COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 1

SOME SUFFICIENT CONDITIONS FOR REGULAR APPROXIMATE DIFFERENTIABILITY

BY

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1. Introduction. In this paper we give sufficient conditions for a measurable map f to be equivalent to one which is regularly approximately differentiable a.e. on an open set G in n-space. We prove (Theorem 1 (a)) that one such sufficient condition is the following: for $j = 1, \ldots, n$, the map f is equivalent to a map g_j which has an ordinary total differential a.e. along almost every hyperplane orthogonal to the j-th coordinate axis. In this connection, we point out that Fadell [4] has proved that if f is continuous on G and has an ordinary total differential a.e. along all hyperplanes orthogonal to the coordinate axes, then f is regularly approximately differentiable a.e. in G.

We also prove (Theorem 1 (b)) that if $p \in [1, \infty]$ and f has an L_p-differential a.e. with respect to all but one variable, then f has what we call a regular approximate L_p-differential a.e. (Definition 1).

These results, in conjunction with a theorem of A. Calderón and A. Zygmund (see [10], p. 242), lead to another sufficient condition which is given in terms of Sobolev norms of restrictions to hyperplanes (Corollary 3). This represents an extension of a theorem of Goffman and Ziemer [7].

In a parallel development, we prove (Theorem 2) that a map is continuous a.e. on almost every hyperplane orthogonal to a coordinate axis if and only if it is what we term regularly approximately continuous a.e. (Definition 2).

2. Notation, definitions and lemmas. For $w \in \boldsymbol{R}^n$, $\eta > 0$ and $E \subset \boldsymbol{R}^n$ let

$$w + \eta E = \{w + \eta u:\ u \in E\}.$$

Let

$$C = \{u \in \boldsymbol{R}^n:\ |u^j| \leqslant 1 \text{ for } j = 1, \ldots, n\}, \qquad K = \partial C,$$
$$P_j = \{u \in \boldsymbol{R}^n:\ u^j = 0\}$$

and let $\{e_j\}_{j=1,\ldots,n}$ be the standard basis for $\boldsymbol{R}^n$. Write $F_j = P_j \cap C$. Let p and r be m-tuples of extended real numbers. We shall write $p \geqslant r$ whenever $p^i \geqslant r^i$ for all $i = 1, \ldots, m$. If $t \in \boldsymbol{R}$, then $p \geqslant t$ means $p^i \geqslant t$ for all i.

Let G be an open subset of $\boldsymbol{R}^n$, and f a measurable map from G into $\boldsymbol{R}^m$. If $S \subset G$, then

$$\|f\|_S = \sup\{|f(u)|: u \in S\}.$$

If S is measurable and $p \geqslant 1$, then

$$\|f\|_{p,S} = \max_i \|f^i\|_{p^i,S},$$

where $\|\cdot\|_{t,S}$ is the usual norm in $L_t(S)$, $t \in [1, \infty]$. If S is k-dimensional, we mean for the norm to be taken with respect to k-dimensional Hausdorff measure.

Definition 1. Suppose that L is a linear map from $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ and that $w \in G$. Denote by A the affine map defined by $A(u) = f(w) + L(u-w)$. Then f is said to have L as

(a) a *regular approximate differential* at w (see [2], [3] and [9]) if

$$\operatorname*{ap\,lim}_{\eta \to 0+} \|(f-A)/\eta\|_{w+\eta K} = 0,$$

(b) a *regular approximate L_p-differential* at w if

$$\operatorname*{ap\,lim}_{\eta \to 0+} \max_i \eta^{-[(n-1)/p^i]} \|(f-A)/\eta\|_{p^i, w+\eta K} = 0.$$

We shall denote by D, L_pD, RAD and RAL_pD the spaces of measurable maps f with the property that some ($\mathscr{L}^n$ a.e.) equivalent map g has, a.e. in G, a differential, an L_p-differential (see [10], p. 242), a regular approximate differential and a regular approximate L_p-differential, respectively.

Again, respectively, we shall denote by HD and HL_pD the spaces of measurable maps f with the property that for each j there exists a g_j equivalent to f having a differential and an L_p-differential, $\mathscr{H}^{n-1}$ a.e. along $\mathscr{L}^1$ almost every hyperplane orthogonal to the j-th coordinate axis.

