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MATRIX TRANSFORMATION METHOD OF APPROXIMATE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction. In this paper we present a numerical method of calculating the approximate values of the function $u(x, y)$ which over the given rectangle

$$D:\{a \leqslant x \leqslant b \leqslant \infty,\ c \leqslant y \leqslant d\}$$

satisfies the differential equation

$$(1.1)\qquad a_1(x)\frac{\partial^4 u}{\partial x^4}+a_2(x)\frac{\partial^4 u}{\partial x^2\partial y^2}+a_3(x)\frac{\partial^4 u}{\partial y^4}+a_4(x)\frac{\partial^3 u}{\partial x^3}+$$

$$+a_5(x)\frac{\partial^3 u}{\partial x\partial y^2}+a_6(x)\frac{\partial^2 u}{\partial x^2}+a_7(x)\frac{\partial^2 u}{\partial y^2}+a_8(x)\frac{\partial u}{\partial x}+a_9(x)u=f(x, y)$$

and the following conditions:

1° over the straight lines $y = c$ and $y = d$ the boundary conditions

$$(1.2_1)\qquad a_{1j}\frac{\partial^2 u}{\partial y^2}\bigg|_{y=c}+a_{2j}\frac{\partial u}{\partial y}\bigg|_{y=c}+a_{3j}u(x, c)=\varphi_j(x),\qquad j=1, 2,$$

$$(1.2_2)\qquad a_{1j}\frac{\partial^2 u}{\partial y^2}\bigg|_{y=d}+a_{2j}\frac{\partial u}{\partial y}\bigg|_{y=d}+a_{3j}u(x, d)=\varphi_j(x),\qquad j=3, 4,$$

where $a_{ij} = \text{const}$;

2° over the straight lines $x = a$ and $x = b$ the boundary conditions (or initial conditions over one of these lines) of the form

$$(1.3)\qquad b_{1j}\frac{\partial^3 u}{\partial x^3}+b_{2j}\frac{\partial^3 u}{\partial x\partial y^2}+b_{3j}\frac{\partial^2 u}{\partial x^2}+b_{4j}\frac{\partial^2 u}{\partial y^2}+b_{5j}\frac{\partial u}{\partial x}+$$

$$+b_{6j}u=\psi_j(y),\qquad j=1, 2, 3, 4,$$

where b_{ij} are constants.

Obviously, we assume that the above-formulated problem has a unique solution over the rectangle D.

The matrix transformation method described here makes it possible to solve this problem. This method is a modification and, at the same time, a generalization of the method given by Polozhiĭ ([1]-[7]) and called *the method of summary representation.*

Polozhiĭ's method reduces a given linear partial differential equation with constant coefficients to n independent difference equations whose solutions determine the approximate values of the function $u(x, y)$ in the knots of a rectangular net covering the rectangle D.

The matrix transformation method reduces the given boundary or mixed problem to n independent linear ordinary differential equations whose analytical or numerical solutions determine the approximate values of the function $u(x, y)$ over the straight lines $y = y_k = c+kh$ $[k = 1, 2, \ldots, n; h = (d-c)/(n+1)]$.

Let us remark that the matrix transformation method can be applied to problems in which both the form of equation (1.1) and the boundary conditions (1.2_1), (1.2_2) and (1.3) may be more general.

2. Matrices of simple structure. In the matrix transformation method important role is played by matrices similar to diagonal ones which are called *matrices of simple structure.*

In the present paper we use the following matrix of simple structure depending on two parameters, α and β, such that $|\mathrm{tg}\,\alpha| = |s| \leqslant 1$ and $|\mathrm{tg}\,\beta| = |t| \leqslant 1$:

$$T = \begin{bmatrix} s & 1 & 0 & & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 1 & 0 \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & 0 & 1 & 0 \; 1 & 0 \\ 0 & \ldots & & 0 & 1 \; 0 & 1 \\ 0 & \ldots & & & 0 \; 1 & t \end{bmatrix}$$

It is easy to see that

$$T^2-2E = \begin{bmatrix} s^2-1 & s\;1\;0 & & \ldots\; 0 \\ s & 0\;0\;1\;0 & & \ldots\; 0 \\ 1 & 0\;0\;0\;1\;0 & & \ldots\; 0 \\ 0 & 1\;0\;0\;0\;1\;0 & & \ldots\; 0 \\ \cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0\;\ldots & & 0\;1\;0\;0\;0\;1 & 0 \\ 0\;\ldots & & 0\;1\;0\;0\;0 & 1 \\ 0\;\ldots & & 0\;1\;0\;0 & t \\ 0\;\ldots & & 0\;1 & t\;\; t^2-1 \end{bmatrix}$$

where E is the unit matrix.

It is also easily seen that (see [1])

$$T^i = P\Lambda^i P^*, \; PP^* = E, \quad i = 1, 2, \ldots, \tag{2.1}$$

where $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is a diagonal matrix formed of the eigenvalues of matrix T, and P is a fundamental matrix whose rows are eigenvectors $\boldsymbol{p}_j$ of the matrix T (P^* denotes the transposed matrix). In order to find eigenvalues of the matrix T of order n we have to solve the equation

$$\sin(n+1)Q + (s+t)\sin nQ + st\sin(n-1)Q = 0, \tag{2.2}$$

whose roots Q_j determine both eigenvalues and eigenvectors of the matrix T. Namely, we get formulae

$$\lambda_j = 2\cos Q_j, \qquad \boldsymbol{p}_j = c_j(p_{1j}, \ldots, p_{nj}),$$

where

$$p_{ij} = \cos a \sin iQ_j - \sin a \sin(i-1)Q_j,$$

$$c_j = \left[\sum_{i=1}^{n} \left(\cos a \sin iQ_j - \sin a \sin(i-1)Q_j\right)^2\right]^{-\frac{1}{2}}.$$

The constants c_j are so chosen that the vectors $\boldsymbol{p}_j$ be normed.

