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ON SOME C-TOTALLY REAL SUBMANIFOLDS IN A SPACE WITH SASAKIAN 3-STRUCTURE

BY

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1. Introduction. Let $\bar{M}(\langle\ ,\ \rangle, \bar{\nabla})$ be a Riemannian manifold. We assume that $\bar{M}$ has a Sasakian 3-structure $(\xi, \varphi; \eta, \psi; \zeta, \theta)$. By M we denote a submanifold of $\bar{M}$. If ξ is tangent to M, and φX is tangent to M for any tangent vector X of M, then (ξ, φ) is called *invariant* on M. If ξ is normal to M, then (ξ, φ) is called *C-totally real* on M. We shall give the same definitions for (η, ψ) and (ζ, θ). If a submanifold M is C-totally real with respect to (ξ, φ) and (η, ψ) and invariant with respect to (ζ, θ), then we can say without loss of generality that M is *2-C-totally real* and *1-invariant*. If M is C-totally real with respect to (ξ, φ), (η, ψ), and (ζ, θ), then M is called *3-C-totally real.*

In the previous paper [2] we studied fundamental properties of submanifolds in a space with Sasakian 3-structure. Especially, we considered an integral formula of 2-C-totally real and 1-invariant submanifolds. In this paper, we shall study the integral formula of 3-C-totally real submanifolds. The notation used here is the same as that of [2].

2. Preliminaries. Let $\bar{M}(\langle\ ,\ \rangle, \bar{\nabla})$ be a Riemannian manifold and M a submanifold isometrically immersed in $\bar{M}$. If we denote the covariant differentiation of M by ∇, then the second fundamental forms B and A are given by

$$\bar{\nabla}_X Y - \nabla_X Y = B(X, Y), \qquad \langle A^N(X), Y\rangle = \langle B(X, Y), N\rangle,$$

where X and Y are tangent vectors of M, and N is a vector normal to M.

If the second fundamental form is identically zero, then M is said to be *totally geodesic*. The *mean curvature vector* μ is defined as $\mu = (\operatorname{Tr} B)/n$, where $\operatorname{Tr} B$ is the trace of B and $n = \dim M$. If $\mu = 0$, then M is said to be *minimal.* If the second fundamental form B is of the form $B(X, Y) = \langle X, Y\rangle \mu$, then M is said to be *totally umbilical.* If $[A^{N_1}, A^{N_2}] = 0$ for any normal vectors N_1 and N_2, then the second fundamental form of M is said to be *commutative.*

The covariant derivatives of the second fundamental form are given by

$$(2.1)\quad \begin{aligned} (\tilde{\nabla}_X B)(Y, Z) &= D_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \\ \langle (\tilde{\nabla}_X B)(Y, Z), N \rangle &= \langle (\nabla_X A)^N(Y), Z \rangle, \end{aligned}$$

where D is the linear connection in the normal bundle.

3. 3-C-totally real submanifolds. First, using calculations similar to those in [2], we can prove the following propositions:

PROPOSITION 1. *Let M be a 3-C-totally real submanifold in a Riemannian manifold with Sasakian 3-structure. Then for any tangent vector X of M the vectors φX, ψX, and θX are normal to M.*

PROPOSITION 2. *Let M be an m-dimensional 3-C-totally real submanifold in a Riemannian manifold $\bar{M}$ ($\dim \bar{M} = 4m+3$) with Sasakian 3-structure. Then*

(i) $\varphi X = D_X \xi$, $\psi X = D_X \eta$, $\theta X = D_X \zeta$;

(ii) $A^{\varphi X}(Y) = A^{\varphi Y}(X)$, $A^{\psi X}(Y) = A^{\psi Y}(X)$, $A^{\theta X}(Y) = A^{\theta Y}(X)$;

(iii) $A^{\xi} = 0$, $A^{\eta} = 0$, $A^{\zeta} = 0$.

