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Continuous isometric semigroups and reflexivity

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Abstract. We consider the reflexivity of a WOT-closed algebra generated by continuous isometric semigroups parametrized by the semigroup of non-negative reals or the semigroup of finite sequences of non-negative reals. It is also proved that semigroups of continuous unilateral multi-parameter shifts are reflexive.

1. Introduction. The following standard notation is used. $B(H)$ denotes the algebra of all (linear, bounded) operators on a Hilbert space H. I_H or I stands for the identity in H. Let S be a semigroup (we only consider the semigroup of non-negative reals $\mathbf{R}_+$, or the semigroup of finite sequences of non-negative reals $\mathbf{R}_+^N$) with unit 0. A mapping $T(\cdot)$: $S \to B(H)$ is called a *semigroup* (*of operators*) if $T(0) = I$ and $T(s+t) = T(s)T(t)$ for all s, $t \in S$. If S is a topological semigroup and $T(\cdot)$ is continuous in SOT(= Strong Operator Topology in $B(H)$), then the semigroup of operators is called *strongly continuous*.

Only closed subspaces of H are considered and by an algebra (of operators) on H we mean a subalgebra of $B(H)$ with unit I_H. If $\mathscr{S} \subset B(H)$, then $\mathfrak{A}(\mathscr{S})$ stands for the WOT(= Weak Operator Topology)-closed algebra generated by $\mathscr{S}$ and $\operatorname{Lat}\mathscr{S}$ stands for the lattice of all invariant subspaces for $\mathscr{S}$. An algebra $\mathscr{A}$ is called *reflexive* if the algebra of all operators on H which leave invariant all subspaces from $\operatorname{Lat}\mathscr{A}$ is equal to $\mathscr{A}$. An operator $A \in B(H)$ (a semigroup of operators $T(s)$, $s \in S$, respectively) is called *reflexive* if the algebra $\mathfrak{A}(A)$ ($\mathfrak{A}(T(s)$: $s \in S)$, respectively) is reflexive.

Deddens [1] proved the reflexivity of an isometry. In [3] the reflexivity of pairs (two-element families) of isometries is considered. In the present paper the reflexivity of continuous isometric semigroups parametrized by $\mathbf{R}_+$ or $\mathbf{R}_+^N$ is considered. It is also proved that semigroups of continuous unilateral shifts are reflexive [Theorem 10]. The idea of this proof refers to the proof of the main theorem in [5].

2. One-parameter isometric semigroups

THEOREM 1. *If $T(s)$, $s \in \mathbf{R}_+$, is a one-parameter strongly continuous semigroup of contractions and T is its cogenerator then* $\mathfrak{A}(T) = \mathfrak{A}(T(s)$: $s \in \mathbf{R}_+)$.

Proof. To prove $\subset$, let us recall [8, III, Theorem 8.1] that

$$T = \text{SOT-}\lim_{s\to 0^+} \phi_s(T(s)) \quad \text{where } \phi_s(z) = \frac{z-1+s}{z-1-s}. \tag{1}$$

Since the Taylor series of ϕ_s, $s \in \mathbf{R}_+$, uniformly converges on the unit disc $\mathbf{D}$, $\phi_s(T(s))$ is a SOT-limit of polynomials in $T(s)$, for all $s \in \mathbf{R}_+$. Now, (1) implies that $T \in \mathfrak{A}(T(s)\colon s \in \mathbf{R}_+)$.

To prove $\supset$, we recall [8, III, Theorem 8.1] that

$$T(s) = e_s(T) \quad \text{where } e_s(z) = \exp\left[s\frac{z+1}{z-1}\right]. \tag{2}$$

[8, III, Theorem 2.3] implies that $e_s(T)$ is a SOT-limit of polynomials in T. Hence $T(s) \in \mathfrak{A}(T)$ for $s \in \mathbf{R}_+$. ∎

Theorem 1 implies

THEOREM 2. *A one-parameter strongly continuous semigroup of contractions is reflexive iff its cogenerator is reflexive.*

Since the cogenerator of a one-parameter strongly continuous semigroup of isometries is an isometry [8, III, Theorem 9.2], we have

