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ON THE EXISTENCE AND REGULARITY OF SOLUTIONS TO THE PROBLEM OF TRANSMISSION

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1. Introduction

In this paper we consider the existence, uniqueness and regularity of solutions of the following problem

$$(1.1)\qquad \begin{aligned} -\nabla A^s(x)\nabla u_s &= f_s \quad \text{in } \Omega_s,\ s=1,2,\\ A^s\cdot\nabla u_s\cdot\bar{n} &= \psi_s \quad \text{on } S_s,\ s=1,2,\\ [A\cdot\nabla u\cdot\bar{n}]\big|_\Gamma &= 0,\quad [u]\big|_\Gamma = 0, \end{aligned}$$

in a bounded domain $\Omega\subset \mathbf{R}^n$, $n\geqslant 2$. The domain Ω is divided into two parts Ω_1,Ω_2 (open subsets of Ω, $\Omega=\Omega_1\cup\Omega_2$) by a surface Γ ($\Gamma=\bar{\Omega}_1\cap\bar{\Omega}_2$) passing through an edge $L\subset\partial\Omega$, so that there are two-surface angles between Γ and $\partial\Omega$. We assume that $\partial\Omega_s=S_s\cup\Gamma$, $s=1,2$, and that the two-surface angles between Γ and S_s are equal to $\vartheta_s=\vartheta_s(x)$, $x\in L$, $s=1,2$. Quantities labelled with index s are considered in the subdomain Ω_s, $s=1,2$. By $[\ ]|_\Gamma$ we denote the jump across Γ and by $\bar{n}$ the unit outward vector normal to the boundary $\partial\Omega=S_1\cup S_2$ (or to Γ). For the ordinary transmission problem L is not an edge on $\partial\Omega$; however, L is an edge for Ω_s, $s=1,2$.

Finally we assume that $A^s=(a^s_{ij})_{i,j=1,\dots,n}$, $s=1,2$, are symmetric matrices such that

$$(1.2)\qquad a^s_0\xi^2\leqslant\sum_{i,j=1}^n a^s_{ij}\xi_i\xi_j\leqslant b^s_0\xi^2,\quad \forall\xi\in\mathbf{R}^n,\ x\in\Omega_s,\ s=1,2,$$

where a^s_0, b^s_0 are positive constants. Let $a_0=\min\{a^1_0,a^2_0\}$, $b_0=\max\{b^1_0,b^2_0\}$. Then the equations $(1.1)_1$ are uniformly elliptic.

Physically Ω may consist of two kinds of elastic media separated by Γ.

The paper is organized in the following way. In Section 2 we introduce necessary notation and auxiliary results (various types of weighted spaces;

embedding theorems). In Section 3 we prove the existence, uniqueness and regularity of solutions of the problem (1.1) on a plane in the case of constant matrices A^s, $s = 1, 2$, in angular domains $\Omega_s = d_{\vartheta_s}$, $s = 1, 2$. First we show the existence in homogeneous spaces H^k_μ (see Theorem 3.3), then in weighted Sobolev spaces W^k_μ (see Theorem 3.7). Using the Fourier transformation with respect to variables in directions parallel to the edge M of dihedral angles $\mathscr{D}_{\vartheta_s}$, $s = 1, 2$ (see Section 2), we extend in Section 4 the statement of Theorem 3.7 to the case of $\Omega_i = \mathscr{D}_{\vartheta_i}$, $i = 1, 2$ (see Theorem 4.2). In Section 5 we prove the existence, uniqueness and regularity of solutions of the problem (1.1) with variable coefficients $a^s_{ij} = a^s_{ij}(x)$, $s = 1, 2$, $i, j = 1, \ldots, n$, in $\Omega_i = \mathscr{D}_{\vartheta_i}$, $i = 1, 2$, by the method of successive approximations (see Theorem 5.2). Finally, in Section 6, using the existence of weak solutions of the problem (1.1) in a bounded domain and a suitable partition of unity we show the existence of solutions of the problem (1.1) in W^k_μ spaces (see Theorem 6.3).

In this paper the coefficients of the asymptotic expansion in a neighbourhood of the edge (see Theorem 3.3) are not given. Results of this type have been obtained for boundary value problems to elliptic equations (see [7], [8]).

The paper mostly relies on methods and results of [10]. The transmission problem with the Dirichlet conditions on S_s, $s = 1, 2$, was considered in [5].

