ANNALES
POLONICI MATHEMATICI
XXXIII (1977)

Orbit properties of functions and "Pre-Abel" equations

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Abstract. Many theorems encountered in iteration theory and in connection with functional equations are algebraic in nature in spite of their analytic appearance. Proofs of such theorems rely on the very important notion of orbit, introduced by K. Kuratowski in 1924. This fact is demonstrated on the case of the "Pre-Abel" equation $\varphi\circ\gamma-\varphi = a$, where the "unknown" function φ takes values in a torsion-free Abelian group and a is invariant under right composition with γ. It is asked under what conditions this Pre-Abel equation is equivalent to the Abel equation $\varphi\circ\gamma-\varphi = \text{const}$. The answer is: if and only if either the orbit graph of γ is connected or every connected component of it contains a cycle. The theorem is then applied to the "torsion-free" analogues of a number of results recently found by Z. Moszner and by M. Kuczma.

1. Orbits as defined by Kuratowski [6], probably named and made more widely known by Whyburn [13], play a central part in iteration theory and in the theory of functional equations connected with iteration [2]. "Pre-Schröder" equations have been defined by the present author [10] as

$$(1.1)\qquad (\Pi_n)\qquad (\varphi\circ\gamma)^n = \varphi^{n-1}(\varphi\circ\gamma_n),\qquad n\geqslant 2$$

(where γ_n denotes the nth iterate of γ, "$\circ$" means composition and the upper indices denote powers) (1.1) has been extensively studied by M. Kuczma, Z. Moszner and the present author: [3]–[5], [7]–[9], [11], [12].

The main question, though not the only one, to be asked is whether, under a given set of conditions on the given function γ and the "unknown" function φ, (Π_2) implies (Π_n) or even implies the Schröder equation

$$(1.2)\qquad (S)\qquad \exists\lambda\ \varphi\circ\gamma = \lambda\varphi.$$

On the other hand, (S) implies (Π_n); in fact this is the way in which the Pre-Schröder equations were introduced [10].

The "Pre-Abel equations" to be defined pose an analogous, but in many respects different problem. In this paper we are going to prove a theorem about Pre-Abel equations and then discuss and prove results partly analogous to a number of theorems recently found by Moszner [9] and by Kuczma [4].

2. Let us now define "Pre-Abel" equations. S is an arbitrary set, $\mathscr{A}$ an (additively written) torsion-free Abelian group, γ a given mapping of (the whole of) S into itself, φ a mapping of (the whole of) S into $\mathscr{A}$. The non-trivial Pre-Abel equations are:

$$(2.1) \qquad (P_n) \qquad \varphi\circ\gamma_n - n(\varphi\circ\gamma)+(n-1)\varphi = 0, \quad n \geqslant 2.$$

Our insistence that $\mathscr{A}$ be torsion free is essential; without this, (P_n) is the additively written form of (Π_n) as interpreted in the case where $\mathscr{A} = \boldsymbol{C}-\{0\}$ or $\mathscr{A} = \boldsymbol{R}-\{0\}$. The multiplicatively written groups $\boldsymbol{C}-\{0\}$ and $\boldsymbol{R}-\{0\}$, however, are not torsion free and thus the generalization cannot be torsion free either.

The first Pre-Abel equation can be written, with $\Gamma\varphi \overset{\text{def}}{=} \varphi\circ\gamma$, in the form

$$(2.2) \qquad (P_2) \qquad (\Gamma - I)^2\varphi = 0.$$

We ask the following question. Under what conditions will (P_2) imply the Abel equation

$$(2.3) \qquad (A) \qquad \exists x_0 \in \mathscr{A} \; \varphi\circ\gamma - \varphi = x_0 ?$$

Here we must point out that (2.3) is more general than the Abel equation in the usual sense where $x_0 \neq 0$ is stipulated; for $\mathscr{A} = \boldsymbol{Z}$ usually $x_0 = 1$. (P_2) can be written in the suggestive form

$$(2.4) \qquad (\gamma) \qquad (\varphi\circ\gamma-\varphi)\circ\gamma = \varphi\circ\gamma - \varphi.$$

since $(\varphi\circ\gamma-\varphi)\circ\gamma = \varphi\circ\gamma_2 - \varphi\circ\gamma$. (γ) means that $\varphi\circ\gamma-\varphi$ is invariant under right composition with γ, that is, an automorphic function. Cf. [2], p. 41.

