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Local stability of the functional equation characterizing polynomial functions

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Abstract. In the paper, we give theorems characterizing functions with locally bounded differences of higher orders as well as some results concerning approximately multiadditive functions on restricted domains.

1. Introduction. In recent years, various aspects of conditional Cauchy functional equations have been studied (see, e.g., [3] and [8]). In particular, if $D \subset \boldsymbol{R}^2$ is an open disc (or more generally, an open and connected subset of the plane) and

$$D_1 := \{x \in \boldsymbol{R}\colon \text{ there exists a } y \in \boldsymbol{R} \text{ such that } (x, y) \in D\},$$

$$D_2 := \{y \in \boldsymbol{R}\colon \text{ there exists a } x \in \boldsymbol{R} \text{ such that } (x, y) \in D\},$$

$$D_3 := \{x+y \in \boldsymbol{R}\colon (x, y) \in D\},$$

then one can show that any function f mapping $D_1 \cup D_2 \cup D_3$ into an Abelian group X and satisfying the equation

$$f(x+y)-f(x)-f(y) = 0 \quad \text{for all } (x, y) \in D$$

differs from an additive function by some constants a, b and $a+b$ on D_1, D_2 and D_3, respectively (cf. [2]).

Furthermore, Skof ([10] and [11]) has pointed out that this result may be combined with the classical Hyers theorem on the stability of the Cauchy functional equation. Namely, if $f\colon D_1 \cup D_2 \cup D_3 \to X$ ($(X, \|\cdot\|)$ being now a Banach space) fulfils the inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$$

for all points (x, y) from a triangle

$$D := \{(x, y) \in \boldsymbol{R}^2\colon x, y \geqslant 0,\ x+y < a\}$$

or from an open disc

$$D := \{(x, y) \in \boldsymbol{R}^2\colon (x-x_0)^2+(y-y_0)^2 < r^2\},$$

then f is close to an additive function.

In what follows, the symbol Δ_y will stand for the difference operator defined by

$$\Delta_y f(x) := f(x+y) - f(x).$$

The superposition of the operators $\Delta_{y_1}, \ldots, \Delta_{y_n}$ will be denoted by $\Delta_{y_1 \ldots y_n}$. If $y_1 = \ldots = y_n = y$, then we simply write Δ_y^n instead of $\Delta_{y \ldots y}$ (with n y's)

A function f mapping an Abelian group G into a linear space X over the rationals is called a generalized polynomial or, alternatively, a *polynomial function* of p-th order if it satisfies the functional equation

$$\Delta_y^{p+1} f(x) = 0$$

for all $x, y \in G$. It is well known (see e.g. [4]) that this equation is equivalent to

$$\Delta_{y_1 \ldots y_{p+1}} f(x) = 0$$

where $x, y_1, \ldots, y_{p+1}$ run over all elements of G.

Moreover, it was shown in [4] that any polynomial function f of p-th order may be uniquely represented in the form

$$f = f_0 + f_1 + \quad + f_p$$

where f_0 is a constant and f_k $(k = 1, \ldots, p)$ is a diagonalization of a symmetric k-additive function $F_k\colon G^k \to X$.

The following two results concerning the functions which satisfy locally the equation of generalized polynomials are due to Székelyhidi [12]:

(A) Let $a > 0$ and p be a positive integer. Let $f\colon [0, a) \to \boldsymbol{R}$ be a function such that

$$\Delta_y^{p+1} f(x) = 0 \quad \text{for all } x, y \geqslant 0,\ x+(p+1)y < a.$$

Then there exists a polynomial function $g\colon \boldsymbol{R} \to \boldsymbol{R}$ of p-th order, i.e., a function satisfying the equation

$$\Delta_y^{p+1} g(x) = 0 \quad \text{for all } x, y \in \boldsymbol{R},$$

such that f is the restriction of g to $[0, a)$.

(B) Let $r > 0$, let p be a positive integer and let

$$f\colon \left(-r\sqrt{1+(p+1)^2},\ r\sqrt{1+(p+1)^2}\right) \to \boldsymbol{R}$$

be a function such that

$$\Delta_y^{p+1} f(x) = 0 \quad \text{for } x^2 + y^2 < r^2.$$

Then f is the restriction of a polynomial function $g\colon \boldsymbol{R} \to \boldsymbol{R}$ of p-th order.

It is also well known (cf. [1], [6], [13] and [14] that the functional equation of polynomial functions is stable in the sense that any solution $f\colon \boldsymbol{R}\to X$ (where $(X, \|\cdot\|)$ is a Banach space) of the inequality

$$\|\Delta_y^{p+1} f(x)\| \leqslant \varepsilon \quad \text{for all } x,\ y\in \boldsymbol{R}$$

is uniformly close to a polynomial function of p-th order.

In fact, the same result remains valid for functions defined on an arbitrary Abelian group. In the present paper, however, we shall restrict ourselves to the case of functions whose domains are subsets of the real line only.

Following Skof [10], we are going to combine the results from [1] and [12] in order to obtain some local versions of the stability theorem for polynomial functions. Among others, we shall give approximative analogues of theorems (A) and (B) quoted above. In the course of our considerations we get some theorems characterizing approximately multiadditive functions on certain restricted domains.

2. Local stability of multiadditive functions. In the sequel, $(X, \|\cdot\|)$ will always denote a Banach space, whereas $\boldsymbol{Z}$ and $\boldsymbol{N}$ will stand for the sets of all integers and positive integers, respectively.

We start with a lemma which is contained implicitly in [10]. For the sake of completeness we include its short proof.

LEMMA 1. *Let $a\in(0, \infty)$, $\eta > 0$, $b := \frac{1}{2}a$ and let $\varphi\colon [0, a)\to X$ satisfy the inequality*

$$\|\varphi(x+y)-\varphi(x)-\varphi(y)\| \leqslant \eta \quad \textit{for } x,\ y,\ x+y\in[0, a).$$

Then the function $\psi\colon \boldsymbol{R}\to X$ defined by

$$\psi(x) := n\varphi(b)+\varphi(r) \quad \textit{for } x = nb+r,\ n\in \boldsymbol{Z},\ r\in[0, b)$$

has the properties:

$$\|\psi(x+y)-\psi(x)-\psi(y)\| \leqslant 2\eta \quad \textit{for } x,\ y\in \boldsymbol{R}; \tag{1}$$

$$\|\varphi(x)-\psi(x)\| \leqslant \eta \quad \textit{for } x\in[0, a). \tag{2}$$

Proof. Let $x = mb+r$, $y = nb+s$ for some $m, n\in \boldsymbol{Z}$ and $r, s\in[0, b)$. We distinguish two cases:

(i) $r+s\in[0, b)$; then

$$\begin{aligned}\|\psi(x+y)-\psi(x)-\psi(y)\| &\\ &= \|(m+n)\varphi(b)+\varphi(r+s)-[m\varphi(b)+\varphi(r)]-[n\varphi(b)+\varphi(s)]\|\\ &= \|\varphi(r+s)-\varphi(r)-\varphi(s)\| \leqslant \eta,\end{aligned}$$

(ii) $r+s \in [b, a)$; then $r+s = b+q$ with some $q \in [0, b)$, whence

$$\begin{aligned}\|\psi(x+y)-\psi(x)-\psi(y)\| \\ &= \|(m+n+1)\varphi(b)+\varphi(q)-[m\varphi(b)+\varphi(r)]-[n\varphi(b)+\varphi(s)]\| \\ &= \|\varphi(b)+\varphi(q)-\varphi(r)-\varphi(s)\| \\ &\leqslant \|\varphi(b)+\varphi(q)-\varphi(b+q)\|+\|\varphi(r+s)-\varphi(r)-\varphi(s)\| \leqslant 2\eta.\end{aligned}$$

For $x \in [0, b)$, the equality $\psi(x) = \varphi(x)$ holds while for $x \in [b, a)$ we have $x = b+r$ with some $r \in [0, b)$ and, consequently,

$$\|\varphi(x)-\psi(x)\| = \|\varphi(b+r)-\varphi(b)-\varphi(r)\| \leqslant \eta$$

which ends the proof.