Example 2.1. Let T_1 be the matrix obtained from T by substituting $\alpha = \beta = 0$. We have

$$T_1 = \begin{vmatrix} 0 & 1 & 0 & & & & \ldots & 0 \\ 1 & 0 & 1 & 0 & & & \ldots & 0 \\ 0 & 1 & 0 & 1 & 0 & & \ldots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots 0 & 1 & 0 & 1 & 0 & & 0 \\ 0 & \ldots & 0 & 1 & 0 & 1 & & 0 \\ 0 & \ldots & & 0 & 1 & 0 & & 1 \\ 0 & \ldots & & & 0 & 1 & & 0 \end{vmatrix}, \qquad T_1^2 - 2E = \begin{vmatrix} -1 & 0 & 1 & 0 & & & & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & & & \ldots & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & & \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots 0 & 1 & 0 & 0 & 0 & 1 & & 1 \\ 0 & \ldots & 0 & 1 & 0 & 0 & 0 & & 1 \\ 0 & \ldots & & 0 & 1 & 0 & 0 & & 0 \\ 0 & \ldots & & & 0 & 1 & 0 & & -1 \end{vmatrix}.$$

For the matrix T_1 we easily get the eigenvalues and the eigenvectors. Equation (2.2) assumes the form

$$\sin(n+1)Q = 0$$

whence

$$Q_j = \frac{j\pi}{n+1}, \qquad j = 1, 2, \ldots, n.$$

Eigenvalues and eigenvectors of the matrix T_1 are given by the following formulae:

(2.3) $$\lambda_j = 2\cos\frac{j\pi}{n+1},$$

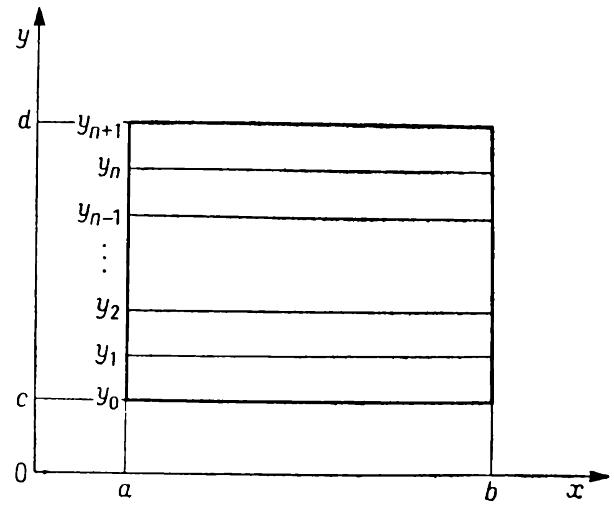
(2.4) $$p_j = \sqrt{\frac{2}{n+1}}(\sin j\gamma, \sin 2j\gamma, \ldots, \sin nj\gamma), \qquad \gamma = \frac{\pi}{n+1}.$$

The formulae

(2.5) $$T_1 = P_1 \Lambda_1 P_1, \qquad P_1^2 = E.$$

are also true.

3. Reduction of a boundary or mixed problem to n independent ordinary differential equations. Divide into $n+1$ strips the rectangle D by the straight lines $y = y_k = c+kh$, $k = 1, 2, \ldots, n; h = (d-c)/(n+1)$.



On each straight line $y = y_k$ we approximate equation (1.1) by the difference-differential equation

(3.1) $$a_1 u_k^{(4)} + a_2 \alpha \delta^2 u_k'' + a_3 \alpha^2 \delta^4 u_k + a_4 u_k''' + a_5 \alpha \delta^2 u_k' + $$
$$+ a_6 u_k'' + a_7 \alpha \delta^2 u_k + a_8 u_k' + a_9 u_k = f_k, \qquad k = 1, 2, \ldots, n,$$

where

$$u_k(x) = u(x, y_k), \qquad f_k(x) = f(x, y_k), \qquad \alpha = 1/h^2,$$

and by $\delta^2 w_k$ and $\delta^4 w_k$ we denote the second and the fourth central differences, respectively.

The system of equations (3.1) thus obtained may also be written as

$$(3.2)\qquad Au_k+B(u_{k-1}+u_{k+1})+C(u_{k-2}+u_{k+2}) = f_k, \qquad k = 1, 2, \dots, n.$$

where A, B and C denote the following linear and homogeneous differential operators:

$$Av = a_1 v^{(4)} + a_4 v''' + (a_6 - 2a_2 a)v'' + (a_8 - 2a_5 a)v' + (a_9 - 2a_7 a + 6a_3 a^2)v,$$

$$Bv = a_2 a v'' + a_5 a v' + (a_7 a - 4a_3 a^2)v,$$

$$Cv = a_3 a^2 v.$$

Let us now successively substitute in (3.2) the values $k = 1, 2, \dots, n$. The corresponding transformation of the two first and two last equations gives the following system of equations:

$$(3.3)\quad \begin{cases} Au_1+B(su_1+u_2)+C[(s^2-1)u_1+su_2+u_3] = r_1 = f_1+B(su_1-u_0)+ \\ \qquad\qquad + C[su_2+(s^2-1)u_1-u_{-1}], \\ Au_2+B(u_1+u_3)+C(su_1+u_4) = r_2 = f_2+(Csu_1-u_0), \\ Au_3+B(u_2+u_4)+C(u_1+u_5) = r_3 = f_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Au_{n-2}+B(u_{n-3}+u_{n-1})+C(u_{n-4}+u_n) = r_{n-2} = f_{n-2}, \\ Au_{n-1}+B(u_{n-2}+u_n)+C(u_{n-3}+tu_n) = r_{n-1} = f_{n-1}+ \\ \qquad\qquad +B(tu_n-u_{n+1}), \\ Au_n+B(u_{n-1}+tu_n)+C[u_{n-2}+tu_{n-1}+(t^2-1)u_n] = r_n = f_n+ \\ \qquad\qquad +B(tu_n-u_{n+1})+C[tu_{n-1}+(t^2-1)u_n-u_{n+2}]. \end{cases}$$