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2. Notation and auxiliary results

At first we introduce necessary notation. Let r, φ be the polar coordinates in the plane; let $d_\vartheta \subset \mathbf{R}^2$ be an infinite angle $r > 0$, $\varphi \in (\varphi_1, \varphi_2)$ of size $\vartheta = \varphi_1 - \varphi_2$; by γ_{φ_1}, γ_{φ_2} we denote the sides of d_ϑ determined by $\varphi = \varphi_1$ and $\varphi = \varphi_2$, respectively; $\mathscr{D}_\vartheta = d_\vartheta \times \mathbf{R}^{n-2}$ is the dihedral angle in $\mathbf{R}^n$, $n > 2$, with sides $\Gamma_{\varphi_i} = \gamma_{\varphi_i} \times \mathbf{R}^{n-2}$, $i = 1, 2$, and with edge $M = \bar{\Gamma}_{\varphi_1} \cap \bar{\Gamma}_{\varphi_2}$.

The points of $\mathscr{D}_\vartheta$ will be denoted by $x = (x', z)$, where $x' \in d_\vartheta$, $z \in \mathbf{R}^{n-2}$.

By $\zeta(x) \in C_0^\infty(\mathbf{R}^n)$ we denote a function depending monotonically on $|x|$ equal to 1 for $|x| \leqslant 1/2$ and 0 for $|x| > 1$.

Now we introduce some function spaces involving functions defined on a domain $\mathscr{D}_\vartheta$ (which can be defined similarly in the case of $d_\vartheta \subset \mathbf{R}^2$) [10], [11]: $\mathscr{H}(\mathscr{D}_\vartheta)$ is the space of functions with the finite Dirichlet integral

$$\|u\|_{\mathscr{H}(D_\vartheta)} = \Big(\int_{\mathscr{D}_\vartheta} |\nabla u|^2\, dx\Big)^{1/2};$$

$W^k_\mu(\mathscr{D}_\vartheta)$, $H^k_\mu(\mathscr{D}_\vartheta)$, $L^k_\mu(\mathscr{D}_\vartheta)$, $0 \leqslant k \in \mathbf{Z}$, $\mu \in \mathbf{R}$, are the closures of smooth functions with compact supports, in the norms (in the case of H^k_μ we only consider smooth functions which vanish in a neighbourhood of the edge M):

$$\|u\|_{W^k_\mu(\mathscr{D}_\vartheta)} = \Big(\sum_{|\alpha|\leqslant k}\int_{\mathscr{D}_\vartheta} |x'|^{2\mu}|D^\alpha_x u|^2\,dx\Big)^{1/2},$$

$$\|u\|_{H^k_\mu(\mathscr{D}_\vartheta)} = \Big(\sum_{|\alpha|\leqslant k}\int_{\mathscr{D}_\vartheta} |x'|^{2(\mu-(k-|\alpha|))}|D^\alpha_x u|^2\,dx\Big)^{1/2},$$

$$\|u\|_{L^k_\mu(\mathscr{D}_\vartheta)} = \Big(\sum_{|\alpha|= k}\int_{\mathscr{D}_\vartheta} |x'|^{2\mu}|D^\alpha_x u|^2\,dx\Big)^{1/2}.$$

For $k=0$ these spaces coincide and are equal to $L_\mu(\mathscr{D}_\vartheta)$.

For $k>0$, elements of $W^k_\mu(\mathscr{D}_\vartheta)$, $H^k_\mu(\mathscr{D}_\vartheta)$, $L^k_\mu(\mathscr{D}_\vartheta)$ leave traces on any $(n-1)$-dimensional plane Γ passing through the edge. These traces belong to the spaces $W^{k-1/2}_\mu(\Gamma)$, $H^{k-1/2}_\mu(\Gamma)$, $L^{k-1/2}_\mu(\Gamma)$, respectively, with the following norms:

$$\|u\|^2_{W^{k-1/2}_\mu(\Gamma)} = \|u\|^2_{L^{k-1/2}_\mu(\Gamma)} + \sum_{|\alpha|\leqslant k-1}\int_\Gamma |D^\alpha u|^2\xi_1^{2\mu}\,d\xi,$$

$$\|u\|_{H^{k-1/2}_\mu(\Gamma)} = \|u\|_{L^{k-1/2}_\mu(\Gamma)} + \sum_{|\alpha|\leqslant k-1}\int_\Gamma |D^\alpha u|^2\xi_1^{2(\mu-k+|\alpha|)+1}\,d\xi,$$