Obviously, for all $p \in [1, \infty]$,

$$D \subset RAD \subset RAL_pD \quad \text{and} \quad D \subset HD \subset HL_pD.$$

By Fubini's theorem,

$$D \subset L_pD \subset RAL_pD,$$

and by Hölder's inequality, if $p > r$, then

$$L_pD \subset L_rD, \quad RAL_pD \subset RAL_rD \quad \text{and} \quad HL_pD \subset HL_rD.$$

It follows from a theorem of Weiss [11], p. 103, that $L_pD \subset HL_pD$. We shall prove (Theorem 1) that

$$HD = HL_\infty D \subset RAL_\infty D = RAD$$

and, in fact, that

$$HL_pD \subset RAL_pD \quad \text{for all } p \in [1, \infty].$$

In this context, the following example may be of interest. Let $F \subset [0,1]$ be a (generalized) Cantor set of positive measure and put $\varphi = \chi_{F\times[0,1]}$. Then φ has an approximate differential a.e. (see [5], p. 212), but $\varphi \notin RAL_1D$.

Definition 2. We say that f is

(a) *regularly approximately continuous* at w if

$$\operatorname*{ap\,lim}_{\eta\to 0+} \|f(w)-f\|_{w+\eta K} = 0,$$

(b) *regularly approximately L_∞-continuous* at w if

$$\operatorname*{ap\,lim}_{\eta\to 0+} \|f(w)-f\|_{\infty,w+\eta K} = 0.$$

Denote by RAC and $RAL_\infty C$ the spaces of maps equivalent to some map which is regularly approximately continuous a.e. and regularly approximately L_∞-continuous a.e., respectively. Clearly,

$$RAD \subset RAC \quad \text{and} \quad RAL_\infty D \subset RAL_\infty C.$$

Denote by HC and $HL_\infty C$ the spaces of measurable maps f such that for every $j = 1, \dots, n$ there exists an equivalent map g_j with the property that, along $\mathscr{L}^1$ almost every hyperplane orthogonal to the j-th axis, g_j is continuous and L_∞-continuous, $\mathscr{H}^{n-1}$ a.e., respectively. That is, for almost every w in almost every hyperplane,

$$\lim_{\eta\to 0+} \|g_j(w)-g_j\|_{w+\eta F_j} = 0 \quad \text{and} \quad \lim_{\eta\to 0+} \|g_j(w)-g_j\|_{\infty,w+\eta F_j} = 0,$$

respectively.

HC contains the $(n-1)$-continuous maps of Goffman and Liu, for example (see [6]).

We shall prove that $HC = HL_\infty C = RAL_\infty C = RAC$ (Theorem 2).

Definition 3. The *upper boundary* $g = (g^1, \dots, g^m)$, in the sense of Blumberg [1], of a measurable map f is defined by

$$g^i(w) = \inf\{a\colon (f^i)^{-1}((-\infty, a]) \text{ has } w \text{ as a point of density}\}.$$

This is also called the *approximate upper limit* of f (see [5], p. 159). Here we use the convention of [5] that $\inf\emptyset = \infty$.

The proofs of the following lemmas are straightforward.

LEMMA 1. *If w is a point of approximate continuity of f, then $f(w) = g(w)$* (see [8]).

LEMMA 2. *Suppose that $w \in G$ and that $\varepsilon > 0$. If $|g^i(w)| < \infty$, then $\{u\colon |g^i(w) - g^i(u)| \leqslant \varepsilon\}$ has positive upper density at w. If $|g^i(w)| = \infty$, then $\{u\colon |g^i(u)| > 1/\varepsilon\}$ has positive upper density at w.*

Lemma 2 is a somewhat strengthened version of a statement in [8].

Enlarging on an argument of Neugebauer [8] (see also [12]), we have

LEMMA 3. *Suppose that $w \in G$.*

(a) *If f has an L_∞-differential at w, then g has a differential at w.*

(b) *If f has a regular approximate L_∞-differential at w and $f(w) = g(w)$, then g has a regular approximate differential at w.*

Proof. We prove only part (b) here, since (a) appears in [8]. For simplicity we take $m = 1$.