Let us now introduce the following vectors:

$$\boldsymbol{u}(x) = \big(u_1(x),\quad u_2(x), \dots, u_n(x)\big),$$

$$\boldsymbol{r}(x) = \big(r_1(x),\quad r_2(x), \dots, r_n(x)\big).$$

Using the matrix T we can give to system (3.3) the matrix form

$$(3.4)\qquad A\boldsymbol{u}+B(T\boldsymbol{u})+C[(T^2-2E)\boldsymbol{u}] = \boldsymbol{r}.$$

We shall show that the two first and the two last unknown components of the vector $\boldsymbol{r}$ can be determined by boundary conditions (1.2_1) and (1.2_2), respectively.

We determine only the components r_1 and r_2 by conditions (1.2_1), since the determination of the components r_{n-1} and r_n by conditions (1.2_2) is analogous.

Let us first remark that r_1 and r_2 depend on functions u_2, u_1, u_0 and u_{-1}. Therefore, boundary conditions (1.2_1) will be approximated by the difference expressions consisting only of those functions. Replacing the derivative of first order by the expressions

$$\frac{u_1-u_0}{h}, \quad \frac{u_1-u_{-1}}{2h}, \quad \frac{u_2-4u_1+3u_0}{2h},$$

and the derivative of second order by

$$\frac{u_2-2u_1+u_0}{h^2}, \quad \frac{u_1-2u_0+u_{-1}}{h^2},$$

we can approximate each of the conditions (1.2_1) by four different, linearly independent, expressions:

$$S_{1j} \equiv \frac{a_{1j}}{h^2}(u_2-2u_1+u_0)+\frac{a_{2j}}{h}(u_1-u_0)+a_{3j}u_0 = \varphi_j(x),$$

$$S_{2j} \equiv \frac{a_{1j}}{h^2}(u_2-2u_1+u_0)+\frac{a_{2j}}{2h}(u_1-u_{-1})+a_{3j}u_0 = \varphi_j(x),$$

$$S_{3j} \equiv \frac{a_{1j}}{h^2}(u_2-2u_1+u_0)+\frac{a_{2j}}{2h}(u_2-4u_1+3u_0)+a_{3j}u_0 = \varphi_j(x),$$

$$S_{4j} \equiv \frac{a_{1j}}{h^2}(u_1-2u_0+u_{-1})+\frac{a_{2j}}{h}(u_1-u_0)+a_{3j}u_0 = \varphi_j(x), \quad j = 1, 2.$$

Take now two systems of numbers $(x_1, x_2, \ldots, x_8)$, such that the expression $x_1S_{11}+x_2S_{21}+x_3S_{31}+x_4S_{41}+x_5S_{12}+x_6S_{22}+x_7S_{32}+x_8S_{42}$ is identically equal to su_1-u_0 for one system of numbers x_i and to $su_2+(s^2-1)u_1-u_{-1}$ for the other one.

To get both systems of numbers x_i we obtain two systems of linear algebraic equations which after corresponding transformations assume the form

$$(3.5)\quad \begin{cases} 2a_{11}(x_1+x_2)+(2a_{11}+ha_{21})x_3+2a_{12}(x_5+x_6)+(2a_{12}+ha_{22})x_7 = p_1, \\ a_{21}(x_1+x_2-x_3+x_4)+a_{22}(x_5+x_6-x_7+x_8) = p_2, \\ a_{31}(x_1+x_2+x_3+x_4)+a_{32}(x_5+x_6+x_7+x_8) = p_3, \\ \quad ha_{21}x_2-2a_{11}x_4+ha_{22}x_6-2a_{12}x_8 = p_4, \end{cases}$$

where in place of (p_1, p_2, p_3, p_4) we substitute for the first system $(0, hs, s-1, 0)$, and for the other one $(2h^2s, h(s^2+2s), s^2+s-2, 2h^2)$.

The Kronecker-Capelli theorem implies that system (3.5) has a solution if its matrix

$$A = \begin{bmatrix} 2a_{11} & 2a_{11} & (2a_{11}+ha_{21}) & 0 & 2a_{12} & 2a_{12} & (2a_{12}+ha_{22}) & 0 \\ a_{21} & a_{21} & -a_{21} & a_{21} & a_{22} & a_{22} & -a_{22} & a_{22} \\ a_{31} & a_{31} & a_{31} & a_{31} & a_{32} & a_{32} & a_{32} & a_{32} \\ 0 & ha_{21} & 0 & -2a_{11} & 0 & ha_{22} & 0 & -2a_{12} \end{bmatrix}$$

and the matrix obtained by adding to A a row of free terms are of the same order. It follows from this that, when solwing system (3.5), in some cases besides of determining the unknowns x_i we have also to determine the values of the parametr s. Moreover, let us remark that for the two systems of equations the value of s must be the same, and the vector $\boldsymbol{r}$ must depend on two boundary conditions. This implies further restrictions on the parameters, as is shown by the following examples.