LEMMA 2.1. *The first non-trivial Pre-Abel equation is equivalent to*

$$(2.5) \qquad (a) \qquad \varphi\circ\gamma - \varphi = a, \quad a\circ\gamma = a.$$

In other words, (P_2) holds if and only if (a) holds, where a is a mapping of (the whole of) S into $\mathscr{A}$ invariant under right composition with γ.

Proof. If (P_2), preferably regarded in the form (γ) holds, then $a = \varphi\circ\gamma - \varphi$ is the desired invariant function. If (a) holds, then $\varphi\circ\gamma-\varphi = a = a\circ\gamma = (\varphi\circ\gamma-\varphi)\circ\gamma$ which is (γ); therefore (P_2).

We now give a number of orbit-theoretical expressions in order to make our terminology clear. A function f of a set S into itself can be represented by its "orbit graph" which is a labelled functional digraph. This is obtained by representing each element of S by a point in a suitably large space (if S has a power not greater than the power of the continuum, the Euclidean plane) and joining two points a and b by an arrow pointing from a to b if and only if $b = f(a)$. This is a directed graph (digraph); moreover, a functional digraph: each point may receive several arrows but issues at most one arrow (exactly one arrow if we indicate fixed points, $a = f(a)$, by an arrow issuing from a and immediately returning to it). The connected components of these functional digraphs are the orbits mentioned in Section 1. Orbits can be shown to be "quasi-in-trees" that is directed functional graphs in which there is no circuit with the possible exception of one directed cycle, in which case every point of the orbit is connected with a point of the cycle by a path. For the graph-theoretical notions we refer the reader to e.g. the book of F. Harary [1]. In special cases, the quasi-in-trees can be ω-chains or $\omega^* + \omega$-chains, with the usual set-theoretical notation for these order types. ω^*-chains, however, are not possible; for the last element of such a chain would have no arrow issuing from it in other words, it would have no image. This is impossible since we insisted that our functions are mappings from the whole set; in the language of logic: total and not partial functions.

Other special cases of quasi-in-trees are pure cycles (k-cycles, where k denotes the number of different elements in the cycle). 1-cycles are (special cases of) fixed points.

LEMMA 2.2 ([2], p. 42). *$a \circ \gamma = a$ if and only if a is constant on the orbits (the connected components of the orbit graph) of γ.*

LEMMA 2.3. *If $a \circ \gamma = a$ where $a = \varphi \circ \gamma - \varphi$, then on the "cyclic orbits" of γ (connected components of the orbit graph which contain a cycle), $a = 0$.*

Proof. Let the cycle be a k-cycle $\xi_0, \gamma(\xi_0), \gamma_2(\xi_0), \ldots, \gamma_k(\xi_0) = \xi_0$. From the conditions

$$(\varphi \circ \gamma - \varphi) \circ \gamma_m = \varphi \circ \gamma_{m+1} - \varphi \circ \gamma_m = a \circ \gamma_m = a \quad \text{for all } 0 \leqslant m \leqslant k.$$

Therefore

(2.6)

$$k a(\xi_0) = \sum_{m=0}^{k-1} (\varphi \circ \gamma_{m+1})(\xi_0) - (\varphi \circ \gamma_m)(\xi_0) = (\varphi \circ \gamma_k)(\xi_0) - \varphi(\xi_0) = 0.$$

Since $\mathscr{A}$ is torsion free, $ka = 0$ implies $a = 0$ on the cycle.

Now for the case of the quasi-tree, that is: a cyclic orbit which is not a pure cycle. As has been mentioned, it is easy to see that there is only one cycle in the orbit, but we do not use this fact now. On the cycle, as can be seen, $a = 0$, and by Lemma 2.2, a is constant on the entire orbit; therefore $a = 0$ on the entire quasi-tree. This concludes the proof.

Let us remark that an analogous statement for the Pre-Schröder equation would be false, since the multiplicatively written group considered there is not torsion free. We return to this problem when considering analogous statements for Pre-Schröder and Pre-Abel equations in Section 4.

