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PARABOLIC NETWORKS AND POLYNOMIAL GROWTH

BY

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1. Introduction and notation. Suppose that Γ is an infinite connected graph, without self-loops and multiple edges, whose vertices x have finite degree d_x.

Denote by V and E the vertex and edge set respectively. If x and y are neighbouring vertices, let us write $x \sim y$. For all p, $1 < p < \infty$, and every real function ϕ defined on V, the *Dirichlet sum* of ϕ *of order* p is defined as

$$D_p(\phi) = \sum_{x \sim y} |\phi(x) - \phi(y)|^p. \tag{1}$$

Choose, once for all, a reference vertex o and let L_0 denote the linear space of all real-valued finitely supported functions on V. We say that Γ is *parabolic of type* p, or p*-parabolic*, if, for some choice of o,

$$\inf\{D_p(\phi);\ \phi \in L_0 \text{ and } \phi(o) = 1\} = 0.$$

Otherwise we say that Γ is *hyperbolic of type* p. Note that if Γ is parabolic of order p, then it is parabolic of order s for all $s \geq p$. The notion of parabolic graph (or network) was introduced by Yamasaki [Y1] in analogy with the classification theory of Riemann surfaces. In fact, for $p = 2$ the sum (1) is the discrete analogue of the energy integral.

Let $d(x, y)$ denote the geodesic distance between two vertices x and y, i.e. the minimal number of edges of a (non-self-intersecting) path joining x and y. Let $E(r)$ denote the subset of all edges whose endpoints x_1 and x_2 satisfy $d(x_i, o) \leq r$ $(i = 1, 2)$. We say that Γ *has polynomial growth of order* p (where $1 < p < \infty$) if there is a constant μ such that, for all r, card $E(r) \leq \mu r^p$. The following theorem is our main result.

THEOREM. *If Γ has polynomial growth of order p, for some $1 < p < \infty$, then Γ is p-parabolic.*

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The proof is based on the study of parabolic and hyperbolic graphs (networks) due to Yamasaki and Kayano and Yamasaki (see [Y1], [Y2], [K-Y] and the references there). We will review some of their results in the next section.

Let us now give an interpretation of the theorem in terms of infinite non-linear networks. Suppose that every edge b of an infinite electrical network, represented by Γ, is assigned the following relation between the voltage v_b and the current i_b flowing in the edge:

$$v_b = \operatorname{sign}(i_b)|i_b|^{q-1} \tag{2}$$

where q is the dual exponent of p (such networks are a particular case of the networks studied in [DM-S] in the more general context of modular sequence spaces). When $q = 2$ the network is called *linear* or *ohmic* (with all the resistances equal to 1). The relation between linear networks, random walks on graphs and Dirichlet finite harmonic functions has been studied by several authors in recent years; see e.g. [C-W], [D], [F], [G], [L], [N-W], [S], [S-W], [T1], [T2], [Z].

An important problem consists in determining for which networks, in absence of current and voltage sources, Kirchhoff's equations (see e.g. [F], [S-W], and [DM-S] for the nonlinear case) do not admit nontrivial solutions in ℓ^q (the so-called "uniqueness" problem).

Let, for every $x \in V$ and every function ϕ defined on V,

$$\Delta_p(\phi)(x) = \sum_{x \sim y} \operatorname{sign}(\phi(x) - \phi(y))|\phi(x) - \phi(y)|^{p-1}.$$

The operator Δ_p is called the *(discrete) laplacian of order p*. If a function ϕ satisfies $\Delta_p(\phi)(x) = 0$ for all $x \in V$, then ϕ is called *p-harmonic*.

In the linear case the laplacian of order p is proportional to the usual laplacian associated with the simple random walk on the graph Γ (see the remark at the end of Section 3), and ϕ is 2-harmonic if and only if it is harmonic on Γ with respect to the latter operator.

It is known that the uniqueness problem has an affirmative answer in linear networks if and only if there are no nonconstant 2-harmonic functions having finite Dirichlet sums. For instance, this happens if Γ has polynomial growth and is vertex transitive (see [S-W]; for other uniqueness results see [S] and [T2]).

In the nonlinear case it follows from Kirchhoff's loop law and [F, Theorem on p. 328] that, if v_b is as in (2), then there exists ϕ on V (the potential) such that, for every edge $b = [x_1, x_2]$,

$$v_b = \phi(x_2) - \phi(x_1).$$

Then, as in the linear case (see [S-W]), it follows from Kirchhoff's node law that "uniqueness" holds (in ℓ^q) if and only if

(3) $$D_p(\phi) < \infty \text{ and } \Delta_p(\phi)(x) = 0 \quad \text{for all } x \in V$$

imply that ϕ is constant. Now, if Γ is p-parabolic it follows from [Y1, Theorem 3.2] (see Proposition 1 below) combined with [Y2, Lemma 2.3] that conditions (3) actually imply $\phi =$ const.

