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# ON KINGMAN'S INTEGRAL INEQUALITIES FOR APPROXIMATIONS OF THE WAITING TIME DISTRIBUTION IN THE QUEUING MODEL $GI/G/1$ WITH AND WITHOUT WARMING-UP TIME

**1. On the model** $GI/G/1$. We consider the waiting model $GI/G/1$. Customer number zero has a waiting time distribution $W_0$. Then, as well known, the waiting time distributions $W_1, W_2, \ldots$ of the following customers are defined recursively by

$$W_n(x) = W_{n-1} * K(x), \quad x > 0, \ n = 1, 2, \ldots,$$

where $K$ is a certain distribution function (d. f.) with $0 < K(0) < 1$. If we have the weak convergence $W_n \Rightarrow W$ $(n \to \infty)$, where $W$ is a proper d. f., then $W$ is determined uniquely by the integral equation

$$W(x) = W*K(x), \quad x > 0,$$

or, equivalently,

$$(1) \qquad \overline{W}(x) = \overline{W}*K(x), \quad x > 0, \ \overline{W} = 1-W.$$

This is a *Wiener-Hopf equation*, and there exists an interesting theory how to solve it. But for the applications it is not very useful. We give two arguments for it:

(i) In practice, $K$ is not given exactly. Therefore, the theory cannot be applied.

(ii) Even if $K$ is known, $W$ often cannot be given explicitly but there is only a formula for the characteristic function of $W$.

The latter argument requires approximative solutions. As will be seen, we obtain a result which is also of interest in the case where $K$ is not known exactly.

Obviously, the d. f. $W_n$ are approximations for $W$. But $W_0$ plays an important role, as it is shown even by very easy considerations.

(i) If we choose $W_0$ in an awkward manner, then we will get a good approximation $W_n$ for $W$ only for $n \gg 1$; but then $W_n$ will be complicated because $n$ convolutions must be carried out.

(ii) The special d. f. $W_0 = \varepsilon_0$, where $\varepsilon_0$ is the unit step function concentrated in the origin, is certain awkward in the sense of (i), but $\varepsilon_0$ has a very pleasant property. In fact, as well known in this case, the approximation $W_n \geqslant W_{n+1} \geqslant W$ is monotonous. Therefore, every further step yields an improvement of the approximation.

The object of this paper is to give general conditions for which we obtain monotonous approximation. It was Kingman who introduced two integral inequalities as an effective means to provide estimations for $W$ (see [2]).

THEOREM 1. *Let $\varphi(x)$ be non-increasing, $\varphi(x) = 1$ $(x < 0)$, $\varphi(\infty) = 0$, satisfying*

$$\varphi(x) \geqslant \varphi * K(x) \quad \text{or} \quad \varphi(x) \leqslant \varphi * K(x), \qquad x > 0; \tag{2}$$

*then we have $\overline{W} \geqslant \varphi$ or $\overline{W} \leqslant \varphi$, respectively.*

If we replace the inequality signs in (2) by equality signs, then from (1) we obtain $\overline{W} = \varphi$. Under the additional assumption

$$\int_{-\infty}^{\infty} e^{\vartheta x} dK(x) = 1$$

for some $\vartheta > 0$ it was found in [2]

$$a e^{-\vartheta x} \leqslant \overline{W}(x) \leqslant e^{-\vartheta x}, \qquad x > 0, \tag{3}$$

where $a \geqslant 0$ is determined only by $K$.

We are of the opinion that Kingman's idea is very ingenious, but he did not draw all the conclusions which can be obtained easily. We show in [3] that the inequalities are of great methodological importance. They yield, for instance,

THEOREM 2. *If $\overline{W}_0$ is a solution of* (2), *then*

$$\overline{W} \leqslant \overline{W}_{n+1} \leqslant \overline{W}_n \quad \text{or} \quad \overline{W} \geqslant \overline{W}_{n+1} \geqslant W_n \tag{4}$$

*and all $\overline{W}_n$ are also solutions of* (2).

For instance, if we start with $\overline{W}_0(x) = e^{-\vartheta x} \geqslant \overline{W}(x)$ (cf. (3)), then by the following step we get

$$\overline{W}_0(x) \geqslant \overline{W}_1(x) = e^{-\vartheta x} \int_{-\infty}^{x} e^{\vartheta u} dK(u) + 1 - K(x) \geqslant \overline{W}(x).$$

Other possibilities of finding favourable initial solutions of (2) are provided by a theorem of Daley-Moran (see [1]). Let us compare three models $GI/G/1$ so that we have three different kernels $K^1$, $K$ and $K^2$ in (2). If we assume

$$K^1 \leqslant K \leqslant K^2, \tag{5}$$

then it follows (cf. [1]) that

$$\overline{W}^1 \geqslant \overline{W} \geqslant \overline{W}^2. \tag{6}$$

This result is useful for applications if $K^1$ and $K^2$ have a simpler form as $K$. It can also be applied, if $K$ is not known exactly, but only $K^1$ and $K^2$ in (5) are given.