THEOREM 3. *A one-parameter strongly continuous semigroup of isometries is reflexive.*

The following example shows a non-reflexive one-parameter strongly continuous semigroup of contractions:

EXAMPLE 4. Let

$$\alpha \geqslant 1 \quad \text{and} \quad T(s) = \exp(-\alpha s)\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \quad \text{for } s \in \mathbf{R}_+.$$

It is easy to see that $T(s)$, $s \in \mathbf{R}_+$, is a strongly continuous semigroup of contractions and its cogenerator is

$$T = \begin{bmatrix} \dfrac{\alpha-1}{\alpha+1} & \dfrac{-2}{(\alpha+1)^2} \\ 0 & \dfrac{\alpha-1}{\alpha+1} \end{bmatrix},$$

and also

$$\mathfrak{A}(T) = \left\{\begin{bmatrix} \lambda & \gamma \\ 0 & \lambda \end{bmatrix}\colon \lambda, \gamma \in \mathbf{C}\right\},$$

but this algebra is not reflexive [6, Example 9.26].

Now, let us note the following important

EXAMPLE 5. If $f \in L^2(0, \infty)$, then we define a one-parameter semigroup $\upsilon(s)$, $s \in \mathbf{R}_+$, of isometries as

$$\big(\upsilon(s)f\big)(x) = f(x-s) \quad \text{for } s \in \mathbf{R}_+,\ x \in \mathbf{R}_+$$

(we define $f(x) = 0$ if $x < 0$). It can be easily shown that $\upsilon(\cdot)$ is strongly continuous. Theorem 3 implies that $\upsilon(s)$, $s \in \mathbf{R}_+$, is reflexive.

The following lemma is a consequence of Theorem 1:

LEMMA 6. *If $T(s)$, $s \in \mathbf{R}_+$, is a one-parameter strongly continuous semigroup of contractions, T is its cogenerator and $A \in B(H)$ then $T(s)A = AT(s)$ for all $s \in \mathbf{R}_+$ iff $TA = AT$.*

3. Multi-parameter semigroups of isometries. In the whole chapter, $S = \mathbf{R}_+^N$, $G = \mathbf{R}^N$, m is the Lebesgue measure on G (we also write m for any restriction of m to simplify the notation) and $p_i\colon \mathbf{R} \to \mathbf{R}^N$, $p_i(t) = (0, \ldots, t, \ldots, 0)$, with t in the ith position. If $T(s)$, $s \in S$, is a semigroup then we set $T_i(t) = T\big(p_i(t)\big)$ for $t \in \mathbf{R}_+$ and $i = 1, \ldots, N$.

THEOREM 7. *If $T(s)$, $s \in S$, is a strongly continuous semigroup of contractions and T_i, $i = 1, \ldots, N$, is the cogenerator for the semigroup $T_i(\cdot)$, then $\mathfrak{A}\big(T(s)\colon s \in S\big) = \mathfrak{A}(T_i\colon i = 1, \ldots, N)$.*

Proof. Note that from Theorem 1 we have

$$\begin{aligned}\mathfrak{A}\big(T(s)\colon s \in S\big) &= \mathfrak{A}\big(T_i(\cdot)\colon i = 1, \ldots, N\big) = \mathfrak{A}\big(\mathfrak{A}(T_i(t)\colon t \in \mathbf{R}_+)\colon i = 1, \ldots, N\big)\\ &= \mathfrak{A}\big(\mathfrak{A}(T_i\colon i = 1, \ldots, N)\big) = \mathfrak{A}(T_i\colon i = 1, \ldots, N). \ \blacksquare\end{aligned}$$

Let us recall that if $A, B \in B(H)$ then A, B are *doubly commuting* iff A, B commute and A, B^* commute.

THEOREM 8. *If $T(s)$, $s \in S$, is a strongly continuous semigroup of isometries such that*

$$(3) \qquad T_i(t_1)T_j^*(t_2) = T_j^*(t_2)T_i(t_1) \quad \textit{for } t_1, t_2 \in \mathbf{R}_+,\ i \neq j,$$

then $\mathfrak{A}\big(T(s)\colon s \in S\big)$ is reflexive.