Therefore the above theorem is a "uniqueness" theorem for nonlinear networks of type (2) whose underlying graph has polynomial growth of order p.

2. Some properties of parabolic graphs. We norm L_0 by

(4) $$\|\phi\|_p = (|\phi(o)|^p + D_p(\phi))^{1/p}, \quad \phi \in L_0,$$

and denote by $\mathbf{D}_0^{(p)}$ the completion of L_0 with the norm (4). We also denote by $\mathbf{D}^{(p)}$ the Banach space of all (real-valued) functions ϕ on V such that the norm (4) is finite. Then $\mathbf{D}_0^{(p)}$ is a closed subspace of $\mathbf{D}^{(p)}$, see [Y1].

PROPOSITION 1 [Y1, Theorem 3.2]. *The following are equivalent:*

(a) *Γ is p-parabolic,*
(b) $1 \in \mathbf{D}^{(p)}$,
(c) $\mathbf{D}_0^{(p)} = \mathbf{D}^{(p)}$.

We come now to the notion of extremal length of order p of a set of path in Γ [K-Y, §2]. Let w be a nonnegative function on the edge set E. Its *energy of order* p $(1 < p < \infty)$, $H_p(w)$, is defined as

$$H_p(w) = \sum_{b \in E} w^p(b).$$

Let $\mathbf{P}$ be a set of one-sided (non-self-intersecting) infinite paths in Γ.

DEFINITION. The *extremal length of order p*, $\lambda_p(\mathbf{P})$, of $\mathbf{P}$ is defined as

(5) $$(\lambda_p(\mathbf{P}))^{-1} = \inf H_p(w),$$

where the infimum in (5) is taken over the set of all nonnegative w such that $H_p(w) < \infty$ and $\sum_{b \in E(\mathbf{p})} w(b) \geq 1$ for all paths $\mathbf{p} \in \mathbf{P}$ (here $E(\mathbf{p})$ denotes the edge set of $\mathbf{p}$).

If a property holds for all paths in $\mathbf{P}$ except for a subset of extremal length ∞ we will say that the property holds for p-almost all paths in $\mathbf{P}$.

For every $x_0 \in V$ let now $\mathbf{P}_{x_0}$ denote the set of all one-sided (non-self-intersecting) infinite paths having x_0 as first vertex. The following proposition characterizes p-parabolic networks.

PROPOSITION 2 (see [Y1, Theorem 4.1]). *Γ is parabolic of type p if and only if there exists $x_0 \in V$ such that $\lambda_p(\mathbf{P}_{x_0}) = \infty$.*

In the proof of our main result we will need the following result due to Kayano and Yamasaki.

PROPOSITION 3 [K-Y, Theorem 3.3]. *Let* $\phi \in \mathbf{D}_0^{(p)}$ *and* $x_0 \in V$. *Then, for p-almost every* $\mathbf{p} \in \mathbf{P}_{x_0}$, $\lim \phi(x) = 0$ *as* $x \to \infty$ *along the vertices of* $\mathbf{p}$.

3. Proof of the Theorem. We start with an elementary lemma.

LEMMA. *Let* $1 < p < \infty$ *and suppose that* Γ *has polynomial growth of order p. Let* e_r *denote the cardinality of* $E(r)$, $r = 0, 1, 2, \ldots$ *Then*

$$\sum_{r=2}^{\infty} \frac{e_{r+1} - e_r}{r^p \log^p r} < \infty.$$

Proof. Assume, without loss of generality, that $\mu = 1$ and $e_2 = 2^p$. Let $y(x)$ $(2 \le x < \infty)$ denote the function whose graph is obtained by joining the points (r, e_r) and $(r+1, e_{r+1})$ by line segments. Then

$$\sum_{r=2}^{\infty} \frac{e_{r+1} - e_r}{r^p \log^p r} \le \text{const} \cdot \int_2^{\infty} \frac{y'(x)}{x^p \log^p x}\, dx.$$

Let $t(x) = x^p - y(x)$ and, for every $M > 2$,

$$F_M(\theta) = \int_2^M \frac{y'(x) + \theta t'(x)}{x^p \log^p x}\, dx, \qquad 0 \le \theta \le 1.$$

Then

$$F'_M(\theta) = \frac{t(M)}{M^p \log^p M} + p \int_2^M \frac{t(x)}{x^{p+1} \log^{p+1} x}(1 + \log x)\, dx > 0.$$

Hence

$$\int_2^{\infty} \frac{y'(x)}{x^p \log^p x}\, dx = \lim_{M\to\infty} F_M(0)$$

$$\le \lim_{M\to\infty} F_M(1) = p \int_2^{\infty} \frac{1}{x^p \log^p x}\, dx < \infty. \ \blacksquare$$

Proof of the Theorem. For every positive real ρ let

$$\alpha(\rho) = (10)^{-1} \log(1 + \log(1 + \rho)), \qquad f(\rho) = \sin \alpha(\rho).$$

For all $x \in V$ let $|x| = d(x, o)$ denote the geodesic distance of x from the reference vertex o and let $\phi(x) = f(|x|)$.