Our further considerations rest on the following:

LEMMA 2. *Let ε, $a \in (0, \infty)$, $p \in N$ and f: $[0, a)^p \to X$. Suppose that for each $i \in \{1, \ldots, p\}$ the inequality*

$$(3) \qquad \begin{aligned}\|f(x_1, \ldots, x_{i-1}, x_i'+x_i'', x_{i+1}, \ldots, x_p)-f(x_1, \ldots, x_{i-1}\ x_i', x_{i+1}, \ldots, x_p) \\ -f(x_1, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\| \leqslant \varepsilon\end{aligned}$$

holds for all $x_1, \ldots, x_{i-1}, x_i', x_i'', x_i'+x_i'', x_{i+1}, \ldots, x_p \in [0, a)$.

Then for every $k \in \{0, 1, \ldots, p\}$ there exists a function h_k: $\mathbf{R}^k \times [0, a)^{p-k} \to X$ with the properties:

(4_k) *if $k > 0$, then h_k is additive in each of the variables $x_1, \ldots, x_k$;*

(5_k) *if $k < p$, then for each $i \in \{k+1, \ldots p\}$,*

$$\begin{aligned}&\|h_k(x_1, \ldots, x_{i-1}, x_i'+x_i'', x_{i+1}, \ldots, x_p) \\ &\quad -h_k(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p)-h_k(x_1, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\| \\ &\qquad \leqslant \varepsilon_k(x_1, \ldots, x_k) := \frac{2|x_1|}{a} \ldots \frac{2|x_k|}{a} \cdot \varepsilon\end{aligned}$$

holds for all $x_1, \ldots, x_k \in \mathbf{R}$, $x_{k+1}, \ldots, x_{i-1}, x_i', x_i'', x_i'+x_i'', x_{i+1}, \ldots, x_p \in [0, a)$;

(6_k) $\|h_k(x_1, \ldots, x_p)-f(x_1, \ldots, x_p)\| \leqslant \delta_k := 3(2^k-1)\varepsilon$

for all $x_1, \ldots, x_p \in [0, a)$.

Remark. If $k = 0$, then the expression $\varepsilon_k(x_1, \ldots, x_k)$ occurring in (5_k) reduces to the constant $\varepsilon_0 := \varepsilon$.

Proof of Lemma 2. In the case where $k = 0$, it suffices to put $h_0 := f$.

Next, let us assume that for a certain $k-1 \in \{0, 1, \ldots, p-1\}$ we have already proved the existence of a function h_{k-1}: $\mathbf{R}^{k-1} \times [0, a)^{p-k+1} \to X$ with the required properties (4_{k-1}), (5_{k-1}), and (6_{k-1}).

Put $b := \frac{1}{2}a$ and define $g_k\colon \boldsymbol{R}^k \times [0, a)^{p-k} \to X$ by the formula

$$g_k(x_1, \ldots, x_p) := $$
$$nh_{k-1}(x_1, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_p) + h_{k-1}(x_1, \ldots, x_{k-1}, r, x_{k+1}, \ldots, x_p)$$

for $x_1, \ldots, x_{k-1} \in \boldsymbol{R}$, $x_k = nb + r$, $n \in \boldsymbol{Z}$, $r \in [0, b)$, $x_{k+1}, \ldots, x_p \in [0, a)$.

Directly from the definition of g_k and from (4_{k-1}) it follows that

(7) g_k is additive in each of the variables $x_1, \ldots, x_{k-1}$ (if $k-1 > 0$).

Fix arbitrarily $x_1, \ldots, x_{k-1} \in \boldsymbol{R}$ and $x_{k+1}, \ldots, x_p \in [0, a)$. On account of (5_{k-1}) we have in particular

$$\begin{aligned} \|h_{k-1}(x_1, \ldots, x_{k-1}, x_k' + x_k'', x_{k+1}, \ldots, x_p) & \\ - h_{k-1}(x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_p) & \\ - h_{k-1}(x_1, \ldots, x_{k-1}, x_k'', x_{k+1}, \ldots, x_p)\| & \\ \leqslant \varepsilon_{k-1}(x_1, \ldots, x_{k-1}) \quad & \text{for all } x_k', x_k'', x_k' + x_k'' \in [0, a), \end{aligned}$$

which, by virtue of Lemma 1, ensures that

$$\begin{aligned} (8) \quad & \|g_k(x_1, \ldots, x_{k-1}, x_k' + x_k'', x_{k+1}, \ldots, x_p) \\ & - g_k(x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_p) - g_k(x_1, \ldots, x_{k-1}, x_k'', x_{k+1}, \ldots, x_p)\| \\ & \leqslant 2\varepsilon_{k-1}(x_1, \ldots, x_{k-1}) \quad \text{for all } x_k', x_k'' \in \boldsymbol{R} \end{aligned}$$

and

$$\begin{aligned} (9) \quad & \|g_k(x_1, \ldots, x_k, \ldots, x_p) - h_{k-1}(x_1, \ldots, x_k, \ldots, x_p)\| \\ & \leqslant \varepsilon_{k-1}(x_1, \ldots, x_{k-1}) \quad \text{for } x_k \in [0, a). \end{aligned}$$

Moreover, if $k < p$ and $i \in \{k+1, \ldots, p\}$, then for $x_1, \ldots, x_{k-1} \in \boldsymbol{R}$, $x_k = nb + r$, $n \in \boldsymbol{Z}$, $r \in [0, b)$, $x_{k+1}, \ldots, x_{i-1}, x_i', x_i'', x_i' + x_i'', x_{i+1}, \ldots, x_p \in [0, a)$, we get

$$\begin{aligned} (10) \quad & \|g_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i' + x_i'', x_{i+1}, \ldots, x_p) \\ & - g_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p) \\ & - g_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\| \\ & \leqslant |n| \cdot \|h_{k-1}(x_1, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_{i-1}, x_i' + x_i'', x_{i+1}, \ldots, x_p) \\ & - h_{k-1}(x_1, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p) \\ & - h_{k-1}(x_1, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\| \\ & + \|h_{k-1}(x_1, \ldots, x_{k-1}, r, x_{k+1}, \ldots, x_{i-1}, x_i' + x_i'', x_{i+1}, \ldots, x_p) \\ & - h_{k-1}(x_1, \ldots, x_{k-1}, r, x_{k+1}, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p) \\ & - h_{k-1}(x_1, \ldots, x_{k-1}, r, x_{k+1}, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\| \\ & \leqslant (|n| + 1)\varepsilon_{k-1}(x_1, \ldots, x_{k-1}). \end{aligned}$$

Now, define the function $h_k\colon R^k \times [0, a)^{p-k} \to X$ as follows:

$$h_k(x_1, \ldots, x_p) := \lim_{m\to\infty} \frac{1}{m} g_k(x_1, \ldots, x_{k-1}, mx_k, x_{k+1}, \ldots, x_p) \tag{11}$$

for $x_1, \ldots, x_k \in R$, $x_{k+1}, \ldots, x_p \in [0, a)$.

