{2}$$

where

$$\Phi = \{\varphi \in C(X)\colon\ \varphi | A = 0,\ |\varphi(x)-\varphi(y)| \leqslant d_a(x, y) \text{ for some } a \in I \text{ and for every } x, y \in X\}. \tag{3}$$

It is easy to see that, for every $\gamma \in J$, q_γ is a pseudonorm on $L(X)$. Putting

$$M = \bigcap_{\gamma \in J} q_\gamma^{-1}(0),$$

we easily see that $M \subset L(A)$.

Finally, let $E = L(X)/M$ and $F = L(A)/M$. Then E and F are locally convex spaces, and F is a subspace of E. Let us show that F is closed in E.

In fact, let $x \notin F$. Then

$$x = \Big[\sum_{i=1}^{n} \lambda_i x_i\Big],$$

where $x_i \in X$ for $1 \leqslant i \leqslant n$, and $\lambda_1, \ldots, \lambda_n$ are different from zero. Since $x \notin F$, we can assume that $x_1 \notin A$. Let us set $B = A \cup \{x_2, \ldots, x_n\}$. Then B is closed in X. Thus there exists an $a_0 \in I$ such that

$$d_{a_0}(x_1, B) = \inf_{y \in B} \{d_{a_0}(x_1, y)\} > 0.$$

Putting $\varphi_0(x) = d_{a_0}(x, B)$ for every $x \in X$, we easily see that $\varphi_0 \in \Phi$ and $\varphi_0(x_1) > 0$.

For every y,

$$y = \Big[\sum_{i=1}^{k} \mu_i y_i\Big] \in F, \qquad y_1, \ldots, y_k \in A,$$

we get, by (2),

$$q_\gamma(x-y) \geqslant \Big|\sum_{i=1}^{n} \lambda_i \varphi_0(x_i) - \sum_{i=1}^{k} \mu_i \varphi_0(y_i)\Big| = |\lambda_1| \varphi_0(x_1) > 0,$$

which shows that $x \notin \bar{F}$. This proves that F is closed in E.

Now we define a linear operator $T: F \to Y$ by

$$T\Big(\Big[\sum_{i=1}^{n} \lambda_i x_i\Big]\Big) = \sum_{i=1}^{n} \lambda_i f(x_i).$$

Then by (1) and (2) we have

$$p_\gamma(T(z)) = p_\gamma\Big(\sum_{i=1}^{n} \lambda_i f(x_i)\Big) = \Big\|H_\gamma\Big(\sum_{i=1}^{n} \lambda_i f(x_i)\Big)\Big\|$$
$$= \Big\|\sum_{i=1}^{n} \lambda_i g_\gamma(x_i)\Big\| \leqslant q_\gamma\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) = q_\gamma(z)$$
$$\text{for every } z = \sum_{i=1}^{n} \lambda_i x_i \in F.$$

Thus T is continuous.

Since Y is injective, there exists a continuous linear operator T': $E \to Y$ such that $T'|F = T$.

Setting $\tilde{F}(x) = T'([x])$ for every $x \in X$, we get a uniformly continuous extension $\tilde{F}$ of f. Thus the theorem is proved.

2. Spaces $\mathrm{AEU}(\mathfrak{M})$ **and** $\mathrm{ARU}(\mathfrak{M})$. The notions of AEU and ARU uniform spaces were introduced and investigated by Isbell [3] and [4]. In this section we consider metric spaces only, hence we use the following definitions which differ slightly from those of Isbell [3] and [4].

2.1. Definition. A metric space Y is called an $\mathrm{AEU}(\mathfrak{M})$ if, whenever X is a metric space and A is a closed subset of X, any uniformly continuous mapping from A into Y can be extended to a uniformly continuous mapping from the whole space X into Y.

2.2. Definition. A metric space Y is said to be an $\mathrm{ARU}(\mathfrak{M})$ if, whenever Y is a closed subset of a metric space X, then there exists a uniformly continuous retraction R from X onto Y.

By $\tilde{Y}$ we denote the completion of a given metric space Y. We have the following

2.3. PROPOSITION. *If Y is an* $\mathrm{AEU}(\mathfrak{M})$ *(respectively, an* $\mathrm{ARU}(\mathfrak{M})$*) then so is* $\tilde{Y}$.