Fix $w \in G$ with $f(w) = g(w)$ and suppose that A is an affine map such that

$$\operatorname*{ap\,lim}_{\eta \to 0+} \|(f - A)/\eta\|_{\infty, w+\eta K} = 0.$$

Let $\varepsilon > 0$. Since f is approximately continuous a.e. (being measurable), by Lemma 1 there exists a set $X \subset (0, 1)$ with right density 1 at 0 and a set

$$N \subset \bigcup_{\eta \in X} (w + \eta K)$$

with the following properties:

(1) all points of X are points of density of X,

(2) for each $\eta \in X$, $\mathscr{H}^{n-1}\big(N \cap (w + \eta K)\big) = 0$,

(3) $|g(u) - A(u)|/\eta < \varepsilon$ for all $u \in \bigcup_{n \in X} [(w + \eta K) \sim N]$.

Since $f(w) = g(w)$, it suffices to show that $N = \emptyset$. Suppose not. Then there exist $\sigma > 0$, $\eta' \in X$ and $u' \in (w + \eta' K) \cap N$ such that

$$|g(u') - A(u')|/\eta' \geqslant \varepsilon + \sigma.$$

We suppose that $|g(u')| < \infty$, the argument in the infinite case being nearly identical. By Lemma 2, for each k,

$$E_k = \{u\colon |g(u') - g(u)| \leqslant 1/k\}$$

has positive upper density at u'. Hence there are a sequence $\xi_l \downarrow 0$ and a $\tau > 0$ such that, for all l,

$$\mathscr{L}^n(u' + \xi_l C) \cdot \tau < \mathscr{L}^n[E_k \cap (u' + \xi_l C)]$$
$$< \mathscr{L}^1\big[\{\eta\colon \mathscr{H}^{n-1}[E_k \cap (w + \eta K)] > 0\} \cap (\eta' - \xi_l, \eta' + \xi_l)\big] \cdot \mathscr{H}^{n-1}(u' + \xi_l K),$$

so that $\{\eta\colon \mathscr{H}^{n-1}[E_k \cap (w + \eta K)] > 0\}$ has positive upper density at η'.

Since η' is a point of density of X, there exists a sequence $\{u_k\}$ converging to u' with

$$\{u_k\} \subset \bigcup_{\eta \in X} [(w+\eta K) \sim N]$$

such that $g(u_k) \to g(u')$. But this would imply that

$$|g(u') - A(u')|/\eta' \leqslant \varepsilon,$$

a contradiction.

Similar reasoning leads to

LEMMA 4. *Suppose that f is measurable on G and $w \in G$.*

(a) *If f is L_∞-continuous at w, then g is continuous at w.*

(b) *If f is regularly approximately L_∞-continuous at w and $f(w) = g(w)$, then g is regularly approximately continuous at w.*

COROLLARY 1. (a) $RAL_\infty C = RAC$.

(b) $RAL_\infty D = RAD$.

Now we need a measurability lemma. For $\sigma = 1, 2, \ldots$ put

$$G_\sigma = \{w \in G\colon \operatorname{dist}(w, \partial G) \geqslant 1/\sigma\}.$$

For all j set

$$A_j(\sigma, k) = \{w \in G_j\colon |f(w) - f(w+y)| \leqslant 1/k \quad \text{for } \mathscr{H}^{n-1} \text{ almost every } y \in (1/\sigma)F_j\}.$$

Let $\operatorname{ap} D_j f^i$ stand for the approximate partial derivatives of f. Set

$$E = \{w \in G\colon \operatorname{ap} D_j f^i(w) \text{ exist at } w \text{ for all } i \text{ and } j\}.$$

For each $w \in E$ denote by L_w the linear map with matrix $[\operatorname{ap} D_j f^i(w)]$, with respect to the standard bases. Note that E is measurable and that, for almost all $w \in E$, $L_w = \operatorname{ap} Df(w)$, the approximate differential of f at w (see [5], p. 214).

Set

$$B_j^\infty(\sigma, k) = \{w \in G_\sigma \cap E\colon |f(w+y) - f(w) - L_w(y)|/|y| \leqslant 1/k \quad \text{for } \mathscr{H}^{n-1} \text{ almost every } y \in (1/\sigma)F_j\}.$$

Also, for each $w \in E$ denote by T_w the affine map defined by

$$T_w(u) = f(w) + L_w(u-w).$$

For $p \in [1, \infty)$ set

$$B_j^p(\sigma, k) = \{w \in G_\sigma \cap E\colon \max_i (\lambda^{-[(n-1)/p^i]-1}) \cdot \|f - T_w\|_{p^i, w+\lambda F_j} \leqslant 1/k \quad \text{for all } \lambda \in (0, 1/\sigma)\}.$$