By (8) and the celebrated Hyers theorem (cf. [5]; see also [9], Theorem 17.1.1 and Corollary 17.1.1), the limit occurring in (11) exists and h_k is an additive function of x_k such that

$$\|g_k(x_1, \ldots, x_p) - h_k(x_1, \ldots, x_p)\| \leqslant 2\varepsilon_{k-1}(x_1, \ldots, x_{k-1}) \tag{12}$$
$$\text{for all } x_1, \ldots, x_k \in R,\ x_{k+1}, \ldots, x_p \in [0, a).$$

Applying (7), one can easily check that h_k remains additive in the variables $x_1, \ldots, x_{k-1}$ (if $k-1>0$). Consequently, h_k fulfils condition (4_k).

If $k<p$ and $i \in \{k+1, \ldots, p\}$, then for $x_1, \ldots, x_{k-1} \in R$, $mx_k = n_m b + r_m$, $m \in N$, $n_m \in Z$, $r_m \in [0, b)$, $x_{k+1}, \ldots, x_{i-1}, x_i', x_i'', x_i'+x_i'', x_{i+1}, \ldots, x_p \in [0, a)$, we infer from (10) that

$$\left\| \frac{g_k(x_1, \ldots, x_{k-1}, mx_k, x_{k+1}, \ldots, x_{i-1}, x_i'+x_i'', x_{i+1}, \ldots, x_p)}{m} \right. \tag{13}$$
$$- \frac{g_k(x_1, \ldots, x_{k-1}, mx_k, x_{k+1}, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p)}{m}$$
$$\left. - \frac{g_k(x_1, \ldots, x_{k-1}, mx_k, x_{k+1}, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)}{m} \right\|$$
$$\leqslant \frac{(|n_m|+1)}{m} \varepsilon_{k-1}(x_1, \ldots, x_{k-1}).$$

Since

$$\frac{|n_m|}{m} \leqslant \frac{|x_k|}{b} + \frac{|r_m|}{mb},$$

we have

$$\frac{|n_m|}{m} \leqslant \frac{|x_k|}{b} + \frac{r_m}{mb} \xrightarrow[m\to\infty]{} \frac{|x_k|}{b}.$$

Thus, passing in (13) to the limit as $m \to \infty$, we obtain

$$\|h_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i'+x_i'', x_{i+1}, \ldots, x_p)$$
$$- h_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_p)$$
$$- h_k(x_1, \ldots, x_k, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_p)\|$$
$$\leqslant \frac{|x_k|}{b} \varepsilon_{k-1}(x_1, \ldots, x_{k-1}) = \varepsilon_k(x_1, \ldots, x_k)$$

for $x_1, \ldots, x_k \in \boldsymbol{R}$, $x_{k+1}, \ldots, x_{i-1}, x_i', x_i'', x_i'+x_i'', x_{i+1}, \ldots, x_p \in [0, a)$, which means that h_k satisfies condition (5_k).

Relations (9) and (12) together yield

$$\|h_k(x_1, \ldots, x_p) - h_{k-1}(x_1, \ldots, x_p)\| \leqslant 3\varepsilon_{k-1}(x_1, \ldots, x_{k-1})$$

for all $x_1, \ldots, x_{k-1} \in \boldsymbol{R}$, $x_k, \ldots, x_p \in [0, a)$. Since $\varepsilon_{k-1}(x_1, \ldots, x_{k-1}) \leqslant 2^{k-1}\varepsilon$ provided that $x_1, \ldots, x_{k-1} \in [0, a)$, we conclude that

$$\|h_k(x_1, \ldots, x_p) - h_{k-1}(x_1, \ldots, x_p)\| \leqslant 3 \cdot 2^{k-1}\varepsilon$$

for $x_1, \ldots, x_p \in [0, a)$. Hence, and from the assumption that h_{k-1} fulfils (6_{k-1}) we derive

$$\begin{aligned}\|h_k(x_1, \ldots, x_p) - f(x_1, \ldots, x_p)\| &\leqslant 3 \cdot 2^{k-1}\varepsilon + 3(2^{k-1}-1)\varepsilon \\ &= 3(2^k-1)\varepsilon = \delta_k \quad \text{for } x_1, \ldots, x_p \in [0, a),\end{aligned}$$

which shows that (6_k) is valid for the function h_k.

Induction on k completes the proof.

THEOREM 1. *Let $\varepsilon, a \in (0, \infty)$, $p \in \boldsymbol{N}$ and $f\colon [0, a)^p \to X$. Suppose that f satisfies inequality (3) for every $i \in \{1, \ldots, p\}$ and for all $x_1, \ldots, x_{i-1}, x_i', x_i'', x_i'+x_i'', x_{i+1}, \ldots, x_p \in [0, a)$. Then there exists a p-additive function $h\colon \boldsymbol{R}^p \to X$ such that*

$$\|h(x_1, \ldots, x_p) - f(x_1, \ldots, x_p)\| \leqslant 3(2^p-1)\varepsilon \tag{14}$$

for $x_1, \ldots, x_p \in [0, a)$.

Moreover, if f is symmetric, then one can require h to be symmetric.

Proof. To prove the first part of our theorem, it is enough to put $h := h_p$ from Lemma 2.

If, additionally, f is symmetric and h is a p-additive function fulfilling (14), then we define a new function $\tilde{h}$ by

$$\tilde{h}(x_1, \ldots, x_p) := \frac{1}{p!}\sum_{\sigma} h(x_{\sigma(1)}, \ldots, x_{\sigma(p)}), \qquad x_1, \ldots, x_p \in \boldsymbol{R},$$

where the summation runs over all permutations σ of the set $\{1, \ldots, p\}$. Then $\tilde{h}$ is a symmetric p-additive function such that

$$\begin{aligned}&\|\tilde{h}(x_1, \ldots, x_p) - f(x_1, \ldots, x_p)\| \\ &\qquad = \left\|\frac{1}{p!}\sum_{\sigma} h(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) - \frac{1}{p!}\sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(p)})\right\| \\ &\qquad \leqslant \frac{1}{p!}\sum_{\sigma} \|h(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) - f(x_{\sigma(1)}, \ldots, x_{\sigma(p)})\| \\ &\qquad \leqslant 3(2^p-1)\varepsilon \quad \text{for } x_1, \ldots, x_p \in [0, a),\end{aligned}$$

so that $\tilde{h}$ has all the properties desired.

THEOREM 2. *Let* ε, $a \in (0, \infty)$, $p \in N$ *and* $f\colon (-a, a)^p \to X$. *Suppose that* f *satisfies inequality* (3) *for every* $i \in \{1, \ldots, p\}$ *and for all* $x_1, \ldots, x_{i-1}, x_i', x_i''$, $x_i'+x_i'', x_{i+1}, \ldots, x_p \in (-a, a)$. *Then there exists a p-additive function* $h\colon \boldsymbol{R}^p \to X$ *such that*

$$\|h(x_1, \ldots, x_p) - f(x_1, \ldots, x_p)\| \leqslant [3(2^p - 1) + 2p]\varepsilon \quad \text{for } x_1, \ldots, x_p \in (-a, a). \tag{15}$$

Moreover, if f *is symmetric, then one can also require* h *to be symmetric.*

Proof. According to Theorem 1, we can find a p-additive function $h\colon \boldsymbol{R}^p \to X$ (symmetric, if f is symmetric) such that (14) holds for $x_1, \ldots, x_p \in [0, a)$.