3. We now give conditions for the equivalence

$$(P_2)\Leftrightarrow(A) \tag{3.1}$$

in terms of the orbit properties of γ.

a) We show that for (3.1), $(P_2)\Rightarrow(A)$ is necessary and sufficient; namely, always $(A)\Rightarrow(P_2)$. This we show in a fashion analogous to the procedure in the proof of Lemma 2.3. We find from (A) that $\varphi\circ\gamma_n-\varphi=nx_0$. But (A) also implies $n(\varphi\circ\gamma)-n\varphi=nx_0$; therefore $\varphi\circ\gamma_n-\varphi=n(\varphi\circ\gamma)-n\varphi$, which is (P_n). Thus $(A)\Rightarrow(P_n)\Rightarrow(P_2)$.

b) We note that $(P_2)\Rightarrow(A)$ is equivalent to $(a)\Rightarrow(A)$ since, according to Lemma 2.1, $(P_2)\Leftrightarrow(a)$. Thus (3.1) will hold if and only if $(a)\Rightarrow(A)$. This implication is more suggestive than the equivalent $(P_2)\Rightarrow(A)$, because it can be worded as follows. If, under a given set of conditions, $\varphi\circ\gamma-\varphi$ is invariant under right composition by γ, then it is a constant.

THEOREM 3.1. *The first Pre-Abel equation (P_2) implies the Abel equation if and only if the orbit graph of γ is either connected or each of its connected components contains a cycle.*

Proof. If the graph is connected, by Lemma 2.2 a is constant and (a) reduces to (A). If all connected components contain cycles, by Lemma 2.3 $a=0$ on the entire graph, again (a) reduces to (A), now with $x_0=0$. This shows that the condition is sufficient. It is also necessary. For, if the condition is not satisfied, the graph contains at least one proper tree T, that is a connected component which contains no cycle (not even a fixed point) and, beside T, at least another connected component, say C. On T, by Lemma 2.2 a is constant, but the constant need not be zero. We put

$$a(\xi)=\begin{cases} x_0\neq 0, & \xi\in T,\\ 0 & \xi\in C.\end{cases} \tag{3.2}$$

Thus a is not a constant and (a) does not imply (A). This concludes the proof.

4. We now apply Theorem 3.1 to situations analogous or similar to those treated by Moszner [9] and by Kuczma [4] in recent papers. Our main objective is to exhibit the "orbit-theoretical" character of the proofs given by Moszner and by Kuczma. Analogous to Moszner's theorem 1°a in [9] is

THEOREM 4.1. $(a)\Rightarrow(A)$ *if γ takes only two values and commutes these.*

Proof. Let the two values be a, b. Then $a=\gamma(b)$, $b=\gamma(a)$. The orbit graph of γ is built from the 2-cycle containing a and b, and a (finite or infinite) number of elements joined by an arrow either to a or to b. The orbit graph is thus connected, and Theorem 3.1 applies.

Next we consider the following theorem of Moszner about Pre-Schröder equations (2° in [9]):

Among the idempotents, it is only for $\gamma = \text{const}$ and $\gamma = I$ that $(a) \Rightarrow (S)$ follows. In our case the situation is different.

THEOREM 4.2. *If γ is idempotent, then $(a) \Rightarrow (A)$.*

Proof. $\gamma_2 = \gamma$ implies that all connected components of the orbit graph contain a fixed point. Each fixed point is either isolated (pure 1-cycle) or has a (finite or infinite) number of elements without predecessors ("initial elements"), that is elements of $S-\gamma(S)$ joined to it; for $\gamma_2 = \gamma$ means that $\gamma[\gamma(\xi)] = \gamma(\xi)$, i.e., the image of every point is a fixed point; the set $\gamma(S)$ consists of fixed points, and every element of $S-\gamma(S)$ is mapped into $\gamma(S)$ in exactly one step(1).

In any case, every connected component of the orbit graph contains a cycle (namely a fixed point) and Theorem 3.1 applies.

We now turn to Moszner's theorem 1°b in [9], which also holds in our case:

THEOREM 4.3. *If γ_2 is a constant, then $(a) \Rightarrow (A)$.*

Proof. Let $\gamma_2(\xi) = \xi_0$. There is one single fixed point, namely ξ_0, and any other element in S is joined to ξ_0 either in one step (if $\xi \in \gamma(S)$) or in two steps (if $\xi \in S-\gamma(S)$). Thus the orbit graph is connected and Theorem 3.1 applies.

Our next theorem corresponds to theorem 3° of Moszner in [9]. It states the following. Among the involutions, only $\gamma = I$ implies $(a) \Rightarrow (S)$. In our case, however, one finds

THEOREM 4.4. *If γ is an involution, then $(a) \Rightarrow (A)$.*

Proof. $\gamma_2 = I$ implies the existence of an inverse γ_{-1} with $\gamma = \gamma_{-1}$. Every connected component of the orbit graph of γ is either a fixed point or a 2-cycle, and Theorem 3.1 applies. In this case, it is easy to see why the statement does not holds in Moszner's case. The group $\boldsymbol{R}-\{0\}$ is not torsion free, in particular $(-1)^2 = 1$. In our notation of course the role of 1 is played by the zero element. Therefore it is possible to choose, on the 1-cycles, $a(x) = 1$ and, on the 2-cycles, $a(x) = -1$; then the analogue of (a) is still satisfied because $(-1)^2 = 1$ holds. (The analogue of $kx_0 = 0$ in the proof of our Lemma 2.3).