THEOREM 1. *Let M be an m-dimensional 3-C-totally real submanifold in a Riemannian manifold $\bar{M}$ ($\dim \bar{M} = 4m+3$) with Sasakian 3-structure. If the second fundamental form A satisfies $(\nabla A)^{\xi} = 0$, $(\nabla A)^{\eta} = 0$, and $(\nabla A)^{\zeta} = 0$, then M is totally geodesic.*

Proof. For any tangent vectors X, Y, and Z of M, we have

$$(3.1)\quad \begin{aligned} \langle (\tilde{\nabla}_Z B)(X, Y), \xi \rangle &= \langle D_Z B(X, Y), \xi \rangle \\ &= -\langle B(X, Y), D_Z \xi \rangle = -\langle B(X, Y), \varphi Z \rangle \end{aligned}$$

by (2.1) and Proposition 2. Similarly, we get the equalities

$$(3.2)\quad \langle (\tilde{\nabla}_Z B)(X, Y), \eta \rangle = -\langle B(X, Y), \psi Z \rangle,$$

$$(3.3)\quad \langle (\tilde{\nabla}_Z B)(X, Y), \zeta \rangle = -\langle B(X, Y), \theta Z \rangle$$

which, by (iii) of Proposition 2, complete the proof.

COROLLARY 1. *Let M be an m-dimensional 3-C-totally real submanifold in a Riemannian manifold $\bar{M}$ ($\dim \bar{M} = 4m+3$) with Sasakian 3-structure. If the second fundamental form A of M is parallel, then M is totally geodesic.*

Let N_a be any unit normal vector of M. Then we write A^a instead of A^{N_a} to simplify the notation. We choose a local field of the orthonormal frame $\{e_1, \ldots, e_m\}$ in $T_p(M)$ $(p \in M)$. Then we can see that the normal space $T_p^{\perp}(M)$ is spanned by

$$\{\varphi e_i, \psi e_i, \theta e_i, \xi, \eta, \zeta\} \quad (i = 1, \ldots, m).$$

Unless otherwise stated, we use the convention that the ranges of indices are the following:

$$i, j = 1, \dots, m, \quad a, b = 1, \dots, 3m,$$
$$I = 1, \dots, 3m,\ 3m+1,\ 3m+2,\ 3m+3.$$

THEOREM 2. *Let M be an m-dimensional minimal 3-C-totally real submanifold in a Riemannian manifold with Sasakian 3-structure. If the second fundamental form of M is commutative, then M is totally geodesic.*

Proof. Since the second fundamental form of M is commutative, choosing a suitable orthonormal frame $\{e_1, \dots, e_m\}$ in $T_p(M)$ we see that the whole second fundamental form can be simultaneously diagonal. Then, by Proposition 2, we have the following matrix representation:

$$A^{\varphi e_i} = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \lambda_i \ \cdots & \\ & & \vdots \ \ddots & \\ & & \vdots & 0 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ \\ \end{matrix}$$
$$\qquad\qquad i$$

Since M is minimal by assumption, $\lambda_i = 0$. Similarly we have $A^{\psi e_i} = 0$ and $A^{\theta e_i} = 0$. On the other hand, we see that $A^{\xi} = 0$, $A^{\eta} = 0$, and $A^{\zeta} = 0$. Thus M is totally geodesic.

COROLLARY 2. *Let M be an m-dimensional 3-C-totally real submanifold in a Riemannian manifold with Sasakian 3-structure and of dimension $4m+3$. If M is totally umbilical, then M is totally geodesic.*

Next we shall study the integral formula of 3-C-totally real submanifolds.