Denote by $\rho_k = \exp(e^{10k\pi} - 1) - 1$ $(k = 1, 2, \ldots)$ the zeros of f. Let

$$\phi_k(x) = \begin{cases} \phi(x), & \text{for } |x| \le \rho_k, \\ 0, & \text{for } |x| \ge \rho_k. \end{cases}$$

Then ϕ_k belongs to L_0. We will show that $D_p(\phi-\phi_k) \to 0$, so that $\phi \in \mathbf{D}_0^{(p)}$.

Set $\psi_k = \phi - \phi_k$ and $S(r) = E(r+1)\backslash E(r)$. Let $[x,y]$ denote the (unoriented) edge having x and y as endpoints. Then

$$D_p(\psi_k) = \sum_{r>\rho_k-1}^{\infty} \sum_{[x,y]\in S(r)} |\psi_k(x) - \psi_k(y)|^p. \tag{6}$$

We have, for $|x| = r \geq \rho_k$, $|y| = r+1$,

$$|\psi_k(x) - \psi_k(y)|^p \leq \int_r^{r+1} |f'(\rho)|^p \, d\rho \leq \int_r^{r+1} h(\rho)\, d\rho \tag{7}$$

where $h(\rho) = (10\rho)^{-p} \log^{-p} \rho$.

If $\rho_k - 1 < r \leq \rho_k$ and $|x| = r$, $|y| = r+1$, then $\psi_k(x) = 0 = f(\rho_k)$, so that

$$|\psi_k(x) - \psi_k(y)|^p \leq \int_{\rho_k}^{r+1} |f'(\rho)|^p \, d\rho \leq \int_r^{r+1} h(\rho)\, d\rho. \tag{8}$$

There exists a positive constant κ such that, for large values of ρ and every ρ^* satisfying $|\rho - \rho^*| \leq 1$, $h(\rho)(h(\rho^*))^{-1} < \kappa$.

Let, as in the preceding Lemma, e_r denote the cardinality of $E(r)$. Since there are $e_{r+1} - e_r$ edges in $S(r)$, we have by (6), (7) and (8)

$$D_p(\psi_k) \leq \kappa (10)^{-p} \sum_{r>\rho_k-1}^{\infty} \frac{e_{r+1} - e_r}{r^p \log^p r},$$

so that $D_p(\psi_k) \to 0$ as $k \to \infty$ by the Lemma. Therefore $\phi \in \mathbf{D}_0^{(p)}$.

Now let $\mathbf{p}$ be any one-sided infinite (non-self-intersecting) path in Γ starting at the reference vertex o. There exist two sequences, say ρ_n and ρ_m, tending to infinity, such that

$$|f(\rho_n)| < 1/5, \qquad |f(\rho_m)| > 4/5.$$

Since $|f(\rho_1) - f(\rho_2)| \leq (10)^{-1}|\rho_1 - \rho_2|$ for all positive ρ_1 and ρ_2, there are two infinite sequences of vertices of $\mathbf{p}$, say $x_{k(n)}$ and $x_{k(m)}$, such that

$$|\phi(x_{k(n)})| < 2/5, \qquad |\phi(x_{k(m)})| > 3/5.$$

Hence $\phi(x)$ does not have a limit as x tends to infinity along the vertices of any path in $\mathbf{P}_o$. By Proposition 3 the extremal length of order p of $\mathbf{P}_o$ is ∞. But then, by Proposition 2, Γ is p-parabolic. This concludes the proof. ∎

Remark. The simple random walk on Γ mentioned in Section 1 is the Markov chain with state space V and probability d_x^{-1} of moving from x to a neighbour y. It is easy to show, on account of a theorem of Lyons [L, p. 394], that Γ is 2-parabolic if and only if the simple random walk is recurrent. The details of the proof are worked out in [So, Theorem 1].

Hence, by our Theorem, if Γ has quadratic growth then Γ is recurrent. This can be also deduced (with an argument similar to the one given in the above Lemma) from Nash-Williams' criterion for recurrence ([N-W, Theorem 2]; see also the paper [McG]).

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