Proof. Let $i \neq j$. Then, from (3), Lemma 6 implies that $T_iT_j^*(t_2) = T_j^*(t_2)T_i$ for $t_2 \in \mathbf{R}_+$, where T_i denotes the cogenerator of $T_i(\cdot)$, $i = 1, \ldots, N$. Thus $T_j(t_2)T_i^* = T_i^*T_j(t_2)$ for $t_2 \in \mathbf{R}_+$ and now, also from Lemma 6, $T_i^*T_j = T_jT_i^*$. In the same way, we can show that $T_iT_j = T_jT_i$ for $i \neq j$. Hence, $\{T_i, i = 1, \ldots, N\}$ is a family of doubly commuting operators. [8, III, Theorem 9.2] shows that each T_i is an isometry. Note that [3, Theorem 1] can be generalized by induction to any finite sequence of doubly commuting isometries, thus $\mathfrak{A}(T_i\colon i = 1, \ldots, N)$ is reflexive, so $\mathfrak{A}\big(T(s)\colon s \in S\big)$ is also reflexive (Theorem 7). $\blacksquare$

Let us generalize Example 5 to the N-dimensional case.

EXAMPLE 9. Consider the space $L^2(S, \boldsymbol{m})$, with a semigroup of isometries

$$(T^0(s)f)(\phi) = f(\phi - s) \quad \text{for } f \in L^2(S, \boldsymbol{m}),\ s \in S,\ \phi \in S$$

(we assume that $f(\phi) = 0$ if $\phi \notin S$). It is easy to see that the semigroup $T^0(s)$ is strongly continuous. Now we show that $T^0(s)$, $s \in S$, fulfills assumption (3). Note that $(T^0(s)^* f)(\phi) = f(\phi + s)$ for ϕ, $s \in S$. If $i \neq j$, $t_1, t_2 \geqslant 0$, then we have for $f \in L^2(S, \boldsymbol{m})$ and $\phi \in S$

$$\begin{aligned}(T_i^0(t_1) T_j^0(t_2)^* f)(\phi) &= (T^0(p_i(t_1)) T^0(p_j(t_2))^* f)(\phi) \\ &= (T^0(p_j(t_2))^* f)(\phi - p_i(t_1)) = f(\phi - p_i(t_1) + p_j(t_2)) \\ &= (T^0(p_i(t_1)) f)(\phi + p_j(t_2)) = (T_j^0(t_2)^* T_i^0(t_1) f)(\phi).\end{aligned}$$

Thus $T_i^0(t_1)$, $T_j^0(t_2)$ fulfill (3). Let T_i^0 denote the cogenerator of the semigroup $T_i^0(t) = T^0(p_i(t))$, $i = 1, \ldots, N$. Now, in the same way as in Theorem 8, we can prove that T_i^0, T_j^0, $i \neq j$, are doubly commuting. Theorem 8 also shows that the algebra $\mathfrak{A}(T^0(s)\colon s \in S) = \mathfrak{A}(T_i^0\colon i = 1, \ldots, N)$ is reflexive.

In view of Examples 5 and 9 we can ask if the Theorem from [5] concerning the discrete situation can be generalized to the continuous case. Let us follow the terminology of [5]. A set $X \subset G$ is called a *diagram* if $g + s \in X$ for $g \in X$, $s \in S$. $\mathscr{X}$ denotes the set of all diagrams. For $g \in G$ we define $E_g = \{X \in \mathscr{X}\colon g \in X\}$ and $\mathscr{B}$ denotes the σ-algebra generated by all sets E_g, $g \in G$. Consider a positive finite measure μ on $(\mathscr{X}, \mathscr{B})$ and the space $\boldsymbol{K}$ of all measurable functions $f\colon G \times \mathscr{X} \to \boldsymbol{H}$ such that $\int \|f(g, X)\|^2\, d\boldsymbol{m} \otimes \mu < \infty$, $f(g, X) = 0$ $\boldsymbol{m} \otimes \mu$-a.e. on $\{(g, X)\colon X \notin E_g\}$ (we identify functions equal $\boldsymbol{m} \otimes \mu$-a.e.). Then $\boldsymbol{K}$ is a Hilbert space with inner product $(f_1, f_2) = \int (f_1(g, X), f_2(g, X))_{\boldsymbol{H}}\, d\boldsymbol{m} \otimes \mu$ for $f_1, f_2 \in \boldsymbol{K}$.