$$\|u\|_{L^{k-1/2}_\mu(\Gamma)} = \Bigg(\sum_{|\alpha|=k-1}\int_\Gamma \xi_1^2\,d\xi\int_{K_+(\xi)} |D^\alpha u(\xi+\eta)-D^\alpha u(\xi)|^2\frac{d\eta}{|\eta|^n}\Bigg)^{1/2},$$

where $\xi=(\xi_1,\xi_2,\ldots,\xi_{n-1})$ are cartesian coordinates on Γ such that $\Gamma=\{\xi\in \mathbf{R}^{n-1}\colon \xi_1>0\}$ and $K_+(\xi)=\{\eta\in\Gamma\colon |\eta|<\xi_1\}$.

Let us first recall theorems about traces for elements from the spaces just introduced. Let $X^k_\mu(\mathscr{D}_\vartheta)$ denote one of the spaces $W^k_\mu(\mathscr{D}_\vartheta)$, $H^k_\mu(\mathscr{D}_\vartheta)$, $L^k_\mu(\mathscr{D}_\vartheta)$. According to [10], [11], we have

THEOREM 2.1. *Let* $u\in X^k_\mu(\mathscr{D}_\vartheta)$, $0\leqslant k\in \mathbf{Z}$, $\mu\in\mathbf{R}$, $|\alpha|<k$. *Then* $D^\alpha u|_\Gamma \in X^{k-|\alpha|-1/2}_\mu(\Gamma)$ *and*

$$\|D^\alpha u\|_{X^{k-|\alpha|-1/2}_\mu(\Gamma)} \leqslant c\,\|u\|_{X^k_\mu(\mathscr{D}_\vartheta)}. \tag{2.1}$$

Let there be given functions $\varphi_j\in X^{k-j-1/2}_\mu(\Gamma)$, $j=0,\ldots,k-1$. *Then there exists a function* $u\in X^k_\mu(\mathscr{D}_\vartheta)$ *such that*

$$\|u\|_{X^k_\mu(\mathscr{D}_\vartheta)} \leqslant c\sum_{j=0}^{k-1}\|\varphi_j\|_{X^{k-j-1/2}_\mu(\Gamma)}. \tag{2.2}$$

We denote $H^k(\mathscr{D}_\vartheta)=W^k_0(\mathscr{D}_\vartheta)$.

Finally, we define the space $W^k_\mu(\Omega)$ by means of the norm

$$\|u\|_{W^k_\mu(\Omega)} = \Big(\sum_{|\alpha|\leqslant k}\int_\Omega \varrho^{2\mu}(x)|D^\alpha u|^2\,dx\Big)^{1/2}, \tag{2.3}$$

where $\varrho(x) = \operatorname{dist}(x, L)$ (L is an edge) and we can introduce, as above, the space of traces $W_\mu^{k-1/2}(\partial\Omega)$.

For convenience we also introduce the space $\Lambda_\mu^{k+2}(\mathscr{D}_\vartheta)$ by the norm

$$\|u\|_{\Lambda_\mu^{k+2}(\mathscr{D}_\vartheta)} = \|D^2 u\|_{W_\mu^k(\mathscr{D}_\vartheta)} + \|u\|_{\mathscr{H}(\mathscr{D}_\vartheta)}.$$

For an arbitrary $v \in H^k(\mathscr{D}_\vartheta)$ the interpolation inequality

$$\int_{\mathscr{D}_\vartheta} |v|^2 |x'|^{-2\mu} dx \leqslant \varepsilon^{2(k-\mu)} \|v\|^2_{L_0^k(\mathscr{D}_\vartheta)} + c\varepsilon^{-2\mu} \|v\|^2_{L_2(\mathscr{D}_\vartheta)} \tag{2.4}$$

is valid for $\mu \in (0, 1)$ and for every $\varepsilon > 0$.

Let us further introduce the space $\mathscr{E}_\mu^k(d_\vartheta)$ by the norm

$$\|u\|^2_{\mathscr{E}_\mu^k(d_\vartheta)} = \sum_{i \leqslant k} \xi^{2i} \|u\|^2_{L_\mu^{k-i}(d_\vartheta)}. \tag{2.5}$$

By the Parseval equality we have

$$\|u\|^2_{L_\mu^k(\mathscr{D}_\vartheta)} = \int_{R^{n-2}} \|\tilde{u}\|^2_{\mathscr{E}_\mu^k(d_\vartheta)} d\xi, \tag{2.6}$$

where $\tilde{u}$ is the Fourier transform of u (see the proof of Theorem 4.2).