Proof. Let Y be an $\mathrm{ARU}(\mathfrak{M})$ and let Z be a metric space containing $\tilde{Y}$. Let us put

$$\mathscr{A} = \{A\colon\ Y \subset A \subset Z \text{ such that } Y \text{ is closed in } A\}.$$

For every $A_1, A_2 \in \mathscr{A}$ set $A_1 \leqslant A_2$ if and only if $A_1 \subset A_2$. Then $\mathscr{A}$ becomes a partially ordered set satisfying the conditions of the Kuratowski-Zorn lemma. Let X be a maximal element of $\mathscr{A}$. It is easy to see that X is dense in Z. Since Y is an $\mathrm{ARU}(\mathfrak{M})$, there is a uniformly continuous retraction R from X onto Y. Since $\tilde{Y}$ is complete, the retraction R can uniquely be extended to a uniformly continuous retraction $\tilde{R}$ from Z onto $\tilde{Y}$. So $\tilde{Y}$ is an $\mathrm{ARU}(\mathfrak{M})$.

The same argument shows that $\tilde{Y}$ is an $\mathrm{AEU}(\mathfrak{M})$ whenever $Y \in \mathrm{AEU}(\mathfrak{M})$. This completes the proof.

Clearly, if Y is an $\mathrm{AEU}(\mathfrak{M})$, then Y is an $\mathrm{ARU}(\mathfrak{M})$.

2.4. THEOREM. *A metric space Y is an* $\mathrm{AEU}(\mathfrak{M})$ *if and only if Y is an* $\mathrm{ARU}(\mathfrak{M})$ *and* $\mathrm{diam}(Y) < \infty$.

Proof. Let Y be an $\mathrm{AEU}(\mathfrak{M})$. Then Y is an $\mathrm{ARU}(\mathfrak{M})$ and we have to show that $\mathrm{diam}(Y) < \infty$.

Assume, on the contrary, that $\mathrm{diam}(Y) = \infty$. Let $\{x_n\}$ be a sequence of points in Y such that

$$d(x_n, x_{n+1}) \geqslant n. \tag{4}$$

Let $X = R^1$ (R^1 being the real line) and let $A = N$ (N being the set of all natural numbers). Define a mapping $f\colon N \to Y$ by $f(n) = x_n$. Then f is uniformly continuous on N and N is closed in R^1. Let $F\colon R^1 \to Y$ be a uniformly continuous extension of f. By a lemma of Lindenstrauss [7] there exists an $L > 0$ such that

$$d\big(f(x), f(y)\big) \leqslant L|x-y|$$

for every $x, y \in R^1$ with $|x-y| \geqslant 1$. Then we get

$$d\big(f(n+1), f(n)\big) \leqslant L$$

for every $n \in N$, a contradiction with (4). This shows that $\operatorname{diam}(Y) < \infty$.

Conversely, assume that Y is an $\mathrm{ARU}(\mathfrak{M})$ and $\operatorname{diam}(Y) < \infty$. Let f be a uniformly continuous mapping from a closed subset A of a metric space X into Y. First we consider a special case where f is an isometric embedding.

Let $Q = X \cup Y$ and let $Z = Q/\!\sim$, where $\sim$ is the equivalence relation on Q defined by $x \sim y$ if and only if $y = f(x)$ or $y = x$. Setting

$$\varrho(x, y) = \begin{cases} d_X(x, y) & \text{if } x, y \in X, \\ d_Y(x, y) & \text{if } x, y \in Y, \\ \inf\limits_{t \in A} \{d_X(x, t) + d_Y(y, f(t))\} & \text{if } x \in X \text{ and } y \in Y, \end{cases}$$

where d_X and d_Y denote metrics on X and Y, respectively, we easily see that ϱ is a metric on Z, and Y is closed in Z. Let R be a uniformly continuous retraction from Z onto Y. Then $F = R \circ i$, where $i\colon X \to Z$ is the natural inclusion, is a uniformly continuous extension of f.

Now, let $f\colon A \to Y$ be an arbitrary uniformly continuous mapping. Putting

$$h(x, y) = d_Y\big(f(x), f(y)\big),$$

we get a bounded uniformly continuous pseudometric on A. By Remark 1.4, there exists a uniformly continuous pseudometric $\tilde{h}$ on X such that $\tilde{h}\,|\,A \times A = h$ and A is closed with respect to the pseudometric $\tilde{h}$ of X. Let $E = X/\tilde{h}$ and $B = A/\tilde{h} \subset E$. Then E is a metric space with the metric $\bar{h}$ induced by $\tilde{h}$, and B is a closed subset in E. It is easy to see that the mapping $g\colon B \to Y$ induced by f is an isometric embedding. Thus, using the proof above, we get a uniformly continuous mapping G from E into Y such that $G|B = g$. Setting $F = G \circ k$, where $k\colon X \to E$ is the quotient mapping, we easily see that F is uniformly continuous with respect to the metric d_X and $F(x) = f(x)$ for every $x \in A$. Thus the theorem is proved.