Example 3.1. Let over the straight line $y = c$ be given the boundary conditions

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=c} = \varphi_1(x), \qquad u(x, c) = \varphi_2(x).$$

We have:

$$a_{11} = 1, \quad a_{21} = 0, \quad a_{31} = 0, \quad a_{12} = 0, \quad a_{22} = 0, \quad a_{32} = 1.$$

Let us add to the matrix A the right-hand sides of two systems:

$$A = \left[\begin{array}{cccccccc|c|c} 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2h^2 s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & hs & h(s^2+2s) \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & s-1 & s^2+s-2 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2h^2 \end{array}\right].$$

It is easy to see that the systems (3.5) have infinitely many solutions for $s = 0$ and are inconsistent for $s \neq 0$. The solutions are easy to find. For example:

System I: $x_1 = x_2 = x_3 = x_4 = 0,\ x_5 = -1,\ x_6 = x_7 = x_8 = 0.$
System II: $x_1 = x_2 = x_3 = 0,\ x_4 = -h^2,\ x_5 = -2,\ x_6 = x_7 = x_8 = 0.$

Other possible solutions bring nothing new.

Expressions $su_1 - u_0$ and $su_2 + (s^2-1)u_1 - u_1$ contained in r_1 and r_2 may be presented in the form

$$-u_0 = -S_{12} = -\varphi_2(x),$$

$$-u_1 - u_{-1} = -h^2 S_{41} - 2S_{12} = -h^2\varphi_1(x) - 2\varphi_2(x)$$

whence

$$r_1(x) = f_1(x) - B\varphi_2(x) - C[h^2\varphi_1(x) + 2\varphi_2(x)],$$

$$r_2(x) = f_2(x) - C\varphi_2(x).$$

Example 3.2 Let us now consider over the straight line $y = c$ the boundary conditions of the form

$$\left.\frac{\partial u}{\partial y}\right|_{y=c} = \varphi_1(x), \qquad u(x, c) = \varphi_2(x).$$

In this case corresponding transformations give

$$A = \left[\begin{array}{cccccccc|c|c} 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 2h^2s \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & hs & h(s^2+4s-2) \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & s-1 & s^2+s-2 \\ 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2h^2 \end{array}\right]$$

whereas for two systems (3.5) we may assume the following solutions:
System I: $x_1 = hs,\ x_2 = x_3 = x_4 = 0,\ x_5 = s-1,\ x_6 = x_7 = x_8 = 0.$
System II: $x_1 = h(s^2+4s-2),\ x_2 = 2h,\ x_3 = 2hs,\ x_4 = 0,$
$x_5 = s^2+s-2,\ x_6 = x_7 = x_8 = 0.$

Hence we find

$$su_1 - u_0 = hs\varphi_1(x)+(s-1)\varphi_2(x),$$

$$su_2+(s^2-1)u_1-u_{-1} = h(s^2+6s)\varphi_1(x)+(s^2+s-2)\varphi_2(x).$$

which implies

$$r_1(x) = f_1(x)+B[hs\varphi_1(x)+(s-1)\varphi_2(x)]+C[h(s^2+6s)\varphi_1(x)+ \\ +(s^2+s-2)\varphi_2(x)],$$

$$r_2(x) = f_2(x)+C[hs\varphi_1(x)+(s-1)\varphi_2(x)].$$

It is easy to see that if we substitute $s = 0$, then neither r_1 nor r_2, and hence r, depend on the function $\varphi_1(x)$. Similarly, for $s = 1$ the vector $\boldsymbol{r}$ does not depend on $\varphi_2(x)$. This implies that instead of s we can take any number different from 0 and 1.

Let us now perform a *matrix transformation* of system (3.4), multiplying the left-hand side of this system by the matrix P. In view of (2.1) we get

$$A(P^*\boldsymbol{u})+B(\Lambda P^*\boldsymbol{u})+C[(\Lambda^2-2E)P^*\boldsymbol{u}] = P^*\boldsymbol{r}.$$

Writing $\boldsymbol{v} = P^*\boldsymbol{u}$ and $\boldsymbol{s} = P^*\boldsymbol{r}$, we obtain for this system the form

$$A\boldsymbol{v}+B(\Lambda\boldsymbol{v})+C[(\Lambda^2-2E)\boldsymbol{v}] = \boldsymbol{s}$$

or the component form

$$Av_k+\lambda_k Bv_k+(\lambda_k^2-2)Cv_k = s_k, \quad k = 1, 2, \ldots, n. \tag{3.6}$$

Taking into account the form of operators A, B and C and performing corresponding transformations, we obtain n independent ordinary linear differential equations

$$a_{1k}(x)v_k^{(4)}(x)+a_{2k}(x)v_k'''(x)+a_{3k}(x)v_k''(x)+a_{4k}(x)v_k'(x)+ \\ +a_{5k}(x)v_k(x) = s_k(x), \quad k = 1, 2, \ldots, n. \tag{3.6}$$

where $a_{ik}(x)$ and $s_k(x)$ are known functions.

From these equations we can uniquely determine the functions $v_k(x)$ using the boundary (or initial) conditions (1.3).

Proceeding analogously to the case of equation (1.1), by reducing to n independent differential equations, we can for fixed $y = y_k$ approximate each of the conditions (1.3) by the expressions

$$Du_k(z)+G[u_{k-1}(z)+u_{k+1}(z)] = \psi_{jk}, \tag{3.7}$$

$$k = 1, 2, \ldots, n; \quad j = 1, 2, 3, 4.$$

where

$$Dv = b_{1j}v'''+b_{3j}v''+(b_{5j}-2b_{2j}a)v'+(b_{6j}-2b_{4j}a)v,$$

$$Gv = b_{2j}av'+b_{4j}av,$$

whereas z is to be replaced by a or b, according to conditions (1.3). Substituting in (3.7) successively $k = 1, 2, \ldots, n$ and transforming the first and the last equations, we obtain

$$\begin{cases} Du_1+G(su_1+u_2) = l_{j1} = \psi_{j1}+G(su_1-u_0), \\ Du_2+G(u_1+u_3) = l_{j2} = \psi_{j2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Du_{n-1}+G(u_{n-2}+u_n) = l_{j,n-1} = \psi_{j,n-1}, \\ Du_n+G(u_{n-1}+tu_n) = l_{jn} = \psi_{jn}+G(tu_n-u_{n+1}), \end{cases} \tag{3.8}$$

where $\psi_{jk} = \psi_j(y_k)$.