We shall prove that for any $k \in \{0, 1, \ldots, p\}$ the following implication holds true:

(16_k) if $x_1, \ldots, x_p \in (-a, a)$ and exactly k of the numbers $x_1, \ldots, x_p$ are negative, then

$$\|f(x_1, \ldots, x_p) - h(x_1, \ldots, x_p)\| \leqslant [3(2^p - 1) + 2k]\varepsilon.$$

Condition (16_0) results immediately from (14). Suppose that (16_{k-1}) is valid for a certain $k-1 \in \{0, 1, \ldots, p-1\}$. Choose numbers $x_1, \ldots, x_p \in (-a, a)$ such that precisely k of them assume negative values. Let for example, $x_i < 0$. Then

$$\begin{aligned}
\|f(x_1, \ldots, x_p) - h(x_1, \ldots, x_p)\| \leqslant{}& \|h(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_p) \\
&- f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_p)\| \\
&+ \|f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_p) \\
&+ f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \\
&- f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p)\| \\
&+ \|2f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p) \\
&- f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p)\| \\
\leqslant{}& [3(2^p - 1) + 2(k-1)]\varepsilon + 2\varepsilon = [3(2^p - 1) + 2k]\varepsilon.
\end{aligned}$$

Induction on k guarantees that (16_k) holds for each $k \in \{0, 1, \ldots, p\}$ and, consequently, the proof of our theorem is completed.

In the remaining part of this section we are going to present similar results concerning approximately multiadditive functions on the following two types of restricted domains:

$$S_p := \{(x_1, \ldots, x_p) \in [0, a)^p\colon x_1 + \ldots + x_p < a\}$$

and

$$T_p := \{(x_1, \ldots, x_p) \in (-a, a)^p\colon |x_1| + \ldots + |x_p| < a\}.$$

LEMMA 3. *Let* ε, $a \in (0, \infty)$, $p \in N$, *and suppose* $f\colon S_p \to X$ *to be a function satisfying inequality* (3) *for every* $i \in \{1, \ldots, p\}$ *and for all* $x_1, \ldots, x_{i-1}, x_i', x_i'', x_{i+1}, \ldots, x_p \in [0, a)$ *such that* $x_1 + \ldots + x_{i-1} + x_i' + x_i'' + \ldots + x_p < a$. *If* $n_1, \ldots, n_p$ *are fixed positive integers, then*

$$\|f(n_1 x_1, \ldots, n_p x_p) - (n_1 \cdot \ldots \cdot n_p) f(x_1, \ldots, x_p)\| \leqslant [(n_1 \cdot \ldots \cdot n_p) - 1]\varepsilon$$

$$\text{for } x_1, \ldots, x_p \in [0, a) \text{ such that } n_1 x_1 + \ldots + n_p x_p < a.$$

Proof. First we shall show that for given $i \in \{1, \ldots, p\}$ and $n \in N$ we have

$$\|f(x_1, \ldots, x_{i-1}, nx_i, x_{i+1}, \ldots, x_p) - nf(x_1, \ldots, x_p)\| \leqslant (n-1)\varepsilon \tag{17}$$

for $x_1, \ldots, x_p \in [0, a)$ such that $x_1 + \ldots + x_{i-1} + nx_i + x_{i+1} + \ldots + x_p < a$.

If $n = 1$, then (17) is trivially fulfilled. Assume (17) true for some $n \in N$ and take $x_1, \ldots, x_p \in [0, a)$ with $x_1 + \ldots + x_{i-1} + (x+1)x_i + x_{i+1} + \ldots + x_p < a$. By the hypothesis,

$$\begin{aligned}
&\|f(x_1, \ldots, x_{i-1}, (n+1)x_i, x_{i+1}, \ldots, x_p) - (n+1)f(x_1, \ldots, x_p)\| \\
&\qquad \leqslant \|f(x_1, \ldots, x_{i-1}, nx_i + x_i, x_{i+1}, \ldots, x_p) \\
&\qquad\qquad - f(x_1, \ldots, x_{i-1}, nx_i, x_{i+1}, \ldots, x_p) - f(x_1, \ldots, x_i, \ldots, x_p)\| \\
&\qquad\qquad + \|f(x_1, \ldots, x_{i-1}, nx_1, x_{i+1}, \ldots, x_p) - nf(x_1, \ldots, x_p)\| \\
&\qquad \leqslant \varepsilon + (n-1)\varepsilon = n\varepsilon
\end{aligned}$$

whence (17) follows by induction on n.

Finally, for arbitrarily chosen $x_1, \ldots, x_p \in [0, a)$ such that $n_1 x_1 + \ldots + n_p x_p < a$ we get

$$\begin{aligned}
&\|f(n_1 x_1, \ldots, n_p x_p) - (n_1 \cdot \ldots \cdot n_p) f(x_1, \ldots, x_p)\| \\
&\leqslant \|f(n_1 x_1, \ldots, n_p x_p) - n_1 f(x_1, n_2 x_2, \ldots, n_p x_p)\| \\
&\quad + n_1 \|f(x_1, n_2 x_2, \ldots, n_p x_p) - n_2 f(x_1, x_2, n_3 x_3, \ldots, n_p x_p)\| \\
&\quad + n_1 n_2 \|f(x_1, x_2, n_3 x_3, \ldots, n_p x_p) - n_3 f(x_1, x_2, x_3, n_4 x_4, \ldots, n_p x_p)\| \\
&\quad + \ldots + (n_1 \cdot \ldots \cdot n_{p-1}) \|f(x_1, \ldots, x_{p-1}, n_p x_p) - n_p f(x_1, \ldots, x_p)\| \\
&\leqslant [(n_1 - 1) + n_1(n_2 - 1) + n_1 n_2 (n_3 - 1) + \ldots + (n_1 \ldots n_{p-1}) \cdot (n_p - 1)]\varepsilon \\
&= [(n_1 \cdot \ldots \cdot n_p) - 1]\varepsilon
\end{aligned}$$

which was to be shown.

LEMMA 4. *If* $n_1, \ldots, n_p$ *are positive integers such that* $n_1 + \ldots + n_p \leqslant 2 \cdot p - 1$, *then* $n_1 \cdot \ldots \cdot n_p \leqslant 2^{p-1}$

Proof. Evidently, the assumptions of our lemma imply that at least one of the integers $n_1, .., n_p$ must be equal to 1. Without loss of generality we may assume that $n_p = 1$. Thus

$$n_1 + \ldots + n_{p-1} \leqslant 2(p-1),$$

and consequently,

$$n_1 \cdot \ldots \cdot n_p = n_1 \cdot \ldots \cdot n_{p-1} \leqslant \left(\frac{n_1 + \ldots + n_{p-1}}{p-1}\right)^{p-1} \leqslant 2^{p-1},$$

which completes the proof.

In what follows, the symbol Q_p will denote the set $[0, a/p)^p$. It is clear that $Q_p \subset S_p$.