2.5. Remark. Let (X, d) be a metric space. For every $\varepsilon > 0, x \in X$ and $n \in N$, put

$$B_1(x, \varepsilon) = \{y \in X: d(x, y) \leqslant \varepsilon\}$$

and define $B_n(x, \varepsilon)$ by induction:

$$B_n(x, \varepsilon) = \{y \in X: \ d(y, z) \leqslant \varepsilon \text{ for some } z \in B_{n-1}(x, \varepsilon)\}.$$

A metric space (X, d) is said to be *uniformly bounded* if there is a point $x_0 \in X$ such that for every $\varepsilon > 0$ there exists an $n \in N$ such that $B_n(x_0, \varepsilon) = X$.

A similar argument as in the proof of Theorem 2.4 shows that every AEU$(\mathfrak{M})$-space is uniformly bounded.

2.6. Remark. It is known (see, e.g., [1]) that every Lipschitz mapping f from a subset A of a metric space X into R^1 can be extended to a Lipschitz mapping F from X into R^1. In particular, we infer that R^1 is an ARU$(\mathfrak{M})$; however, it is not an AEU$(\mathfrak{M})$.

2.7. Remark. Let us put $\varrho(x, y) = \min\{1, |x-y|\}$ for every $x, y \in R^1$. Then from Theorem 2.4 we infer that (R^1, ϱ) is not an ARU$(\mathfrak{M})$; however, (R^1, ϱ) and $(R^1, |\cdot|)$ are uniformly equivalent.

2.8. Remark. Isbell [4] showed that, for $-\infty < a < b < \infty$, (a, b) is an ARU$(\mathfrak{M})$. Thus from Theorem 2.4 we see that (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are AEU$(\mathfrak{M})$-spaces.

3. The union of two AEU$(\mathfrak{M})$-spaces.

3.1. Theorem. *Let (X, ϱ) be a metric space and let X_0, X_1, X_2 be closed subsets of X such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2 \neq \emptyset$. Assume that $X_1, X_0, X_2 \in$ AEU$(\mathfrak{M})$. Then $(X, \varrho) \in$ AEU$(\mathfrak{M})$ if and only if the metric d on X defined by*

$$d(x, y) = \begin{cases} \varrho(x, y) & \text{if } (x, y) \in X_i \times X_i \text{ for } i = 1, 2, \\ \inf\limits_{t \in X_0} \{\varrho(x, t) + \varrho(y, t)\} & \text{otherwise} \end{cases}$$

is uniformly equivalent to ϱ.

Proof. First we assume that $X_0, X_1, X_2 \in$ AEU$(\mathfrak{M})$. Then, by Theorem 2.4, $\operatorname{diam} \varrho(X) < \infty$. By the definition of d, $\operatorname{diam}_d(X) < \infty$. In order to prove that $(X, \varrho) \in$ AEU$(\mathfrak{M})$ it now suffices to show that (X, d) is an ARU$(\mathfrak{M})$.

Let Z be a metric space containing (X, d) isometrically as a closed subset. By a theorem of Kuratowski and Wojdysławski (see, e.g., [2],

[6], [8]) we may assume without loss of generality that Z is a convex set lying in a normed space. Let us put

$$Z_0 = \{z \in Z:\ d(z, X_1) = d(z, X_2)\},$$
$$Z_1 = \{z \in Z:\ d(z, X_1) < d(z, X_2)\},$$
$$Z_2 = \{z \in Z:\ d(z, X_1) > d(z, X_2)\}.$$

Clearly, $Z = Z_0 \cup Z_1 \cup Z_2$ and $X_i \cap Z_0 = X_0$ for $i = 1, 2$. Since X_0 is closed in Z_0, there exists a uniformly continuous retraction R_0 from Z_0 onto X_0. Let

$$R_i:\ X_i \cup Z_0 \to X_i \cup X_0 = X_i$$

be a mapping defined by

$$R_i(z) = \begin{cases} z & \text{for } z \in X_i, \\ R_0(z) & \text{for } z \in Z_0. \end{cases}$$

Observe that R_i is uniformly continuous for $i = 1, 2$.

Indeed, to show this it is enough to prove that if $x \in X_1$ and $y \in Z_0$ are sufficiently close, then $R_1(x) = x$ is closed to $R_1(y) = R_0(y)$.

Given any $\varepsilon > 0$, let $\delta \in (0, \varepsilon/6)$ be such that if $x, y \in Z_0$ and $d(x, y) < 4\delta$, then $d(R_0(x), R_0(y)) < \varepsilon/2$. Let $x \in X_1$ and $y \in Z_0$ be such that $d(x, y) < \delta$. We shall show that $d(R_1(x), R_1(y)) < \varepsilon$. By the definition of Z_i there exists a $z \in X_2$ such that $d(z, y) < \delta$. By the definition of d there is a $t \in X_0$ such that

$$d(x, t) + d(z, t) \leqslant d(x, z) + \delta \leqslant d(x, y) + d(y, z) + \delta < 3\delta.$$

Thus we have

$$d(x, t) < 3\delta \quad \text{and} \quad d(z, t) < 3\delta.$$

Therefore

$$d(t, y) \leqslant d(t, x) + d(x, y) < 3\delta + \delta = 4\delta.$$

Hence

$$d(R_0(t), R_0(y)) < \frac{\varepsilon}{2}\,.$$

Consequently,

$$d(R_1(x), R_1(y)) = d(x, R_0(y)) \leqslant d(x, t) + d(t, R_0(y))$$
$$= d(x, t) + d(R_0(t), R_0(y)) < 3\frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon.$$

The uniform continuity of R_i is established.