The first and the last unknown components of vectors $\boldsymbol{l}_j = l_{j1}, l_{j2}, \ldots, l_{jn})$ may be determined from boundary conditions (1.2_1) and (1.2_2) analogously to the case of independent components of the vector $\boldsymbol{r}(x)$, and then we substitute $x = z$.

The system (3.8) may be written in the matrix form:

$$D\boldsymbol{u}(z)+G[T\boldsymbol{u}(z)] = \boldsymbol{l}_j, \quad j = 1, 2, 3, 4.$$

Let us now perform the left-hand multiplication of these equations by the matrix P^*. We have

$$D[P^*\boldsymbol{u}(z)]+G[\Lambda P^*\boldsymbol{u}(z)] = P^*\boldsymbol{l}_j, \quad j = 1, 2, 3, 4.$$

Taking into account the form of operators D and G and writing $\boldsymbol{m}_j = P^*\boldsymbol{l}_j$, we get

$$\beta_{1jk}v_k'''(z)+\beta_{2jk}v_k''(z)+\beta_{3jk}v_k'(z)+\beta_{4jk}v_k(z) = m_{jk}, \tag{3.9}$$

$$k = 1, 2, \ldots, n; \quad j = 1, 2, 3, 4;$$

where β_{ijk} are known constants.

In this way the boundary or mixed problem, formulated in the introduction, is reduced to n independent ordinary linear differential equations (3.6) whose each solution $v_k(x)$ would satisfy four boundary or initial conditions (3.9). Solving in an exact or approximate manner problem (3.6), (3.9), we find functions $v_1(x), v_2(x), \ldots, v_n(x)$, i. e. the vector $\boldsymbol{v}(x)$

$$\boldsymbol{v}(x) = P^* \boldsymbol{u}(x).$$

The left-hand multiplication of this equality by the matrix P, in view of (2.1), implies

$$u(x) = P\boldsymbol{v}(x). \tag{3.10}$$

The components of the vector $\boldsymbol{u}(x)$ determine the approximate solution of the problem, formulated in the introduction, over the straight lines $y = y_k$, $(k = 1, 2, \ldots, n)$.

4. An example of the boundary problem. Find the function $u(x, y)$ which on the square $D: \{0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ satisfies the Dirichlet equation (see [7]),

$$\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 4y \tag{4.1}$$

and the boundary conditions

$$u(x, 0) = 0, \quad u(x, 1) = x^2 - 1, \tag{4.2}$$

$$u(0, y) \neq \infty, \quad u(1, y) = 0. \tag{4.3}$$

According to our method, equation (4.1) is replaced by the system of equations

$$u_k'' + \frac{2}{x} u_k' - 2au_k + a(u_{k-1} + u_{k+1}) = 4y_k, \tag{4.4}$$

$$k = 1, 2, \ldots, n, \quad \text{where} \quad y_k = kh, \quad h = \frac{1}{n+1}, \quad a = 1/h^2.$$

Boundary conditions (4.2) and (4.3) are approximated by

$$u_0 = 0, \quad u_{n+1} = x^2 - 1, \tag{4.5}$$

$$u_k(0) \neq \infty, \quad u_k(1) = 0, \quad k = 1, 2, \ldots, n. \tag{4.6}$$

Introduce the operator

$$Av = v'' + \frac{2}{x} v' - 2av.$$

For system (4.4) we then have

$$Au_k + a(u_{k-1} + u_{k+1}) = 4y_k, \quad k = 1, 2, \ldots, n,$$

or, in view of (4.5), the developed form

$$(4.7)\qquad \begin{cases} Au_1 + au_2 = 4k = r_1, \\ Au_2 + a(u_1 + u_3) = 4\cdot 2k = r_2, \\ \dots\dots\dots\dots\dots\dots \\ Au_{n-1} + a(u_{n-2} + u_n) = 4(n-1)k = r_{n-1}, \\ Au_n + au_{n-1} = 4nk = r_n. \end{cases}$$

Now, applying the matrix T_1 defined in example 2.1, for system (4.7) we obtain the matrix form:

$$\boldsymbol{A u} + aT_1\boldsymbol{u} = \boldsymbol{r}.$$

Performing a matrix transformation of this equation, substituting $P_1\boldsymbol{u} = \boldsymbol{v}$ and taking into account formulae (2.3) and (2.4), we obtain u independent differential equations

$$(4.8)\qquad v_k'' + \frac{2}{x}v_k' - \frac{2}{h^2}\left(1 - 2\cos\frac{k\pi}{n+1}\right)v_k = s_k, \qquad k = 1, 2, \dots, n,$$

where

$$s_k = 4h\sum_{i=1}^{n} i\sin\frac{ik\pi}{n+1} + (x^2 - 1)\sin\frac{nk\pi}{n+1}.$$