We have the following

LEMMA 5. *Let f be measurable. For all j, σ, k and p,*

(a) *$A_j(\sigma, k)$ is measurable,*

(b) *$B_j^p(\sigma, k)$ is measurable.*

Proof. We shall prove only (b), the proof of (a) being similar. For simplicity we assume that $m = 1$. First, let $p = \infty$ and fix j, σ and k. Let

$$Y = \{(w, y) \in E \times (F_j \sim \{0\}): w + y \in G\}$$

and define ζ on Y by

$$\zeta(w, y) = |f(w+y) - f(w) - L_w(y)| / |y|.$$

Then Y and ζ are $(\mathscr{L}^n \times \mathscr{H}^{n-1})$-measurable. Let

$$S = \{(w, y) \in Y: w \in G_\sigma, y \in (1/\sigma)F_j \text{ and } \zeta(w, y) \leqslant 1/k\}.$$

Define a function h on E by

$$h(w) = \mathscr{H}^{n-1}(\{y \in (F_j \sim \{0\}): (w, y) \in S\}).$$

Then S (and hence h) is $\mathscr{L}^n$-measurable. Therefore,

$$B_j^\infty(\sigma, k) = h^{-1}(\{(2/\sigma)^{n-1}\})$$

is $\mathscr{L}^n$-measurable as required.

For the case $p \in [1, \infty)$, fix $\lambda \in (0, 1/\sigma)$ and define a measurable function on $S \times (\lambda F_j)$ by

$$\xi^\lambda(w, y) = \begin{cases} |f(w+y) - f(w) - L_w(y)| / \lambda & \text{if } (w, y) \in Y, \\ 0 & \text{if } (w, y) \notin Y. \end{cases}$$

Then, by Fubini's theorem, the function on S defined by

$$\theta^\lambda(w) = \lambda^{-[(n-1)/p]} \Big[\int_{\lambda F_j} |\xi^\lambda(w, y)|^p \, d\mathscr{H}^{n-1}(y) \Big]^{1/p}$$

is measurable. Hence, by the dominated convergence theorem,

$$B_j^p(\sigma, k) = \bigcap_{\lambda \in (0, 1/\sigma)} (\theta^\lambda)^{-1}([0, 1/k]) = \bigcap_{r \in Q \cap (0, 1/\sigma)} (\theta^r)^{-1}([0, 1/k])$$

is measurable.

Set

$$M_j = \bigcap_{k=1}^{\infty} \bigcup_{\sigma=1}^{\infty} A_j(\sigma, k) \quad \text{and} \quad N_j^p = \bigcap_{k=1}^{\infty} \bigcup_{\sigma=1}^{\infty} B_j^p(\sigma, k).$$

Then M_j is the set of w at which $f|_{w+F_j}$ is L_∞-continuous and N_j^p is the set of $w \in E$ at which the L_p-differential of $f|_{w+F_j}$ exists and is given by the approximate partial derivatives of f at w.

Set

$$M = \bigcap_{j=1}^{n} M_j \quad \text{and} \quad N^p = \bigcap_{j=1}^{n} N_j^p.$$

We have

COROLLARY 2. (a) *If* $f \in HL_\infty C$, *then* $\mathscr{L}^n(G \sim M) = 0$.

(b) *If* $p \in [1, \infty]$ *and* $f \in HL_p D$, *then* $\mathscr{L}^n(G \sim N^p) = 0$.

Proof. First, in connection with part (b), we note that if a map has l as an L_p-differential at a point, then l is also the approximate differential at this point (see [12]). In turn, if a map has an approximate differential a.e., then it has approximate partial derivatives a.e. (This follows, for example, from a linear density argument and the fact that the map may be approximated in a Lusin sense by continuously differentiable maps — see [5], p. 228.) Furthermore, the approximate differential is given a.e. by the approximate partial derivatives.

Let $f \in HL_p D$ and apply the reasoning above to restrictions of f to hyperplanes. In particular, it follows that f has approximate partial derivatives a.e. in almost every hyperplane. Hence $\mathscr{L}^n(G \sim E) = 0$ since the domains of the approximate partial derivatives of any measurable map are measurable (see [5], p. 214).

The rest of the proof is an easy application of Fubini's theorem and Lemma 5.