But a is not a constant, and thus (a) does not imply (A).

(1) One should feel free to draw diagrams when following reasonings of this type. This is (at least) part of the heuristic or intuitive approach to "orbit" arguments.

We now come to two theorems which are more special inasmuch as S is $\boldsymbol{R}$ and $\mathscr{A}$ is also $\boldsymbol{R}$ with addition as the group operation. The following is analogous to a theorem of Moszner (also 3° in [9]).

THEOREM 4.5. *Among the strictly increasing real functions, it is only for $\gamma(x) = x$ that the implication* $(a) \Rightarrow (A)$ *holds.*

For the proof we need

LEMMA 4.1. *Let γ be a strictly increasing real function. Any x_1 which is not a fixed point is an element of either an ω-chain or an $\omega^* + \omega$-chain.*

Proof. We have $\gamma(x_1) \neq x_1$. Let $x_1 < \gamma(x)_1$. The proof follows the same steps if $x_1 > \gamma(x_1)$. From $x_1 < \gamma(x_1)$ one finds by induction $\gamma_n(x_1) < \gamma_{n+1}(x_1)$; thus

$$x_1 < \gamma(x_1) < \ldots < \gamma_n(x_1) < \ldots$$

Since γ is strictly increasing, every element in the orbit containing x_1 has at most one predecessor, and thus the orbit is either an ω-chain or an $\omega^* + \omega$-chain. This concludes the proof.

Implicit in the proof of Lemma 4.1 is

LEMMA 4.2. *The orbit graph of a strictly increasing real function can have no other connected components than isolated fixed points, ω-chains and $\omega^* + \omega$-chains.*

Now for the proof of Theorem 4.5. If γ is not the identity, according to Lemma 4.1 the orbit graph contains at least one proper tree (namely, in this case an ω-chain or an $\omega^* + \omega$-chain) which thus contains no cycle. In order that Theorem 3.1 should still apply, the orbit graph should be connected, that is consist only of the chain containing x_1. This, however, is impossible, since the chain is countably infinite and $\boldsymbol{R}$ is of the power of the continuum. Thus there are other connected components and the necessary and sufficient conditions of Theorem 3.1 are not satisfied. This concludes the proof of Theorem 4.5.

This type of reasoning, however, need not be restricted to real functions. In fact, the following theorem of Kuczma ([5]) can be considered as a generalization of the part of Theorem 3° of Moszner in [9] which we quoted as a preliminary to Theorem 4.5. Kuczma's theorem is worded in our terminology. Let the orbit graph of γ have more than one connected component and let one component be a $2k$-cycle (a cycle consisting of an even number of elements); then (a) does not imply (S). In our case this does not follow. Again, in Kuczma's case the construction is possible because on the elements of the $2k$-cycle $a(x) = -1$ can be chosen, leading to $\varphi[\gamma_{2k}(x)] = (-1)^{2k}\varphi(x)$. In our case an analogous construction is impossible because our group $\mathscr{A}$ is torsion free.

We conclude this section with a generalization of Theorem 4.5 and a special case of Theorem 3.1

THEOREM 4.6. *Let S be a non-countably infinite set. Then, if the orbit graph of γ has a connected component which is an ω-chain or an $\omega^* + \omega$-chain,* (a) *does not imply* (A).

Proof. As in the proof of Theorem 4.5, for cardinality reasons the chain cannot be the only connected component; also, it contains no cycle. Thus the necessary and sufficient conditions of Theorem 3.1 are not fulfilled; this furnishes the proof.

5. It would seem that there are three levels on which one can formulate conditions for the type of theorems we have discussed.

I. Properties of the orbit graph of γ: $S \to S$.

II. Group-theoretical properties of $\mathscr{A}(a : S \to \mathscr{A})$.

III. Conditions depending on the particular choice of the mapping a: $S \to \mathscr{A}$.

The third type of conditions, however, did not occur in our discussion. We did use particular assignment of values from $\mathscr{A}$ to the elements of S, as a method of proof, but we did not formulate any condition in terms of any properties of the mapping a.

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Reçu par la Rédaction le 20. 5. 1974