Now consider the following semigroup $T(s)$, $s \in S$, of operators on $\boldsymbol{K}$:

$$(T(s)f)(g, X) = f(g - s, X) \quad \text{for } f \in \boldsymbol{K}.$$

The space $\boldsymbol{K}$ is invariant for all $T(s)$, $s \in S$, since if we take $f \in \boldsymbol{K}$ and $g \notin X$ then $(g - s) + s \notin X$ and $g - s \notin X$, because X is a diagram. Thus $(T(s)f)(g, X) = f(g - s, X) = 0$ $\boldsymbol{m} \otimes \mu$-a.e.

THEOREM 10. *The algebra* $\mathfrak{A}(T(s)\colon s \in S)$ *is reflexive.*

Because of Theorem 7, to prove the above theorem it is convenient to use the cogenerators T_i of the semigroups $T_i(\cdot) = T(p_i(\cdot))$, $i = 1, \ldots, N$. However, we cannot use the Theorem of [5], since the spaces $\boldsymbol{K}$ considered here and in [5] are different, but we try to adapt the idea to our situation.

If $g \in G$, then we write $S_g = g + S$ and $L_g = L^2(S_g \times E_g, \boldsymbol{m} \otimes \mu|_{E_g}, \boldsymbol{H})$. The elements of L_g can be considered as functions defined on the whole $G \times \mathscr{X}$ if we define them to be 0 off $S_g \times E_g$. Then it is easy to see that L_g is a subspace of $\boldsymbol{K}$.

Remark 11. $K = \overline{\operatorname{span}\{L_g\colon g \in G\}}$.

Proof. Let $f \in K$ and $f \perp \operatorname{span}\{L_g\colon g \in G\}$. We denote by χ_g the characteristic function of the set $S_g \times E_g$. Let $g \in G$. Then $\chi_g f \in L_g$, thus $0 = (f, \chi_g f) = \int \|f|_{S_g \times E_g}\|^2 \, dm \otimes \mu$. Hence $f|_{S_g \times E_g} = 0$ $m \otimes \mu$-a.e. for all $g \in G$. It is easy to see that $\{(g, X) \in G \times \mathscr{X}\colon X \in E_g\} \subset \bigcup_{g \in G} S_g \times E_g$. So $f(g, X) = 0$ $m \otimes \mu$-a.e. on $\{(g, X) \in G \times \mathscr{X}\colon X \in E_g\}$. Since $f \in K$, $f(g, X) = 0$ $m \otimes \mu$-a.e. on $\{(g, X) \in G \times \mathscr{X}\colon X \notin E_g\}$. Hence $f = 0$ $m \otimes \mu$-a.e.

LEMMA 12. *The subspace L_g is invariant for $T(s)$, for all $s \in S$, $g \in G$.*

Proof. Let $f \in L_g$. Then $f(t, X) = 0$ if $(t, X) \notin S_g \times E_g$. We should show that, for all s, $(T(s)f)(t, X) = 0$ whenever $(t, X) \notin S_g \times E_g$. If $X \notin E_g$ then $(T(s)f)(t, X) = f(t-s, X) = 0$. If $t \notin S_g$ then $t-s \notin S_g$ and $(T(s)f)(t, X) = f(t-s, X) = 0$. ∎

Let $g \in G$. Then the subspace L_g is unitarily equivalent to $L^2(S \times E_g, m \otimes \mu|_{E_g}, H)$ (which is isomorphic to $L^2(S, m) \otimes L^2(E_g, \mu|_{E_g}, H)$ by a natural isomorphism), and let U_g define this unitary equivalence, i.e.