We can now prove

LEMMA 6. (a) $HL_\infty C = HC$.

(b) $HL_\infty D = HD$.

Proof. By previous remarks we need only to prove that $HL_\infty C \subset HC$ and $HL_\infty D \subset HD$. Fix j. Let $f \in HL_\infty C$ (respectively, $f \in HL_\infty D$). Define a map g_j on C by letting $g_j(w)$ be the value of the upper boundary of $f|_{w+F_j}$ at w. By Corollary 2 (a), $f|_{w+F_j}$ is $\mathscr{L}^{n-1}$ approximately continuous at w for $\mathscr{L}^n$ almost every $w \in G$. Hence, by Lemma 1, $g_j = f$ $\mathscr{L}^n$ a.e. We are done by Lemmas 3 and 4, since $g_j|_{w+F_j}$ is continuous (respectively, has a differential) whenever $f|_{w+F_j}$ is L_∞-continuous (respectively, has an L_∞-differential).

3. The main theorems. Corollary 2 also leads to the following, the proof of which was originally inspired by a similar argument, for $n = 2$, in [3], p. 408.

THEOREM 1. (a) $HD \subset RAD$.

(b) *If* $p \in [1, \infty]$, *then* $HL_p D \subset RAL_p D$.

Proof. By Corollary 1 and Lemma 6, it suffices to prove part (b). Let $m = 1$ and fix $p \in [1, \infty]$. Let $f \in HL_p D$ so that $\mathscr{L}^n(G \sim N^p) = 0$. We show that f has L_w (see the remarks before Lemma 5) as a regular approximate L_p-differential a.e. in N^p. Note that, as in the proof of Corollary 2, the approximate partial derivatives $\operatorname{ap} D_j f$ of f exist a.e. in G and are measurable. Let $\varepsilon, \tau > 0$.

Choose k' such that $1/k' < \tau$. By Lemma 5, $\{B_j^p(\sigma, k')\}_{\sigma=1}^{\infty}$ is an increasing sequence of measurable sets and, by Corollary 2,

$$\mathscr{L}^n\Big(G \sim \bigcup_{\sigma=1}^{\infty} B_j^p(\sigma, k')\Big) = 0 \quad \text{for each } j.$$

Hence there exists σ' such that

$$\mathscr{L}^n\big(G \sim B_j^p(\sigma', k')\big) < \varepsilon/2n \quad \text{for each } j.$$

Thus, by Lusin's theorem, there exist a (closed) set

$$H \subset \bigcap_{j=1}^{n} B_j^p(\sigma', k')$$

and a $\delta \in (0, 1/\sigma)$ such that $\mathscr{L}^n(G \sim H) < \varepsilon$ and

$$|\operatorname{ap} D_j f(v) - \operatorname{ap} D_j f(w)| < \tau \quad \text{for all } j,$$

whenever $v, w \in H$ and $|v - w| < \delta$. Let S be the set of points of H at which H has linear density 1 in each coordinate direction.

For all $w \in S$ let

$$P_w = \{\eta \in (0, \delta)\colon w \pm \eta e_j \in S \text{ and } |f(w \pm \eta e_j) - f(w) - \operatorname{ap} D_j f(w)\cdot \eta| < \tau\eta \quad \text{for all } j\}.$$

Then P_w has right density 1 at 0 for every $w \in S$.

Fix $w \in S$, $\eta \in P_w$ and j. Let $v = w \pm \eta e_j$. Then, for all $u \in v + \eta F_j$,

$$|u - v| \leqslant \eta\sqrt{n-1}/2$$

and

$$|f(u) - f(w) - L_w(u - w)| \leqslant |f(u) - f(v) - L_v(u - v)| + |f(v) - f(w) - L_w(v - w)| + |L_v(u - v) - L_w(u - v)|.$$

Hence, by Minkowski's inequality,

$$(2n)^{-1/p}\|f - T_w\|_{p, w+\eta K} \leqslant (2\eta)^{(n-1)/p}[\tau\eta + \tau\eta + \tau\eta(n-1)\sqrt{n-1}/2],$$

where $T_w(u) = f(w) + L_w(u - w)$, as before. Thus there exists a constant $\gamma > 0$ (depending only on n and p) such that, for all $w \in S$ and all $\eta \in P_w$,

$$\eta^{-[(n-1)/p]}\|(f - T_w)/\eta\|_{p, w+\eta K} \leqslant \gamma\tau.$$

This proves the theorem since $\mathscr{L}^n(G \sim S) < \varepsilon$, and ε and τ are arbitrary.

