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JOINT CONTINUITY OF FUNCTION SPACES

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In this note (1) we extend a well known result (e.g., see Kelley [2] regarding the compact-open topology by removing all assumptions on domain and range spaces and prove, using elementary methods,

THEOREM A. *Let (X, T) and (Y, S) be topological spaces and let F be the family of all functions $f: (X, T) \to (Y, S)$ which are continuous on each compact subset of (X, T). Then the compact-open topology for F is jointly continuous on compacta (in the sense of Kelley [2]).*

We use Pervin's axioms [3], in the notation of Davis [1], to prove a more general theorem concerning the joint continuity of a broad class of function space topologies.

In Davis' notation, Pervin's result can be stated:

The pair (X, T), where X is a set and T is a family of subsets of X, is a topological space iff there exists a family $[N_a: a \in I]$ of functions which assign to each $x \in X$ a subset $N_a(x) \subseteq X$ such that:

(i) for each $x \in X$, and each $a \in I$, $x \in N_a(x)$;

(ii) for each pair $a \in I$, $b \in I$ there exists $c \in I$ such that $N_c(x) \subseteq N_a(x) \cap N_b(x)$ for all $x \in X$;

(iii) for each $a \in I$ there exists $b \in I$ such that for any $x \in X$, if $z \in N_b(x)$ and $y \in N_b(z)$, then $y \in N_a(x)$;

(iv) given $a \in I$, $x \in X$, and $y \in N_a(x)$, there exists $b \in I$ such that $N_b(y) \subseteq N_a(x)$;

(v) $G \in T$ iff for each $x \in G$ there exists $a \in I$ such that $N_a(x) \subseteq G$.

In proving this theorem, Pervin showed that every topological space admits the following structure which is easily shown to satisfy (i) - (v). Let I be the set of all finite subfamilies of T. For each $a \in I$, let

$$N_a = \bigcap \left[(G \times G) \cup \big((X \sim G) \times X\big): G \in a\right].$$

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We note that $N_a(x) \in T$ for all $a \in I$ and all $x \in X$. Also, $N_a(x) = \bigcap [G \in a: x \in G]$. A structure on a space X which satisfies (i) - (iv) (and which, by means of (v), uniquely determines a topology T) is called an *Indexed System of Open Neighborhoods* or, briefly, an *ISON*. The particular ISON described above in terms of a topology T will be called the *Pervin ISON* for the space (X, T).

Let F be a set of functions defined on a set X with values in a topological space (Y, S) and let Q be a non-void family of subsets of X closed under finite unions. If $[N_a: a \in I]$ is the Pervin ISON for (Y, S), then $[R_d: d \in L]$ is easily seen to satisfy ISON properties (i) - (iv), where $L = Q \times I$, and for any $f \in F$ and any $d = (A, a) \in L$,

$$R_d(f) = \big[g \in F: g(x) \in N_a\big(f(x)\big) \text{ for each } x \in A\big].$$

The topology uniquely determined by this new ISON will be called the Q-topology for F.

THEOREM B. *Let (X, T) and (Y, S) be topological spaces, let Q be a non-void family of subsets of X which is closed under finite unions, and let F be a set of functions f: $(X, T) \to (Y, S)$. Then the Q-topology for F is jointly continuous on the sets of Q iff each $f \in F$ is continuous on each $A \in Q$.*

Proof. Let $[N_a: a \in I]$ and $[M_b: b \in J]$ be the Pervin ISONs, respectively, for (Y, S) and (X, T). Suppose that each $f \in F$ is continuous on each $A \in Q$. Fix $A \in Q$. We show that the function $\theta: F \times A \to Y$, where $\theta(f, x) = f(x)$, is continuous. Let $[(f_n, x_n): n \in D]$ be a net in $F \times A$ which converges to $(f, x) \in F \times A$. By ISON property (iii), for a given $a \in I$ there exists $b \in I$ such that $z \in N_b\big(f(x)\big)$ and $y \in N_b(z)$ imply $y \in N_a\big(f(x)\big)$. Since $[f_n: n \in D]$ converges to f in the Q-topology for F, there exists $m_1 \in D$ such that $n > m_1$ and $d = (A, b)$ imply that

$$f_n \in R_d(f) = \big[g \in F: g(x) \in N_b\big(f(x)\big) \text{ for all } x \in A\big].$$

In particular, $f_n(x_n) \in N_b\big(f(x_n)\big)$ if $n > m_1$. Since $[x_n: n \in D]$ converges to x and f is continuous, there exists $m_2 \in D$ such that $n > m_2$ implies $f(x_n) \in N_b\big(f(x_n)\big)$. Letting $m = \sup[m_1, m_2]$, we obtain that $n > m$ implies $f_n(x_n) \in N_b\big(f(x_n)\big)$ and $f(x_n) \in N_b\big(f(x)\big)$. Hence, by our choice of b, $n > m$ implies $f_n(x_n) \in N_a\big(f(x)\big)$, proving the continuity of θ on $F \times A$ from that of the functions $f \in F$ on A. The converse is obvious.

We now prove theorem A by showing that if Q is the set of all compact subsets of (X, T) then the Q-topology for F is the compact-open topology for F. Indeed, let $B = (C_1, 0_1) \cap \ldots \cap (C_n, O_n)$ be a base element of the compact-open topology for F, where $(C_i, O_i) = [f \in F: f(C_i) \subseteq O_i]$, and let $f \in B$. For each i, $1 \leqslant i \leqslant n$, let $d_i = (C_i, [O_i]) \in L$. By ISON property (ii) there exists $d \in L$ such that

$$R_d(f) \subseteq \bigcap [R_{d_i}(f): i = 1, \ldots, n] = B.$$

Thus, each open set in the compact-open topology is also open in the Q-topology (in conformity with Kelley [2]). Conversely, let $f \in F$ and $d = (C, a) \in L$ be given. If $[a_1, \ldots, a_k]$ denotes the set of all the subfamilies of a, for each i, $1 \leqslant i \leqslant k$, let $U_i = \bigcap [G: G \in a_i]$. Then for each $x \in C$, $N_a(f(x)) = U_i$ for some i, by the definition of Pervin ISON. The sets $C_i = [x \in C: N_a(f(x)) = U_i]$ are closed in C and thus compact since f is continuous on C and

$$C_i = C \sim f^{-1}(\bigcup [G: G \in a \sim a_i]).$$

Observe now that if $g \in R_d(f)$, then $g(C_i) \subseteq \bigcup [N_a(f(x)): x \in C_i] = U_i$ and thus

$$R_d(f) \subseteq (C_1, U_1) \cap \ldots \cap (C_k, U_k).$$

Moreover, if $g \in (C_1, U_1) \cap \ldots \cap (C_k, U_k)$ and $x \in C$, then $x \in C_i$ for some i and $g(x) \in U_i = N_a(f(x))$; that is, $(C_1, U_1) \cap \ldots \cap (C_k, U_k) \subseteq R_d(f)$. Thus each set $R_d(f)$ is open in the compact open topology.

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