Since $X_i \in \mathrm{AEU}(\mathfrak{M})$ and $X_i \cup Z_0$ is closed in $Z_i \cup Z_0$, we infer that the mapping R_i can be extended to a uniformly continuous mapping f_i from $Z_i \cup Z_0$ into X_i.

Now define a retraction R from Z onto X by

$$R(z) = f_i(z) \quad \text{if } z \in Z_i \cup Z_0 \text{ for } i = 1, 2.$$

Since f_i is uniformly continuous for every $i = 1, 2$, to show that R is uniformly continuous it is enough to prove that if $x \in Z_1 \cup Z_0$ and $y \in Z_2 \cup Z_0$ are sufficiently close, then $R(x)$ is closed to $R(y)$.

Indeed, let $\varepsilon > 0$ be given. Since the function f_i is uniformly continuous, there exists a $\delta > 0$ such that if $x, y \in Z_i \cup Z_0$ and $d(x, y) < \delta$, then $d(f_i(x), f_i(y)) < \varepsilon/2$ for every $i = 1, 2$.

Let $x \in Z_1 \cup Z_0$ and $y \in Z_2 \cup Z_0$ be such that $d(x, y) < \delta$. It follows from the definition of Z_i that there is an $a \in [0, 1]$ such that

$$z = ax + (1-a)y \in Z_0.$$

Since $d(x, z) \leqslant d(x, y) < \delta$ and $d(y, z) \leqslant d(x, y) < \delta$, we infer that

$$d\big(R(x), R(y)\big) \leqslant d\big(R(x), R(z)\big) + d\big(R(z), R(y)\big)$$
$$= d\big(f_1(x), f_1(z)\big) + d\big(f_2(z), f_2(y)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus R is a uniformly continuous retraction.

Conversely, assume that $(X, \varrho) \in \mathrm{AEU}(\mathfrak{M})$. Let us prove that ϱ and d are uniformly equivalent.

Again, we can assume that (X, ϱ) is a closed subset of a convex set Z lying in a normed space. Let $R\colon Z \to X$ be a uniformly continuous retraction. To show that d and ϱ are uniformly equivalent it suffices to prove that if $\{x_n\} \subset X_1$ and $\{y_n\} \subset X_2$ are such that $\varrho(x_n, y_n) \to 0$, then $d(x_n, y_n) \to 0$.

In fact, let

$$[x_n, y_n] = \{z \in Z\colon z = tx_n + (1-t)y_n,\ 0 \leqslant t \leqslant 1\}.$$

We easily see that, for every $n \in N$, there exists a $z_n \in [x_n, y_n]$ such that $R(z_n) \in Z_0$. Since

$$\varrho\big(x_n, R(z_n)\big) \leqslant \operatorname{diam}(R[x_n, y_n]) \quad \text{and} \quad \varrho\big(y_n, R(z_n)\big) \leqslant \operatorname{diam}(R[x_n, y_n]),$$

we infer that

$$d(x_n, y_n) \leqslant \varrho\big(x_n, R(z_n)\big) + \varrho\big(R(z_n), y_n\big) \leqslant 2\operatorname{diam}(R[x_n, y_n]) \to 0.$$

Thus the theorem is proved.

3.2. Example. Let ABC be a triangle in the plane R^2 and let

$$X = [AB) \cup (BC] \cup [CA], \quad X_1 = [AC] \cup [CB), \quad X_2 = [CA] \cup [AB).$$

Then X_1, X_2 and $X_1 \cap X_2 \in \mathrm{AEU}(\mathfrak{M})$ (see Remark 2.8). Moreover, X_1 and X_2 are closed in X, but the completion of X is not an $\mathrm{AR}(\mathfrak{M})$ (in the sense of Borsuk [2]). Therefore, by Proposition 2.3, X is not an $\mathrm{ARU}(\mathfrak{M})$.

3.3. COROLLARY. *In the notation of Theorem* 3.1, *if at least one of the subsets* X_1, X_2 *is compact, then* $(X, \varrho) \in \mathrm{AEU}(\mathfrak{M})$.

Indeed, it is easy to see that if one of the subsets X_1, X_2 is compact, then the metric d is uniformly equivalent to ϱ.

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