THEOREM 3. *Under* (5), $\overline{W}^1$ *is a solution of* (2) *and* $\overline{W}^1$ *can be used as the initial approximation which can be improved successively.*

**2. The model $GI/G/1$ with warming-up time.** We now consider a generalization of the model $GI/G/1$ a description of which is given in [4]. Let $C$ be the d. f. of the warming-up time. The stationary waiting time distribution $W'$ in this model satisfies the integral equation

$$W'(x) = W' * K(x) - \beta\bar{C}(x), \quad x > 0, \quad \beta = W' * K(+0), \quad \bar{C} = 1 - C.$$

Analogously to theorem 1 we proved in [3]

THEOREM 4. *Let* $\psi(x)$ *be of bounded variation on* $\overline{0, \infty}$, $\psi(x) = 1$ $(x < 0)$, $\psi(\infty) = 0$, *satisfying*

$$\psi(x) \geqslant \psi * K(x) + \beta\bar{C}(x) \quad \text{or} \quad \psi(x) \leqslant \psi * K(x), \quad x > 0; \tag{7}$$

*then* $\psi(x) \geqslant \overline{W}'(x)$ *or* $\psi(x) \leqslant W'(x)$, *respectively, where* $\overline{W}' = 1 - W'$.

We can also generalize theorem 3 for this model.

THEOREM 5. *If the stationary d. f.* $W'$ *exists and* $\psi_0$ *satisfying the assumptions in theorem* 4, *then the sequence of functions*

$$\psi_n(x) = \begin{cases} \psi_{n-1} * K(x) + \beta\bar{C}(x), & x > 0, \\ 1, & x \leqslant 0, \end{cases} \qquad n = 1, 2, \ldots,$$

*tends weakly to* $\overline{W}'$ *for* $n \to \infty$. *Further, if* $\psi_0$ *is a solution of* (7), *then*

$$\psi_n \geqslant \psi_{n+1} \geqslant \overline{W}' \quad \text{or} \quad \psi_n \leqslant \psi_{n+1} \leqslant W', \quad n = 0, 1, \ldots,$$

*and all* $\psi_n$ *are solutions of* (7).

Note that it is not necessary that the functions $1 - \psi_n$ $(n = 0, 1, \ldots)$ are d. f. Some interesting initial solutions satisfying (7) were also given in [3].

**References**


[1] D. J. Daley and P. A. P. Moran, *Two-sided inequalities for waiting-time and queue-size distribution in GI/G/1*, Teor. Verojatnost. 13 (1968), p. 338-341.

[2] J. F. C. Kingman, *Inequalities in the theory of queues*, J. Royal Statist. Soc. B 32 (1970), p. 102-110.

[3] H.-J. Rossberg and G. Siegel, *Die Bedeutung von Kingmans Integralungleichungen bei der Approximation der stationären Wartezeitverteilung im Modell GI/G/1 mit und ohne Verzögerung beim Beginn einer Beschäftigungsperiode*, Operationsforschung u. Math. Statistik, submitted.

[4] — *The GI/G/1 model with warming-up time*, Lecture at the symposium on queuing theory, Karpacz 1973, this fascicle, p. 17-26.


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## O NIERÓWNOŚCIACH CAŁKOWYCH KINGMANA APROKSYMUJĄCYCH ROZKŁAD CZASU CZEKANIA W SYSTEMIE OBSŁUGI MASOWEJ *GI/G/1* Z ROZGRZEWANIEM I BEZ ROZGRZEWANIA


STRESZCZENIE

Praca zawiera najważniejsze wyniki dotyczące oszacowania stacjonarnych rozkładów czekania w systemie *GI/G/1* i oparte na nierównościach całkowych Kingmana (bez dowodów). Podkreślono, że istnieją także inne rozwiązania tych nierówności całkowych, nie podane przez Kingmana [2], że nierówności te mogą być wykorzystane do polepszenia otrzymanych oszacowań oraz że metoda daje się przenieść na ogólniejszy model *GI/G/1* z rozgrzewaniem.