LEMMA 5. *For any $(y_1, \ldots, y_p) \in S_p$ there exists a point $(x_1, \ldots, x_p) \in Q_p$ and integers $n_1, \ldots, n_p$ such that $y_1 = n_1 x_1, \ldots, y_p = n_p x_p$ and $n_1 \cdot \ldots \cdot n_p \leqslant 2^{p-1}$*

Proof. Let $[z]$ stand for the integral part of a number z and put $b := a/p$. With this notation we define

$$n_i := [y_i/b] + 1 \quad \text{and} \quad x_i := y_i/n_i \quad \text{for } i = 1, \ldots, p.$$

Notice that $y_i = n_i x_i$ and $(x_1, \ldots, x_p) \in Q_p$. Indeed,

$$[y_i/b] \cdot b \leqslant y_i < ([y_i/b] + 1)b = n_i b,$$

so that $x_i < b$ for $i = 1, \ldots, p$.

Moreover, since $(y_1, \ldots, y_p) \in S_p$, we have $y_1 + \ldots + y_p < a$ which means that

$$[(y_1 + \ldots + y_p)/b] \leqslant p - 1.$$

Hence

$$n_1 + \ldots + n_p = [y_1/b] + \ldots + [y_p/b] + p \leqslant [(y_1 + \ldots + y_p)/b] + p \leqslant 2p - 1$$

which on account of Lemma 4 implies that

$$n_1 \cdot \ldots \cdot n_p \leqslant 2^{p-1}$$

This finishes the proof.

Theorem 1 jointly with Lemmas 3 and 5 allow us to prove the following theorem.

THEOREM 3. *Let $\varepsilon, a \in (0, \infty)$, $p \in N$ and let $f: S_p \to X$ be a function satisfying inequality* (3) *for every $i \in \{1, \ldots, p\}$ and for all $x_1, \ldots, x_{i-1}, x_i', x_i'', x_{i+1}, \ldots, x_p \in [0, a)$ such that $x_1 + \ldots + x_{i-1} + x_i' + x_i'' + x_{i+1} + \ldots + x_p < a$. Then there exists a p-additive function $h: R^p \to X$ such that*

$$\|h(x_1, \ldots, x_p) - f(x_1, \ldots, x_p)\| \leqslant k_p \varepsilon \quad \text{for } (x_1, \ldots, x_p) \in S_p,$$

where $k_p := 3 \cdot 2^{2p-1} - 2^p - 1$.

Moreover, one can require h to be symmetric, provided so is f.

Proof. Since $O_p \subset S_p$, Theorem 1 assures that there exists a p-additive function $h\colon \mathbf{R}^p \to X$ (symmetric, if f is symmetric) such that

$$(18) \qquad \|h(x_1,\ldots,x_p)-f(x_1,\ldots,x_p)\| \leqslant 3(2^p-1)\varepsilon \quad \text{for } (x_1,\ldots,x_p)\in Q_p.$$

In view of Lemma 5, to an arbitrarily chosen point $(y_1,\ldots,y_p)\in S_p$ one can assign a point $(x_1,\ldots,x_p)\in Q_p$ and numbers $n_1,\ldots,n_p\in N$ such that $y_1 = n_1x_1,\ldots,y_p = n_px_p$ and $n_1\cdot\ldots\cdot n_p \leqslant 2^{p-1}$. Then by Lemma 3, we have

$$\|f(y_1,\ldots,y_p)-(n_1\cdot\ldots\cdot n_p)f(x_1,\ldots,x_p)\| \leqslant [(n_1\cdot\ldots\cdot n_p)-1]\varepsilon \leqslant (2^{p-1}-1)\varepsilon.$$

Hence and from (18) we derive

$$\begin{aligned}\|f(y_1,\ldots,y_p)-h(y_1,\ldots,y_p)\| &\leqslant \|f(y_1,\ldots,y_p)-(n_1\cdot\ldots\cdot n_p)f(x_1,\ldots,x_p)\| \\ &\quad +(n_1\cdot\ldots\cdot n_p)\|f(x_1,\ldots,x_p)-h(x_1,\ldots,x_p)\| \\ &\leqslant (2^{p-1}-1)\varepsilon+2^{p-1}\cdot 3(2^p-1)\varepsilon = (3\cdot 2^{2p-1}-2^p-1)\varepsilon\end{aligned}$$

so the proof is finished.

We complete this section with a theorem establishing an analogue of Theorem 3 in which the domain of the function in question is T_p instead of S_p. One can easily prove this result using Theorem 2 and repeating (with slight alterations) the reasoning which has led us from Theorem 1 to Theorem 3. Therefore, we shall only formulate the result omitting the detailed proof.

THEOREM 4. *Let ε, $a\in(0,\infty)$, $p\in N$ and let $f\colon T_p\to X$ be a function satisfying inequality* (3) *for every $i\in\{1,\ldots,p\}$ and for all $x_1,\ldots,x_{i-1},x_i',x_i'',x_{i+1},\ldots,x_p\in(-a,a)$ such that $|x_1|+\ldots+|x_{i-1}|+|x_i'|+|x_i''|+|x_{i+1}|+\ldots+|x_p|<a$. Then there exists a p-additive function $h\colon \mathbf{R}^p\to X$ such that*

$$\|h(x_1,\ldots,x_p)-f(x_1,\ldots,x_p)\| \leqslant k_p'\varepsilon \quad \textit{for } (x_1,\ldots,x_p)\in T_p$$

where $k_p' := 3\cdot 2^{2p}+(p-1)2^{p+1}-1$.

Moreover, one can require h to be symmetric, provided so is f.

3. Functions with locally bounded $(p+1)$-th differences.

THEOREM 5. *Let ε, $a\in(0,\infty)$ and $p\in N\cup\{0\}$. If a function $f\colon [0,a)\to X$ satisfies the condition*

$$(19) \qquad \|\Delta_{h_1\ldots h_{p+1}}f(x)\| \leqslant \varepsilon \quad \textit{for all } x,h_1,\ldots,h_{p+1}\in[0,a)$$

such that $x+h_1+\ldots+h_{p+1}<a$, then there exists a polynomial function $g\colon \mathbf{R}\to X$ of p-th order with the property

$$(20) \qquad \|f(x)-g(x)\| \leqslant l_p\varepsilon \quad \textit{for } x\in[0,a)$$

where

$$l_0 := 1 \quad \textit{and} \quad l_p := \prod_{i=1}^{p} (k_i + 1) \quad \textit{for } p \geqslant 1,$$

k_i being the constant defined in Theorem 3.

Proof. The proof runs by induction on p and is based on ideas similar to those of [1].

If $p = 0$, then $\|\Delta_h f(0)\| \leqslant \varepsilon$ for all $h \in [0, a)$, i.e.,

$$\|f(x) - f(0)\| \leqslant \varepsilon \quad \text{for } x \in [0, a).$$

Setting $g := f(0) = \text{const}$, we are through.

Now suppose that the theorem holds true for a $p = q - 1 \in N \cup \{0\}$ and let $f\colon [0, a) \to X$ satisfy

$$(21) \qquad \|\Delta_{h_1 \dots h_{q+1}} f(x)\| \leqslant \varepsilon \quad \text{for all } x, h_1, \dots, h_{q+1} \in [0, a)$$

such that $x + h_1 + \quad + h_{q+1} < a$.