Let M be an m-dimensional 3-C-totally real minimal submanifold in a unit sphere $S^{4m+3}(1)$. Then we have the Simons' type formula

$$\nabla^2 A = mA - A \circ \tilde{A} - \underset{\sim}{A} \circ A, \tag{3.4}$$

where the operators $\tilde{A}$ and $\underset{\sim}{A}$ are defined as follows:

$$\tilde{A} = {}^t A \circ A \quad \text{and} \quad \underset{\sim}{A} - \sum_I (\operatorname{ad} A^I) \operatorname{ad} A^I.$$

We obtain

$$\underset{\sim}{A} - \sum_a (\operatorname{ad} A^a) \operatorname{ad} A^a$$

by Proposition 2.

LEMMA. *We have $\langle \nabla A, \nabla A \rangle - \|A\|^2 \geqslant 0$.*

Proof. From (3.1)-(3.3) it follows that

$$
\begin{aligned}
\langle \nabla A, \nabla A\rangle &= \sum_{i,j}\sum_{I} \langle (\nabla_{e_i}A)^I(e_j), (\nabla_{e_i}A)^I(e_j)\rangle \\
&= \sum_{i,j}\sum_{a} \langle (\nabla_{e_i} A)^a (e_j), (\nabla_{e_i} A)^a (e_j)\rangle + \\
&\quad + \sum_{i,j} [\langle (\nabla_{e_i} A)^{\xi} (e_j), (\nabla_{e_i} A)^{\xi} (e_j)\rangle + \langle (\nabla_{e_i} A)^{\eta} (e_j), (\nabla_{e_i} A)^{\eta} (e_j)\rangle + \\
&\quad + \langle (\nabla_{e_i} A)^{\zeta} (e_j), (\nabla_{e_i} A)^{\zeta} (e_j)\rangle] \\
&= \sum_{i,j}\sum_{a} \langle (\nabla_{e_i} A)^a (e_j), (\nabla_{e_i} A)^a (e_j)\rangle + \\
&\quad + \sum_{i,j} [\langle A^{\varphi e_i} (e_j), A^{\varphi e_i} (e_j)\rangle + \langle A^{\psi e_i} (e_j), A^{\psi e_i} (e_j)\rangle + \\
&\quad + \langle A^{\theta e_i} (e_j), A^{\theta e_i} (e_j)\rangle] \\
&= \sum_{i,j}\sum_{a} \langle (\nabla_{e_i} A)^a (e_j), (\nabla_{e_i} A)^a (e_j)\rangle + \|A\|^2,
\end{aligned}
$$

which completes the proof.

We put $S_{ab} = \mathrm{Tr}(A^a A^b)$ and $S_a = S_{aa}$. Since S_{ab} is a symmetric $(3m, 3m)$-matrix, we can assume it is diagonal for a suitable frame. We have $\sum_a S_a = \|A\|^2$. From (3.4) we get

$$
\begin{aligned}
\langle \nabla^2 A, A\rangle &= m\|A\|^2 - \langle A\circ \tilde{A} + \underset{\sim}{A}\circ A, A\rangle \\
&= m\|A\|^2 - \sum_a (S_a)^2 + \sum_{a\neq b} \mathrm{Tr}(A^a A^b - A^b A^a)^2.
\end{aligned}
$$

Hence, using the well-known inequality from [1], we have

$$
\begin{aligned}
(3.5)\qquad -\langle \nabla^2 A, A\rangle &= -m\|A\|^2 + \sum_a (S_a)^2 - \sum_{a\neq b} \mathrm{Tr}(A^a A^b - A^b A^a)^2 \\
&\leqslant -m\|A\|^2 + 2\sum_{a\neq b} S_a S_b + \sum_a (S_a)^2 \\
&= \left[\left(2-\frac{1}{3m}\right)\|A\|^2 - m\right]\|A\|^2 - \frac{1}{3m}\sum_{a>b} (S_a - S_b)^2.
\end{aligned}
$$

Using the Lemma and (3.5), we have the following theorems:

THEOREM 3. *Let M be an m-dimensional compact minimal 3-C-totally real submanifold in a unit sphere $S^{4m+3}(1)$. Then*

$$
\int_M \left[\left\{\left(2-\frac{1}{3m}\right)\|A\|^2 - (m+1)\right\}\|A\|^2 - \frac{1}{3m}\sum_{a>b}(S_a - S_b)^2\right] \geqslant 0.
$$

THEOREM 4. *Let M be an m-dimensional compact minimal 3-C-totally real submanifold in a unit sphere $S^{4m+3}(1)$. If the second fundamental form A satisfies the condition*

$$\|A\|^2 < 3m(m+1)/(6m-1),$$

then M is totally geodesic.