Each of the equations (4.8) can be exactly solved and for every function $v_k(x)$ we have the general solution

$$v_k(x) = A_k\frac{shr_k x}{x} + B_k\frac{chr_k x}{x} + x^2\frac{\sin n\gamma_k}{r_k^2} + \frac{4\sin n\gamma_k}{r_k^4} - \frac{4\sum_{i=1}^{n} i\sin i\gamma_k + \sin n\gamma_k}{r_k^2}$$

where

$$r_k = \frac{2}{h}\sin\frac{\gamma_k}{2}, \qquad \gamma_k = \frac{k\pi}{n+1},$$

Constants A_k and B_k are determined by boundary conditions (4.6). These conditions determine the vectors $\boldsymbol{u}(0)$ and $\boldsymbol{u}(1)$. The left-hand multiplication of those vectors by the matrix P_1 gives

$$\boldsymbol{v}(0) = P_1\boldsymbol{u}(0) \neq \infty, \qquad \boldsymbol{v}(1) = P_1\boldsymbol{u}(1) = 0.$$

The first of these conditions is satisfied for $B_k = 0$. The second one implies that

$$A_k = \frac{r_k^2(4h\sum_{i=1}^{n} i\sin i\gamma_k + \sin n\gamma_k) - \sin n\gamma_k}{r_k^4 shr_k}.$$

Thus we have determined the vector $\boldsymbol{v}(x) = (v_1(x), v_2(x), \ldots, v_n(x))$. Finitely, we see that

$$\boldsymbol{u}(x) = P_1 \boldsymbol{v}(x).$$

5. Solution of 3-dimensional problems. Many physical and technical questions may be reduced to 3-dimensional boundary problems. The solution of problems by the method of olifference equations leads to enormous systems of algebraic equations. It turns out that the matrix transformation method may be easily applied to these problems.

Solving the 3-dimensional problem consists in a reduction of the problem to m independent 2-dimensional boundary problems, each of which can be solved by the matrix transformation method.

Let in the parallelepiped $S: \{a \leqslant x \leqslant b, c \leqslant y \leqslant d, e \leqslant z \leqslant f\}$ the function $u(x, y, z)$ satisfy the differential equation

$$(5.1) \quad a_1(x)\frac{\partial^4 u}{\partial x^4} + a_2(x)\frac{\partial^4 u}{\partial y^4} + a_3(x)\frac{\partial^4 u}{\partial z^4} + a_4(x)\frac{\partial^4 u}{\partial x^2 \partial y^2} + a_5(x)\frac{\partial^4 u}{\partial x^2 \partial z^2} + a_6(x)\frac{\partial^4 u}{\partial y^2 \partial z^2} + a_7(x)\frac{\partial^3 u}{\partial x^3} + a_8(x)\frac{\partial^3 u}{\partial x \partial y^2} + a_9(x)\frac{\partial^3 u}{\partial x \partial z^2} + a_{10}(x)\frac{\partial^2 u}{\partial x^2} + a_{11}(x)\frac{\partial^2 u}{\partial y^2} + a_{12}(x)\frac{\partial^2 u}{\partial z^2} + a_{13}(x)\frac{\partial u}{\partial x} + a_{14}(x) u = f(x, y, z)$$

and

1° over each of the planes $z = e$ and $z = f$ two boundary conditions

$$(5.2) \quad a_{1j}\frac{\partial^2 u}{\partial z^2} + a_{2j}\frac{\partial u}{\partial z} + a_{3j} u = \varphi_j(x, y), \quad j = 1, 2, 3, 4;$$

2° over each of the planes $y = c$ and $y = d$ two boundary conditions

$$(5.3) \quad b_{1j}\frac{\partial^3 u}{\partial y \partial z^2} + b_{2j}\frac{\partial^2 u}{\partial y^2} + b_{3j}\frac{\partial^2 u}{\partial z^2} + b_{4j}\frac{\partial u}{\partial y} + b_{5j} u = \psi_j(x, z), \quad j = 1, 2, 3, 4;$$

3° over each of the planes $x = a$ and $x = b$ boundary conditions (or initial conditions over one of them)

$$(5.4) \quad c_{1j}\frac{\partial^3 u}{\partial x \partial z^2} + c_{2j}\frac{\partial^2 u}{\partial x^2} + c_{3j}\frac{\partial^2 u}{\partial y^2} + c_{4j}\frac{\partial^2 u}{\partial z^2} + c_{5j}\frac{\partial u}{\partial x} + c_{6j} u = \chi_j(y, z), \quad j = 1, 2, 3, 4,$$

where a_{ij}, b_{ij} and c_{ij} are constant, be given.

Obviously, we assume that this problem has a unique solution in S.

We shall show that the above-given problem may be reduced to m independent 2-dimensional boundary problems.

Indeed, let us divide the parallelepiped S by the planes $z = z_i = e + hi[i = 1, 2, \ldots, m;\ h = (f-e)/(m+1)]$ into m sections. Write: $u(x, y, z_i) = u_i(x, y), f(x, y, z_i) = f_i(x, y)$.

On each of the planes $z = z_i$ we can approximate equation (5.1) by the difference-differential equation

$$a_1 \frac{\partial^4 u_i}{\partial x^4} + a_2 \frac{\partial^4 u_i}{\partial y^4} + a_3 a^2 \delta^4 u_i + a_4 \frac{\partial^4 u_i}{\partial x^2 \partial y^2} + a_5 a \delta^2 \frac{\partial^2 u_i}{\partial x^2} +$$

$$+ a_6 a \delta^2 \frac{\partial^2 u_i}{\partial y^2} + a_7 \frac{\partial^3 u_i}{\partial x^3} + a_8 \frac{\partial^3 u_i}{\partial x \partial y^2} + a_9 a \delta^2 \frac{\partial u}{\partial x} + a_{10} \frac{\partial^2 u_i}{\partial x^2} + a_{11} \frac{\partial^2 u_i}{\partial y^2} +$$

$$+ a_{12} a \delta^2 u_i + a_{13} \frac{\partial u_i}{\partial x} + a_{14} u_i = f_i(x, y), \qquad i = 1, 2, \ldots, m.$$