The formula

$$F(x_1, \dots, x_q) := \Delta_{x_1 \dots x_q} f(0) \quad \text{for } (x_1, \dots, x_q) \in S_q$$

defines a symmetric function $F\colon S_q \to X$ such that

$$\begin{aligned} &\|F(x_1, \dots, x_{i-1}, x_i' + x_i'', x_{i+1}, \dots, x_q) \\ &\quad - F(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_q) - F(x_1, \dots, x_{i-1}, x_i'', x_{i+1}, \dots, x_q)\| \\ &\qquad = \|\Delta_{x_1 \dots x_{i-1}, x_{i+1} \dots x_q}[\Delta_{x_i' + x_i''} f(0) - \Delta_{x_i'} f(0) - \Delta_{x_i''} f(0)]\| \\ &\qquad = \|\Delta_{x_1 \dots x_{i-1}, x_i', x_i'', x_{i+1} \dots x_q} f(0)\| \leqslant \varepsilon \end{aligned}$$

for every $i \in \{1, \dots, q\}$ and all $x_1, \dots, x_{i-1}, x_i', x_i'', x_{i+1}, \dots, x_q \in [0, a)$ such that $x_1 + \dots + x_{i-1} + x_i' + x_i'' + x_{i+1} + \dots + x_q < a$.

Applying Theorem 3, we can find a q-additive symmetric function $H\colon R^q \to X$ fulfilling

$$(22) \qquad \|F(x_1, \dots, x_q) - H(x_1, \dots, x_q)\| \leqslant k_q \varepsilon \quad \text{for } (x_1, \dots, x_q) \in S_q.$$

Put

$$G_q(x_1, \dots, x_q) := \frac{1}{q!} H(x_1, \dots, x_q), \quad (x_1, \dots, x_q) \in R^q,$$

and let $g_q\colon R \to X$ be the diagonalization of G_q, i.e.,

$$g_q(x) := G_q(x, \dots, x), \quad x \in R.$$

Further, let $f^*(x) := f(x) - g_q(x)$ for $x \in [0, a)$ and choose arbitrarily $x, h_1, \dots, h_q \in [0, a)$ with $x + h_1 + \dots + h_q < a$. Then by (21), (22) and Lemma 2 from [4] (see also [9], Lemma 15.9.2) we have

$$\begin{aligned}\|\Delta_{h_1 \dots h_q} f^*(x)\| &\leqslant \|\Delta_{h_1 \dots h_q} f(x) - \Delta_{h_1 \dots h_q} f(0)\| + \|\Delta_{h_1 \dots h_q} f(0) - \Delta_{h_1 \dots h_q} g_q(x)\| \\ &= \|\Delta_{h_1 \dots h_q, x} f(0)\| + \|F(h_1, \dots, h_q) - q!\, G_q(h_1, \dots, h_q)\| \\ &\leqslant \varepsilon + \|F(h_1, \dots, h_q) - H(h_1, \dots, h_q)\| \leqslant (k_q + 1)\varepsilon.\end{aligned}$$

From the induction hypothesis we deduce that there exists a polynomial function $g^*\colon \boldsymbol{R} \to X$ of order $q-1$ such that

$$\|f^*(x) - g^*(x)\| \leqslant l_{q-1}(k_q + 1)\varepsilon \quad \text{for } x \in [0, a),$$

and consequently,

$$\|f(x) - g_q(x) - g^*(x)\| \leqslant l_q \varepsilon \quad \text{for } x \in [0, a).$$

Since $g := g_q + g^*$ is a polynomial function of q-th order, we arrive at (20) for $p = q$ and the proof is finished by induction.

With the aid of Theorem 4 in a similar manner one can also prove the following result concerning functions defined on a symmetric neighbourhood of zero.

THEOREM 6. *Let $\varepsilon, a \in (0, \infty)$ and $p \in \boldsymbol{N} \cup \{0\}$. If a function $f\colon (-a, a) \to X$ satisfies condition (19) for all $x, h_1, \dots, h_{p+1} \in (-a, a)$ such that $|x| + |h_1| + \dots + |h_{p+1}| < a$, then there exists a polynomial function $g\colon \boldsymbol{R} \to X$ of p-th order with the property*

$$\|f(x) - g(x)\| \leqslant l'_p \varepsilon \quad \textit{for } x \in (-a, a),$$

where

$$l'_0 := 1 \quad \textit{and} \quad l'_p := \prod_{l=1}^{p} (k'_l + 1) \quad \textit{for } p \geqslant 1.$$

In what follows, we are going to strengthen the preceding results in such a way that (19) will be postulated only for equal values of $h_1, \dots, h_{p+1}$. We begin with the following lemma.

LEMMA 6. *Let $\varepsilon, a \in (0, \infty)$, $p \in \boldsymbol{N}$, and suppose that $f\colon (-a, a) \to X$ satisfies the condition:*

$$(23) \quad \|\Delta_h^{p+1} f(x)\| \leqslant \varepsilon \quad \textit{for all } x, h \in (-a, a) \quad \textit{such that } x + (p+1)h \in (-a, a).$$

If $b := (p+1)p^{-1}a$, then there exists a function $F\colon (-b, b) \to X$ such that

(i) *$\|\Delta_h^{p+1} F(x)\| \leqslant (2^{p+1} - 1)\varepsilon$ for $x, h \in (-b, b)$ provided $x + (p+1)h \in (-b, b)$;*

(ii) *$\|f(x) - F(x)\| \leqslant \varepsilon$ for $x \in (-a, a)$.*

Proof. Since $kx/(p+1)\in(-a, a)$ whenever $x\in(-b, b)$ and $k\in\{0,\dots, p\}$, the function F: $(-b, b)\to X$ is well defined by the formula

$$F(x):=\sum_{k=0}^{p}(-1)^{p-k}\binom{p+1}{k}f(kx/(p+k)), \qquad x\in(-b, b).$$

If we confine ourselves to x's from $(-a, a)$, then

$$\begin{aligned}\|f(x)-F(x)\| &= \left\|f(x)+\sum_{k=0}^{p}(-1)^{p+1-k}\binom{p+1}{k}f(kx/(p+1))\right\| \\ &= \|\Delta_{x/(p+1)}^{p+1}f(0)\| \leqslant \varepsilon\end{aligned}$$

which yields (ii).

Further choose x, $h\in(-b, b)$ such that $x+(p+1)h\in(-b, b)$. Then for every $k\in\{0,\dots, p\}$ the points

$$k\frac{x}{p+1}, \quad k\frac{h}{p+1} \quad \text{and} \quad k\frac{x}{p+1}+(p+1)k\frac{h}{p+1}=\frac{k}{p+1}(x+(p+1)h)$$

lie in the interval $(-a, a)$, and therefore

$$\begin{aligned}\|\Delta_h^{p+1}F(x)\| &= \left\|\sum_{j=0}^{p+1}(-1)^{p+1-j}\binom{p+1}{j}F(x+jh)\right\| \\ &= \left\|\sum_{j=0}^{p+1}(-1)^{p+1-j}\binom{p+1}{j}\sum_{k=0}^{p}(-1)^{p-k}\binom{p+1}{k}f\left(k\frac{x+jh}{p+1}\right)\right\| \\ &= \left\|\sum_{k=0}^{p}(-1)^{p-k}\binom{p+1}{k}\sum_{j=0}^{p+1}(-1)^{p+1-j}\binom{p+1}{j}f(k\cdot x/(p+1)+jk\cdot h/(p+1))\right\| \\ &= \left\|\sum_{k=0}^{p}(-1)^{p-k}\binom{p+1}{k}\Delta_{k\cdot h/(p+1)}^{p+1}f(k\cdot x/(p+1))\right\| \\ &\leqslant \sum_{k=0}^{p}\binom{p+1}{k}\varepsilon=(2^{p+1}-1)\varepsilon.\end{aligned}$$

This implies (i) and accomplishes the proof.