THEOREM 5. *Let M be an m-dimensional minimal 3-C-totally real submanifold in a unit sphere $S^{4m+3}(1)$. If the sectional curvature of M is constant, say C, then either $C = 1$ (in this case, M is totally geodesic) or $C \leqslant 0$.*

Proof. Since M is of constant sectional curvature C, we have

$$(3.6)\quad (1-C)(\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle) = \langle B(X, Z), B(Y, W)\rangle - \langle B(X, W), B(Y, Z)\rangle$$

by the Gauss equation. Putting $X = e_i$ and $W = A^N(e_i)$ in (3.6) and summing over i, we get

$$(3.7)\quad -(1-C)\langle A^N(Y), Z\rangle = \sum_a \langle A^a A^N A^a(Y), Z\rangle - \langle A^{\tilde{A}(N)}(Y), Z\rangle.$$

Hence

$$(3.8)\quad (C-1)\|A\|^2 = \sum_{a,b}\sum_i \langle A^b A^a(e_i), A^a A^b(e_i)\rangle - \langle A\circ\tilde{A}, A\rangle.$$

If we put $Y = Z = e_i$ in (3.6) and sum over i, we have

$$(3.9)\quad (1-C)(m-1)\langle X, W\rangle = \sum_a \langle A^a A^a(X), W\rangle.$$

Hence we obtain

$$(3.10)\quad (1-C)(m-1)\|A\|^2 = \sum_{a,b}\sum_i \langle A^a A^a A^b A^b(e_i), e_i\rangle.$$

On the other hand, from the definition of $\underset{\sim}{A}$ we get

$$(3.11)\quad \langle \underset{\sim}{A}\circ A, A\rangle = 2\sum_{a,b}\sum_i [\langle A^a A^a A^b A^b(e_i), e_i\rangle - \langle A^a A^b A^a A^b(e_i), e_i\rangle].$$

Putting (3.10) in (3.11), we obtain

$$(3.12)\quad \langle \underset{\sim}{A}\circ A, A\rangle = 2(1-C)(m-1)\|A\|^2 - 2\sum_{a,b}\sum_i \langle A^a A^b A^a A^b(e_i), e_i\rangle.$$

From (3.8) and (3.12) it follows that

$$(3.13)\quad \langle \underset{\sim}{A}\circ A + A\circ\tilde{A}, A\rangle = 2(1-C)m\|A\|^2 - \langle A\circ\tilde{A}, A\rangle.$$

By (3.9) we have

$$(3.14)\quad \langle A\circ\tilde{A}, A\rangle = (1-C)(m-1)\|A\|^2.$$

Putting (3.14) in (3.13), we get

$$\langle \underset{\sim}{A} \circ A + A \circ \tilde{A}, A \rangle = (1-C)(m+1)\|A\|^2. \tag{3.15}$$

Therefore, (3.6) and (3.15) imply

$$\langle \nabla A, \nabla A \rangle - \|A\|^2 = -Cm(m^2-1)(1-C),$$

which, by the Lemma, completes the proof.

Remark. Let i be the natural isomorphic immersion of the m-dimensional unit sphere $S^m(1)$ into $S^{4m+3}(1)$, that is, for any point $p \in S^m(1)$ the immersion i is given by $i(p) = (p, 0, 0, 0)$. Then i is a standard example of a 3-C-totally real submanifold.

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