We obtain a stronger result in the case of continuity, namely:

THEOREM 2. $HC = RAC$.

The analogous proposition for the case of differentiability remains to be investigated.

In view of Corollary 1 and Lemma 6, Theorem 2 is an immediate consequence of

LEMMA 7. $HL_\infty C = RAL_\infty C$.

Proof. $HL_\infty C \subset RAL_\infty C$ follows from a simplification of the proof of Theorem 1.

To prove the reverse inclusion, we suppose that $f \notin HL_\infty C$. Then $\mathscr{L}^n(G \sim M) > 0$, so there exists j such that

$$0 < \mathscr{L}^n\Big(G \sim \bigcap_{k=1}^{\infty} \bigcup_{\sigma=1}^{\infty} A_j(\sigma, k)\Big) = \mathscr{L}^n\Big(\bigcup_{k=1}^{\infty} \bigcap_{\sigma=1}^{\infty} \big(G \sim A_j(\sigma, k)\big)\Big).$$

Hence, by Lemma 5, there exists k' such that

$$\mathscr{L}^n\Big(\bigcap_{\sigma=1}^{\infty}\big(G \sim A_j(\sigma, k')\big)\Big) > 0.$$

Set

$$X = \bigcap_{\sigma=1}^{\infty}\big(G \sim A_j(\sigma, k')\big).$$

There is a closed set $Y \subset X$, $\mathscr{L}^n(Y) > 0$, such that f is continuous on Y. Let Z be the set of points of Y at which Y has linear density 1 in the j-th coordinate direction and let $w \in Z$. Then there exists $H_w \subset (0, 1)$ with right density 1 at 0 such that, for every $\eta \in H_w$,

$$|f(w+\eta e_j) - f(w)| < 1/2k' \quad \text{and} \quad \|f(w+\eta e_j) - f\|_{\infty, w+\eta e_j+\eta F_j} \geqslant 1/k'.$$

Hence

$$\|f(w) - f\|_{\infty, w+\eta K} > 1/2k' \quad \text{for all } \eta \in H_w,$$

so f is not regularly approximately L_∞-continuous at w. The lemma follows by contraposition since $\mathscr{L}^n(Z) > 0$.

4. A corollary. We now mention some related results regarding Sobolev maps. Denote by $W_p^{\mathrm{loc}}(G)$ the space of equivalence classes of maps in $L_p^{\mathrm{loc}}(G)$ whose distributional partial derivatives are functions in $L_p^{\mathrm{loc}}(G)$ (see [10]). We define $HW_p^{\mathrm{loc}}(G)$ to be the space of (equivalence classes of) maps whose restrictions to $\mathscr{L}^1$ almost every hyperplane P, orthogonal to a coordinate axis, are in $W_p^{\mathrm{loc}}(G \cap P)$. By Fubini's theorem,

$$W_p^{\mathrm{loc}}(G) \subset HW_p^{\mathrm{loc}}(G).$$

Let V be the interior of C. Then, for $p \in (1, \infty)$, the map $g(u) = (|u^1| + |u^2|)^{-1/p}$ is easily seen to be in $HW_p(V) \sim W_p(V)$. Accordingly, the following is a slight generalization of a theorem in [7].

COROLLARY 3. (a) *If* $p > n-1$, *then*

$$HW_p^{\mathrm{loc}}(G) \subset RAD.$$

(b) *If* $1 \leqslant p < n-1$ *and* $\mu = p(n-1)/[(n-1)-p]$, *then*

$$HW_p^{\mathrm{loc}}(G) \subset RAL_\mu D.$$

Proof. By [10], p. 242, if $p > n-1$, then $HW_p^{\mathrm{loc}} \subset HD$, and if $1 \leqslant p < n-1$, then $HW_p^{\mathrm{loc}} \subset HL_\mu D$. The corollary follows from Theorem 1.

Parts of this paper are based on work done at Purdue University in the author's Ph. D. Thesis. The author wishes to express his deep gratitude to Professors E. Silverman, C. Goffman, C. Neugebauer and R. Kaufman for the helpful suggestions and criticisms.

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Reçu par la Rédaction le 13. 4. 1977