$$U_g\colon L^2(S_g \times E_g, m \otimes \mu|_{E_g}, H) \to L^2(S \times E_g, m \otimes \mu|_{E_g}, H),$$

$$(U_g f)(s, X) = f(s+g, X) \quad \text{for } f \in L^2(S_g \times E_g, m \otimes \mu|_{E_g}, H),\ s \in S,\ X \in \mathscr{X}.$$

Then for $s, t \in S$, $X \in \mathscr{X}$ and $f \in L^2(S_g \times E_g, m \otimes \mu|_{E_g}, H)$ we have

$$\begin{aligned}(U_g T(t) f)(s, X) &= (T(t)f)(s+g, X) = f(s-t+g, X) = (U_g f)(s-t, X)\\ &= \big((T^0(t) \otimes I_g) U_g f\big)(s, X) \quad \text{where } I_g = I|_{L^2(E_g, \mu|_{E_g}, H)}.\end{aligned}$$

Hence

$$U_g T(t) U_g^* = T^0(t) \otimes I_g \quad \text{for } t \in S,\ g \in G. \tag{4}$$

LEMMA 13. *If $g \leqslant h$, $g, h \in G$, then $U_h T(h-g)|_{L_g} = U_g$.*

Proof. Let $f \in L_g$, $\alpha \in S$, $X \in E_g$. Then

$$\begin{aligned}(U_h T(h-g) f)(\alpha, X) &= (T(h-g)f)(\alpha+h, X) = f(\alpha+h-h+g)\\ &= f(\alpha+g, X) = (U_g f)(\alpha, X). \ ∎\end{aligned}$$

LEMMA 14. *If $g \leqslant h$, $g, h \in G$, then*

$$U_g U_h^*|_{L^2(S,m) \otimes L^2(E_g, \mu|_{E_g}, H)} = T^0(h-g) \otimes I_g.$$

Proof. Let $f \in L^2(S \times E_g, m \otimes \mu|_{E_g}, H)$ and $s \in S$, $X \in \mathscr{X}$. Then

$$\begin{aligned}(U_g U_h^* f)(s, X) &= (U_h^* f)(s+g, X)\\ &= f(s+g-h, X) = \big(T^0(h-g) \otimes I_g\big)(s, X). \ ∎\end{aligned}$$

LEMMA 15. *If* $\operatorname{Lat}\{T(s)\colon s \in S\} \subset \operatorname{Lat} A$ *then* $AT(s) = T(s)A$ *for* $s \in S$.

Proof. Let $g \in G$. Then, by Lemma 12, $L_g \in \operatorname{Lat}\{T(s)\colon s \in S\} \subset \operatorname{Lat} A$, thus $\operatorname{Lat}\{T(s)|_{L_g}\colon s \in S\} \subset \operatorname{Lat} A|_{L_g}$. Now, (4) implies $\operatorname{Lat}\{T^0(s) \otimes I_g\colon s \in S\} = \operatorname{Lat}\{U_g T^0(s) U_g^*\colon s \in S\} \subset \operatorname{Lat} U_g A|_{L_g} U_g^*$. Since the algebra $\mathfrak{A}(T^0(s)\colon s \in S)$ is reflexive (Example 9), the algebra $\mathfrak{A}(T^0(s) \otimes I_g\colon s \in S)$ is also reflexive [6, Theorem 9.18, Corollary 9.19]. Hence $U_g A|_{L_g} U_g^* \in \mathfrak{A}(T^0(s) \otimes I_g\colon s \in S)$ and so $A|_{L_g} \in \mathfrak{A}(T(s)|_{L_g}\colon s \in S)$. Thus $A|_{L_g} T(s)|_{L_g} = T(s)|_{L_g} A|_{L_g}$. Remark 11 finishes the proof. ∎

Now for $g \in G$ consider the subspace $M_g = \{f \in K\colon f(h, X) = 0\ m \otimes \mu\text{-a.e. on } G \times (\mathscr{X} - E_g)\}$. A sequence $\{g_n\} \subset G$ is called *strongly increasing* if and only if $g_n^{(i)} < g_{n+1}^{(i)}$, $i = 1, \ldots, N$, where the $g_k^{(i)}$ are the coordinates of g_k.