The system of equations obtained may be written as

$$(5.5) \qquad A u_i(x, y) + B[u_{i-1}(x, y) + u_{i+1}(x, y) + C[u_{i-2}(x, y) + u_{i+2}(x, y)] = f_i(x, y), \qquad i = 1, 2, \ldots, m,$$

where A, B and C denote the following linear and homogeneous differential operators:

$$A v = a_1 \frac{\partial^4 v}{\partial x^4} + a_2 \frac{\partial^4 v}{\partial y^4} + a_4 \frac{\partial^4 v}{\partial x^2 \partial y^2} + a_7 \frac{\partial^3 v}{\partial x^3} + a_8 \frac{\partial^3 v}{\partial x \partial y^2} +$$

$$+ (a_{10} - 2a_5 a) \frac{\partial^2 v}{\partial x^2} + (a_{11} - 2a_6 a) \frac{\partial^2 v}{\partial y^2} + (a_{13} - 2a_9 a) \frac{\partial^2 v}{\partial x} +$$

$$+ (a_{14} - 2a_{12} a + 6a_3 a^2) v,$$

$$B v = a_5 a \frac{\partial^2 v}{\partial x^2} + a_6 \frac{\partial^2 v}{\partial y^2} + a_9 a \frac{\partial v}{\partial x} + (a_{12} - 4a_3 a^2) v,$$

$$C v = a_3 a^2 v.$$

Substituting now in (5.5) successively $i = 1, 2, \ldots, m$ and transforming the two first and the two last equations, we have

$$(5.6)\quad \begin{cases} Au_1+B(su_1+u_2)+C[(s^2-1)u_1+su_2+u_3] = r_1 \\ \qquad = f_1+B(su_1-u_0)+C[su_2+(s^2-1)u_1-u_{-1}], \\ Au_2+B(u_1+u_3)+C(su_1+u_2) = r_2 = f_2+C(su_1-u_0), \\ Au_3+B(u_2+u_4)+C(u_1+u_5) = r_3 = f_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Au_{m-2}+B(u_{m-3}+u_{m-1})+C(u_{m-4}+u_m) = r_{m-2} = f_{m-2}, \\ Au_{m-1}+B(u_{m-2}+u_m)+C(u_{m-3}+tu_m) = r_{m-1} = f_{m-1}+ \\ \qquad +C(tu_m-u_{m+1}), \\ Au_m+B(u_{m-1}+tu_m)+C([u_{m-2}+tu_{m-1}+(t^2-1)u_m] = r_m = \\ \qquad = f_m+B(tu_m-u_{m+1})+C[tu_{m-1}+(t^2-1)u_m-u_{m+2}]. \end{cases}$$

Let us introduce the vectors

$$\boldsymbol{u}(x,y) = \big(u_1(x,y),\quad u_2(x,y),\ldots,u_m(x,y)\big),$$
$$\boldsymbol{r}(x,y) = \big(r_1(x,y),\quad r_2(x,y),\ldots,r_m(x,y)\big).$$

Using the matrix T, we can write the system of equations (5.6) in matrix form

$$(5.7)\qquad A\boldsymbol{u}+B(T\boldsymbol{u})+C[(T^2-2E)\boldsymbol{u}] = \boldsymbol{r}.$$

The two first and two last unknown components of vector $\boldsymbol{r}$ are determined from boundary conditions (5.2) analogously to the above-described 2-dimensional case.

Now we perform the *matrix transformation* of equation (5.7). In view of (2.1) we have

$$A(P^*\boldsymbol{u})+B(\Lambda P^*\boldsymbol{u})+C[(\Lambda^2-2E)P^*\boldsymbol{u}] = P^*\boldsymbol{r},$$

or, substituting $\boldsymbol{v}(x,y) = P^*\boldsymbol{u}(x,y)$, $\boldsymbol{s}(x,y) = P^*\boldsymbol{r}(x,y)$,

$$(5.8)\qquad A\boldsymbol{v}+B(\Lambda\boldsymbol{v})+C[(\Lambda^2-2E)\boldsymbol{v}] = \boldsymbol{s}.$$

Hence, writing (5.8) for the components of vector $\boldsymbol{v}(x,y)$ and taking into account forms of the operators A, B and C, we obtain m independent 2-dimensional partial differential equations

$$(5.9)\qquad a_{1i}(x)\frac{\partial^4 v_i}{\partial x^4}+a_{2i}(x)\frac{\partial^4 v_i}{\partial x^2\partial y^2}+a_{3i}(x)\frac{\partial^4 v_i}{\partial y^4}+a_{4i}(x)\frac{\partial^3 v_i}{\partial x^3}+$$
$$+a_{5i}(x)\frac{\partial^3 v_i}{\partial x\partial y^2}+a_{6i}(x)\frac{\partial^2 v_i}{\partial x^2}+a_{7i}(x)\frac{\partial^2 v_i}{\partial y^2}+a_{8i}(x)\frac{\partial v_i}{\partial x}+a_{9i}(x)v_i = s_i(x,y),$$

where $a_{1i}(x)$ and $s_i(x,y)$ are known functions.