COROLLARY 1. *Assume that* ε, $a\in(0, \infty)$, $p\in N$ *and* f: $(-a, a)\to X$ *is a function satisfying* (23). *Let* $b\geqslant a$ *and*

$$n:=\min\{k\in N\cup\{0\}:\ ((p+1)/p)^k\cdot a\geqslant b\}.$$

Then there exists a function F: $(-b, b)\to X$ *such that*

(i)$_n$ $\|\Delta_h^{p+1}F(x)\|\leqslant(2^{p+1}-1)^n\varepsilon$ *for* x, $h\in(-b, b)$ *with* $x+(p+1)h\in(-b, b)$;
(ii)$_n$ $\|f(x)-F(x)\|\leqslant n\varepsilon$ *for* $x\in(-a, a)$.

The corollary above results on applying Lemma 6 n times. Let us recall here a fact (cf. [9], Theorem 15.1.2, and [4], Theorem 2) which turns out to be useful in the proof our next theorem.

LEMMA 7. *Let $f: \boldsymbol{R} \to X$ be an arbitrary function and fix $h_1, \ldots, h_{p+1} \in \boldsymbol{R}$. For any $\varepsilon_1, \ldots, \varepsilon_{p+1} \in \{0, 1\}$ put*

$$h'_{\varepsilon_1 \ldots \varepsilon_{p+1}} := -\sum_{j=1}^{p+1} \varepsilon_j h_j/j, \qquad h''_{\varepsilon_1 \ldots \varepsilon_{p+1}} := \sum_{j=1}^{p+1} \varepsilon_j h_j. \tag{24}$$

Then for every $x \in \boldsymbol{R}$ we have

$$\Delta_{h_1 \ldots h_{p+1}} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_1 + \ldots + \varepsilon_{p+1}} \Delta^{p+1}_{h'_{\varepsilon_1 \ldots \varepsilon_{p+1}}} f(x + h''_{\varepsilon_1 \ldots \varepsilon_{p+1}}). \tag{25}$$

THEOREM 7. *Let ε, $a \in (0, \infty)$, $p \in \boldsymbol{N}$ and suppose $f: (-a, a) \to X$ to be a function satisfying condition (23). Then there exists a polynomial function $g: \boldsymbol{R} \to X$ of p-th order such that*

$$\|f(x) - g(x)\| \leqslant l''_p \varepsilon \quad \textit{for } x \in (-a, a)$$

where

$$l''_p := n_p + 2^{p+1}(2^{p+1} - 1)^{n_p} \cdot l'_p, \quad n_p := \min\{k \in \boldsymbol{N} \cup \{0\}: ((p+k)/p)^k \geqslant p\}.$$

Proof. Denoting $p \cdot a$ by b, one can easily see that

$$n_p = \min\{k \in \boldsymbol{N} \cup \{0\}: ((p+k)/p)^k \cdot a \geqslant b\}.$$

In virtue of Corollary 1 there exists a function $F: (-b, b) \to X$ fulfilling conditions $(\mathrm{i})_{n_p}$ and $(\mathrm{ii})_{n_p}$.

Now, let us choose arbitrarily x, $h_1, \ldots, h_{p+1} \in (-a, a)$ such that $|x| + |h_1| + \ldots + |h_{p+1}| < a$, and for a system of indices $\varepsilon_1, \ldots, \varepsilon_{p+1} \in \{0, 1\}$ let elements $h'_{\varepsilon_1 \ldots \varepsilon_{p+1}}$ and $h''_{\varepsilon_1 \ldots}$ be given by (24). Since the following relations hold:

$$|x + h''_{\varepsilon_1 \ldots \varepsilon_{p+1}}| \leqslant |x| + \sum_{j=1}^{p+1} |h_j| < a \leqslant b,$$

$$|h'_{\varepsilon_1 \ldots \varepsilon_{p+1}}| \leqslant \sum_{j=1}^{p+1} |h_j|/j \leqslant \sum_{j=1}^{p+1} |h_j| < a \leqslant b,$$

$$|x + h''_{\varepsilon_1 \ldots \varepsilon_{p+1}} + (p+1) h'_{\varepsilon_1 \ldots \varepsilon_{p+1}}| = \Big|x + \sum_{j=1}^{p+1} \big(1 - (p+1)/j\big) \varepsilon_j h_j\Big| \leqslant |x| + p \sum_{j=1}^{p+1} |h_j|$$

$$\leqslant p\Big(|x| + \sum_{j=1}^{p+1} |h_j|\Big) < p \cdot a = b,$$

it becomes apparent that the points $x+h''_{\varepsilon_1\ldots\varepsilon_{p+1}}$, $h'_{\varepsilon_1\ldots}$ and $x+h''_{\varepsilon_1\ldots\varepsilon_{p+1}}+(p+1)h'_{\varepsilon_1\ldots\varepsilon_{p+1}}$ belong to the interval $(-b, b)$.

Bearing in mind (25) and the fact that F fulfils $(\mathrm{i})_{n_p}$, we infer that

$$\|\Delta_{h_1\ldots h_{p+1}}F(x)\| \leqslant \sum_{\varepsilon_1,\ldots,\varepsilon_{p+1}=0}^{1} \|\Delta^{p+1}_{h'_{\varepsilon_1\ldots\varepsilon_{p+1}}}F(x+h''_{\varepsilon_1\ldots\varepsilon_{p+1}})\| \leqslant 2^{p+1}(2^{p+1}-1)^{n_p}\varepsilon.$$

Hence and from Theorem 6 it follows that we can find a polynomial function $g\colon \boldsymbol{R}\to X$ of p-th order such that

$$\|F(x)-g(x)\| \leqslant l'_p 2^{p+1}(2^{p+1}-1)^{n_p}\varepsilon \quad \text{for } x\in(-a, a).$$

Finally, taking into account $(\mathrm{ii})_{n_p}$, we obtain for $x\in(-a, a)$,

$$\|f(x)-g(x)\| \leqslant \|f(x)-F(x)\|+\|F(x)-g(x)\| \leqslant n_p\varepsilon+l'_p 2^{p+1}(2^{p+1}-1)^{n_p}\varepsilon,$$

which completes the proof.

The subsequent theorem shows that a corresponding result is valid for functions whose domain is a neighbourhood of an arbitrary point $x_0\in\boldsymbol{R}$.