Lemma 16. *If $g \leqslant h$ then $M_g \subset M_h$. If a sequence $\{g_n\} \subset G$ is strongly increasing then $\overline{\bigcup_{n \in \mathbb{N}} M_{g_n}} = K$.*

Proof. The first assertion is an immediate consequence of the inclusion $E_g \subset E_h$. To prove the other one, let $f \in K$ and $f \perp M_{g_n}$ for all n. Since g_n is strongly increasing, for any $g \in G$ there is g_n such that $g \leqslant g_n$. Then $L_g \subset M_g \subset M_{g_n}$. Remark 11 implies that $K \subset \overline{\bigcup_{g \in G} L_g} \subset \overline{\bigcup_{n \in \mathbb{N}} M_{g_n}}$. ∎

Proof of the Theorem. Let $g \in G$. Then, from Lemma 12, $L_g \in \operatorname{Lat}\{T(s)\colon s \in S\} \subset \operatorname{Lat} A$, thus $\operatorname{Lat}\{T(s)|_{L_g}\colon s \in S\} \subset \operatorname{Lat} A|_{L_g}$. Now, (4) implies

$$\operatorname{Lat}\{T^0(s) \otimes I_g\colon s \in S\} = \operatorname{Lat}\{U_g T^0(s) U_g^*\colon s \in S\} \subset \operatorname{Lat} U_g A|_{L_g} U_g^*. \tag{5}$$

The algebra $\mathfrak{A}(T^0(s)\colon s \in S) = \mathfrak{A}(T_i^0\colon i = 1, \ldots, N)$ is reflexive but we also need the following condition:

(6) For any $A \in \mathfrak{A}(T^0(s)\colon s \in S)$ there is $c \geqslant 0$ and a sequence of polynomials q_m such that $\|q_m(T_1^0, \ldots, T_N^0)\| \leqslant c$ and $q_m(T_1^0, \ldots, T_N^0)$ WOT-converges to A as $m \to \infty$.

The semigroups $T_i^0(\cdot)$, $i = 1, \ldots, N$, are completely non-unitary [8, III, §9.3], so the T_i^0, $i = 1, \ldots, N$, are shifts. Hence, the algebra $\mathfrak{A}(T_i^0\colon i = 1, \ldots, N)$ is unitarily equivalent to the WOT-closed algebra generated by the multiplication operators on the Hardy space $H^2(\Gamma^N)$ [7, Theorem 1]. Thus (6) is a consequence of the proof of the main theorem in [4].

(5) and (6) imply that there is $C > 0$ and a sequence of operators $\sigma_n^g \otimes I_g$ WOT-converging to $U_g A|_{L_g} U_g^*$, where σ_n^g is a polynomial in the operators T_i^0, $i = 1, \ldots, N$, and $\|\sigma_n^g \otimes I_g\| < C$. The above convergence also shows that $U_g A|_{L_g} U_g^* = A_g \otimes I_g$, where A_g is chosen appropriately.

Remark 11 implies that there is g_0 such that $L_{g_0} \neq \{0\}$ and we set $\sigma_n = \sigma_n^{g_0}$. The sequence $\sigma_n \otimes I_{g_0}$ WOT-converges to $U_{g_0} A|_{L_{g_0}} U_{g_0}^* = A_{g_0} \otimes I_{g_0}$, thus the sequence $U_{g_0}^*(\sigma_n \otimes I_{g_0}) U_{g_0}$ WOT-converges to $A|_{L_{g_0}}$. Let $h \geqslant g_0$. Then, like in [5] (p. 413) we can prove that $\sigma_n \otimes I_h$ WOT-converges to $A_h \otimes I_h = U_h A|_{L_{h\psi}} U_h^*$ on $L^2(s, m) \otimes L^2(E_h, \mu|_{E_h}, H)$.