Now we transform boundary conditions (5.3). For fixed $z = z_i$, we approximate each of these conditions by the following expressions:

$$b_{1j} a\delta^2 \frac{\partial u_i}{\partial y} + b_{2j} \frac{\partial^2 u_i}{\partial y^2} + b_{3j} a\delta^2 u_i + b_{4j} \frac{\partial u_i}{\partial y} + b_{5j} u_i = \psi_{ij}(x),$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, 3, 4.$$

We write the systems obtained in the form

(5.10) $\quad Du_i + G(u_{i-1} + u_{i+1}) = \psi_{ji}(x), \quad i = 1, 2, \ldots, m; \quad j = 1, 2, 3, 4,$

where D and G are the operators of the form

$$Dv = b_{2j} \frac{\partial^2 v}{\partial y^2} + (b_{4j} - 2b_{1j} a) \frac{\partial v}{\partial y} + (b_{5j} - 2b_{3j} a) v,$$

$$Gv = b_{1j} a \frac{\partial v}{\partial y} + b_{3j} av.$$

Substituting now in (5.10) successively $i = 1, 2, \ldots, m$, we get

$$(5.11) \quad \begin{cases} Du_1 + G(su_1 + u_2) = l_{j1} = \psi_{j1} + G(su_1 - u_0), \\ Du_2 + G(u_1 + u_3) = l_{j2} = \psi_{j2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Du_{m-1} + G(u_{m-2} + u_m) = l_{j,m-1} = \psi_{j,m-1}, \\ Du_m + G(u_{m-1} + tu_m) = l_{jm} = \psi_{jm} + G(tu_m - u_{m+1}), \end{cases}$$

$j = 1, 2, 3, 4.$

The first and the last unknown components of the vectors $\boldsymbol{l}_j$ are determined by boundary conditions (5.2). We then write systems (5.11) in the matrix form

$$\boldsymbol{Du} + G(\boldsymbol{Tu}) = \boldsymbol{l}_j, \quad j = 1, 2, 3, 4.$$

whence, performing the matiix transformation and writing $\boldsymbol{m}_j = P^* \boldsymbol{l}_j$, we find for every function $v_i(x, y)$ two boundary conditions over each of the straight lines $y = c$ and $y = d$ in the form

$$(5.12) \quad \beta_{1j} \frac{\partial^2 v_i}{\partial y^2} + \beta_{2j} \frac{\partial v_i}{\partial y} + \beta_{3j} v_i = m_{ji}(x),$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, 3, 4.$$

Transforming analogously boundary (or initial) conditions (5.4), we obtain for every function $v_i(x, y)$ four conditions over the straight lines $x = a$ and $x = b$, the boundary or the initial ones, of the form

$$\gamma_{1j}\frac{\partial^2 v_i}{\partial x^2}+\gamma_{2j}\frac{\partial^2 v_i}{\partial y^2}+\gamma_{3j}\frac{\partial v_i}{\partial x}+\gamma_{4j}v_i = n_{ji}(y), \tag{5.13}$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, 3, 4.$$

Equations (5.9) and boundary conditions (5.12) and (5.13) determine m independent 2-dimensional boundary problems each of which may be solved by the method described in Section 3 of this paper.

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METODA TRANSFORMACJI MACIERZOWYCH PRZYBLIŻONEGO ROZWIĄZYWANIA RÓWNAŃ RÓŻNICZKOWYCH CZĄSTKOWYCH

STRESZCZENIE

Opisana w pracy metoda numeryczna jest modyfikacją i zarazem uogólnieniem metody opracowanej przez G. N. Położija [3]. Umożliwia ona znajdowanie przybliżonych rozwiązań następującego zadania:

Niech funkcja $u(x, y)$ spełnia 1^0 na prostokącie D: $\{a \leqslant x \leqslant b \leqslant \infty, c \leqslant y \leqslant d\}$ równanie różniczkowe (1.1), 2^0 na prostych $y = c$ i $y = d$ warunki brzegowe (1.2_1) i (1.2_2), 3^0 na prostych $x = a$ i $x = b$ warunki brzegowe (lub na jednej z nich warunki początkowe) postaci (1.3). Zakłada się, że zagadnienia określone równaniem (1.1) oraz warunkami (1.2_1), (1.2_2) i (1.3) ma jednoznaczne rozwiązanie na prostokącie D.

Metoda transformacji macierzowych sprowadza określone wyżej zagadnienie brzegowe lub mieszane do n niezależnych równań różniczkowych zwyczajnych, których analityczne lub numeryczne rozwiązania określają przybliżone wartości szukanej funkcji $u(x, y)$ na prostych $y = y_k = c + kh$ (h = const, $k = 1, 2, \ldots, n$).

Metodę transformacji macierzowych można zastosować również do zagadnień trójwymiarowych, jak to pokazano w precy.

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МЕТОД МАТРИЧНЫХ ПРЕОБРАЗОВАНИЙ ПРИБЛИЖЕННОГО РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ

РЕЗЮМЕ

Представленный в этой статье метод является модификацией и одновременно обобщением метода суммарных представлений, разработанного Г. Н. Положим [3]. Этот метод делает возможным решать следующую задачу:

Пусть функция $u(x, y)$ удовлетворяет 1^0 дифференциальному уравнению (1.1) в прямоугольнике D:$\{a \leqslant x \leqslant b \leqslant \infty, c \leqslant y \leqslant d\}$, 2^0 краевым условиям (1.2_1) и (1.2_2) на прямых $y = c$ и $y = d$, 3^0 краевым условиям (1.3) на прямых $x = a$ и $x = b$ или начальным условиям на одной из этий прямых. Предполагается, что сформулированная выше задача имеет единственное решение в прямоугольнике D.

Метод матричных преобразований приводит краевую или смешанную задачу к n независимым обыкновенным дифференциальным уравнениям, которых аналитические или численные решения определяют приближенные значения искаемой функции на прямых $y = y_k = c + kh$ (h = const, $k = 1, 2, \ldots, n$).

В статье показано применение метода матричных преобразований к трехмерным задачам.