THEOREM 8. *Let $x_0\in\boldsymbol{R}$, ε, $a\in(0, \infty)$, $p\in N$ and assume that a function $f\colon (x_0-a, x_0+a)\to X$ satisfies the condition*:

$$\|\Delta_h^{p+1}f(x)\| \leqslant \varepsilon \quad \textit{for all } x\in(x_0-a, x_0+a) \textit{ and } h\in(-a, a)$$

such that $x+(p+1)h\in(x_0-a, x_0+a)$. Then there exists a polynomial function $g\colon \boldsymbol{R}\to X$ of p-th order such that

$$\|f(x)-g(x)\| \leqslant l''_p\varepsilon \quad \textit{for } x\in(x_0-a, x_0+a).$$

Proof. Define a function $f^*\colon (-a, a)\to X$ by

$$f^*(t):=f(x_0+t), \quad t\in(-a, a).$$

Take $t, h\in(-a, a)$ such that $t+(p+1)h\in(-a, a)$ and put $x:=x_0+t$. Then x and $x+(p+1)h$ belong to (x_0-a, x_0+a), which yields

$$\|\Delta_h^{p+1}f^*(t)\| = \|\Delta_h^{p+1}f(x_0+t)\| = \|\Delta_h^{p+1}f(x)\| \leqslant \varepsilon.$$

Let $g^*\colon \boldsymbol{R}\to X$ be a polynomial function of p-th order, whose existence results from Theorem 7, such that

$$\|f^*(t)-g^*(t)\| \leqslant l''_p\varepsilon \quad \text{for } t\in(-a, a).$$

The formula

$$g(x):=g^*(x-x_0), \quad x\in\boldsymbol{R},$$

determines a polynomial function $g\colon \boldsymbol{R}\to X$ of p-th order. Moreover, if $x\in(x_0-a, x_0+a)$, then $t:=x-x_0\in(-a, a)$ and

$$\|f(x)-g(x)\| = \|f^*(t)-g^*(t)\| \leqslant l''_p\varepsilon,$$

which was to be proved.

Performing a suitable change of variables in Theorem 8 applied for $p = 1$, we are able to give an affirmative answer to a question of Kominek [7] concerning the stability of Jensen's equation on an interval.

COROLLARY 2. *If $a, b \in \boldsymbol{R}$, $a < b$ and $f\colon (a, b) \to X$ is a function such that*

$$\|2f(\tfrac{1}{2}(x+y)) - f(x) - f(y)\| \leqslant \varepsilon \quad \text{for } x, y \in (a, b),$$

then there exists an affine function $g\colon \boldsymbol{R} \to X$, i.e., the sum of a constant and an additive transformation, such that

$$\|f(x) - g(x)\| \leqslant 48\varepsilon \quad \text{for } x \in (a, b).$$

THEOREM 9. *Let $\varepsilon, r \in (0, \infty)$, $p \in \boldsymbol{N}$ and suppose $f\colon \left(-r\sqrt{1+(p+1)^2}, r\sqrt{1+(p+1)^2}\right) \to X$ is a function such that*

$$\|\Delta_y^{p+1} f(x)\| \leqslant \varepsilon \quad \text{for all } (x, y) \in \boldsymbol{R}^2 \text{ satisfying } x^2 + y^2 < r^2.$$

Then there exists a polynomial function $g\colon \boldsymbol{R} \to X$ of p-th order such that

$$\|f(x) - g(x)\| \leqslant m_p \varepsilon \quad \text{for } x \in \left(-r\sqrt{1+(p+1)^2}, r\sqrt{1+(p+1)^2}\right),$$

with a constant m_p depending on p only.

Proof. We apply the method used in the proof of Theorem 3 from [12]. Let $r_1 := r$ and select a $\varrho_1 \in (0, r_1)$ such that

$$\{(x, y) \in (-\varrho_1, \varrho_1)^2\colon x + (p+1)y \in (-\varrho_1, \varrho_1)\} \subset \{(x, y) \in \boldsymbol{R}^2\colon x^2 + y^2 < r_1^2\}.$$

By Theorem 7 there exists a polynomial function $g\colon \boldsymbol{R} \to X$ of p-th order such that

$$\|f(x) - g(x)\| \leqslant m_{p,1}\varepsilon \quad \text{for } x \in (-\varrho_1, \varrho_1).$$

Setting

$$r_2 := \frac{\varrho_1}{\sqrt{1+p^2}}, \qquad \varrho_2 := r_2\sqrt{1+(p+1)^2} = \varrho_1 \frac{\sqrt{1+(p+1)^2}}{\sqrt{1+p^2}},$$

we derive the following implication: if (x, y) varies in the disc $\{(x, y) \in \boldsymbol{R}^2\colon x^2 + y^2 < r_2^2\}$, then $x + ky$ falls into $(-\varrho_1, \varrho_1)$ for $k = 0, 1, \ldots, p$, whereas $x + (p+1)y$ runs over the whole interval $(-\varrho_2, \varrho_2)$.

Consequently, if the antecedent of the implication above is fulfilled, then

$$\begin{aligned}
&\|f(x+(p+1)y) - g(x+(p+1)y)\| \\
&\leqslant \|f(x+(p+1)y) - \Delta_y^{p+1} f(x) + \Delta_y^{p+1} g(x) - g(x+(p+1)y)\| + \|\Delta_y^{p+1} f(x)\| \\
&\leqslant \left\| \sum_{k=0}^{p} (-1)^{p+1-k} \binom{p+1}{k} g(x+ky) - \sum_{k=0}^{p} (-1)^{p+1-k} \binom{p+1}{k} f(x+ky) \right\| + \varepsilon \\
&\leqslant \sum_{k=0}^{p} \binom{p+1}{k} \|f(x+ky) - g(x+ky)\| + \varepsilon \leqslant (2^{p+1} - 1) m_{p,1}\varepsilon + \varepsilon =: m_{p,2}\varepsilon
\end{aligned}$$

which means that

$$\|f(x)-g(x)\| \leqslant m_{p,2}\varepsilon \quad \text{for } x\in(-\varrho_2, \varrho_2).$$

Similarly, setting

$$r_3 := \frac{\varrho_2}{\sqrt{1+p^2}}, \quad \varrho_3 := \varrho_2 \frac{\sqrt{1+(p+1)^2}}{\sqrt{1+p^2}} = \varrho_1 \left(\frac{\sqrt{1+(p+1)^2}}{\sqrt{1+p^2}}\right)^2$$

one can show by the same argument that there exists a constant $m_{p,3} > 0$ such that

$$\|f(x)-g(x)\| \leqslant m_{p,3}\varepsilon \quad \text{for } x\in(-\varrho_3, \varrho_3).$$

Since

$$\frac{\sqrt{1+(p+1)^2}}{\sqrt{1+p^2}} > 1,$$

after a finite number of analogous steps we arrive at

$$\|f(x)-g(x)\| \leqslant m_p\varepsilon \quad \text{for } x\in\left(-r\sqrt{1+(p+1)^2},\ r\sqrt{1+(p+1)^2}\right)$$

with some constant $m_p\in(0, \infty)$. This completes the proof.

Remarks. We have to admit that the constants k_p, k'_p, l_p, l'_p, and l''_p determined in Theorems 3–7 increase very fast as $p\to\infty$. We do not claim that our estimation of these constants is sharp and cannot be improved. What appears to be important is the fact that all these constants are independent of the size of intervals forming local domains of the functions we deal with. On the other hand, the stability of a functional equation is very often understood in the following way: if a function satisfies the equation only with a certain accuracy (up to a constant $\varepsilon > 0$), then it differs from a solution of the equation by less than a constant $\delta > 0$. In this formulation the relation between ε and δ is not of primary importance.

Finally, let us note that majority of our results can easily be extended from the case of the real line to that of an arbitrary finite-dimensional Euclidean space.

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