Let $\{g_m\} \subset G$ be a strongly increasing sequence with first element g_0. For any m let us define a sequence of operators

$$\eta_n^m = U_{g_m}(\sigma_n \otimes I_{g_m}) U_{g_m}^*. \tag{7}$$

Hence, η_n^m WOT-converges to $A|_{L_{g_m}}(n \to \infty)$ and $\|\eta_n^m\| \leqslant C$. Let us extend η_n^m to the whole M_{g_m} in the following way. Set $M_{g_m}^l = M_{g_m} \cap L^2(S_{-l} \times E_{g_m}, \boldsymbol{m} \otimes \mu|_{E_{g_m}}, H)$ for all non-negative integers l and $\boldsymbol{l} = (l, \ldots, l)$. It is easy to see that $M_{g_m} = \overline{\bigcup_l M_{g_m}^l}$. We define

$$\eta_n^m f = T(\boldsymbol{l}+g_m)^* \eta_n^m T(\boldsymbol{l}+g_m) f \quad \text{for } f \in M_{g_m}^l. \tag{8}$$

Since $\alpha \in S_{g_m}$ if and only if $\alpha - \boldsymbol{l} - g_m \in S_{-l}$, thus $f \in M_{g_m}^l$ implies $T(\boldsymbol{l}+g_m) f \in L_{g_m}$, hence the definition (8) is correct. The operators σ_n are polynomials in the cogenerators T_i^0, $i = 1, \ldots, N$, thus the η_n^m are polynomials in the cogenerators $T_i|_{L_{g_m}}$ of the semigroups $T_i(\cdot)|_{L_{g_m}}$, by (7). Hence, η_n^m commutes with $T(s)$ on L_{g_m}. Therefore (8) gives a well-defined extension (independent of l) of η_n^m to $\bigcup_l M_{g_m}^l$. Also $\|\eta_n^m\| \leqslant C$ on L_{g_m}, and so, from (8), $\|\eta_n^m\| \leqslant C$ on $M_{g_m}^l$ for all l. We have thus extended η_n^m to the whole M_{g_m} with $\|\eta_n^m\| \leqslant C$. By (8), the extended η_n^m are polynomials in the cogenerators of the semigroups $T_i(\cdot)|_{M_{g_m}}$, $i = 1, \ldots, N$.

The following lemma, with the proof like that of Lemma 10 in [5], is needed:

LEMMA 17. *If* $m \leqslant k$, $n \in \mathbf{N}$, *then* $\eta_n^m = \eta_n^k|_{M_{g_m}}$.

Lemma 17 implies that we can define the operators $\bigcup_m \eta_n^m$ on $\bigcup_m M_{g_m}$ and $\|\bigcup_m \eta_n^m\| \leqslant C$. Thus, there is a sequence of operators $\beta_n \overset{\text{df}}{=} \overline{\bigcup_m \eta_n^m}$ on $K = \overline{\bigcup_m M_{g_m}}$ (Lemma 16) and $\|\beta_n\| \leqslant C$. Lemma 17 and the properties of closure imply that the β_n are polynomials in the cogenerators of the semigroups $T_i(\cdot)$, since the η_n^m have the analogous property. Using commutativity of A and $T(s)$, $s \in S$ (Lemma 15) we can prove in the same way as in [5] (p. 411) that β_n WOT-converges to A. Thus $A \in \mathfrak{A}(T_i\colon i = 1, \ldots, N) = \mathfrak{A}(T(s)\colon s \in S)$. ∎

We note an interesting

EXAMPLE 18. Consider $L^2(\Omega, \boldsymbol{m})$, where $\Omega = \{(x, y) \in \mathbf{R}^2\colon x \geqslant 0 \text{ or } y \geqslant 0\}$ and $\boldsymbol{m}$ is the Lebesgue measure on Ω, and the isometric semigroup $(T(s)f)(g) = f(g-s)$ for $s \in S, f \in L^2(\Omega, \boldsymbol{m}), g \in \Omega$ (we define $f(g) = 0$ if $g \notin \Omega$). If we take the measure $\mu = \delta_\Omega$ (the point mass at Ω) then Theorem 10 shows that the above semigroup is reflexive.

Example 18 shows that a natural unilateral translation (for the definition see [2]) on the semigroup of finite sequences of non-negative reals is reflexive. But a modified unilateral translation is unitarily equivalent to a natural unilateral translation (see [2]), hence we have

COROLLARY 19. *A modified unilateral translation on the semigroup of finite sequences of non-negative reals is reflexive.*

References

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