We shall need the following Hardy inequality (see [3], § 4, [1], Ch.1). Let $f \in L_\nu^1(d_\vartheta)$ be such that $f(0) = 0$ and $\nu < 0$; then

$$\left(\int_0^\infty |f|^2 r^{2\nu-1} dr\right)^{1/2} \leqslant \frac{1}{-\nu}\left(\int_0^\infty |f_r|^2 r^{2\nu+1} dr\right)^{1/2}; \tag{2.7}$$

and if $f(\infty) = 0$ and $\nu > 0$, then

$$\left(\int_0^\infty |f|^2 r^{2\nu-1} dr\right)^{1/2} \leqslant \frac{1}{\nu}\left(\int_0^\infty |f_r|^2 r^{2\nu+1} dr\right)^{1/2} \tag{2.8}$$

Let $f(x)$, $x \in d_\vartheta$, be an arbitrary function. By $f^{(j)}(x)$ we denote the partial sum of the Taylor series with respect to x':

$$f^{(j)}(x) = \sum_{|\alpha| = \alpha_1 + \alpha_2 \leqslant j} D_{x'}^\alpha f(x)|_{x=0} \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1! \alpha_2!}. \tag{2.9}$$

Similarly, for a function φ given on γ_i we define

$$\varphi^{(j)}(r) = \sum_{k=0}^{j} \frac{1}{k!} \frac{\partial^k}{\partial r^k} \varphi \bigg|_{r=0} r^k. \tag{2.10}$$

By the Hardy inequalities (2.7), (2.8) and in view of (2.9), (2.10) we have (see [14], Ch.2, Lemma 2.1)

LEMMA 2.2. *Let $u \in L_\mu^k(d_\vartheta)$, $\mu \in \boldsymbol{R}$, $1 \leqslant k \in \boldsymbol{Z}$ and $\mu + 1 \geqslant 0$.*

(a) *Let* $s = \mu+1$ *be a noninteger. Then there exists* $j \in \mathbf{Z}$ *such that* $k-\mu-2 < j < k-\mu-1$, $u-u^{(j)} \in H^k_\mu(d_\vartheta)$ *and*

$$\|u-u^{(j)}\|_{H^k_\mu(d_\vartheta)} \leqslant c\,\|u\|_{L^k_\mu(d_\vartheta)}. \tag{2.11}$$

(b) *Let* $s = \mu+1$ *be an integer. Let* $D^{k-s}u \in L_{\mu-s}(d_\vartheta)$, $j = k-s$. *Then* $u-u^{(j-1)} \in H^k_\mu(d_\vartheta)$ *and*

$$\|u-u^{(j-1)}\|_{H^k_\mu(d_\vartheta)} \leqslant c\big(\|u\|_{L^k_\mu(d_\vartheta)} + \|D^{k-s}u\|_{L_{\mu-s}(d_\vartheta)}\big). \tag{2.12}$$

We will also need the following type of the Hardy inequality (see [3], § 4):

$$\|u\|_{L_{\mu-k}(d_\vartheta)} \leqslant c\,\|u\|^2_{L^k_\mu(d_\vartheta)}, \qquad \mu-k > -1. \tag{2.13}$$

3. Existence of solutions to the problem (1.1) with constant coefficients in a cone on a plane

In this section we show the existence, uniqueness and regularity of solutions of the problem (1.1) which we rewrite in the following form:

$$(3.1)\qquad \begin{aligned} &L_s u_s \equiv -A^s \nabla' \nabla' u_s = f_s && \text{in } d_{\vartheta_s},\ s = 1, 2,\\ &F_s u_s \equiv A^s \cdot \nabla' u_s \cdot \bar n|_{\varphi=(-1)^s\vartheta_s} = \psi_s && \text{on } \gamma_s,\ s = 1, 2,\end{aligned}$$

$$A^1 \cdot \nabla' u_1 \cdot \bar n|_{\varphi=0} = A^2 \cdot \nabla' u_2 \cdot \bar n|_{\varphi=0}, \qquad u_1|_{\varphi=0} = u_2|_{\varphi=0}, \qquad \text{on } \gamma_0,$$

where

$$A^s = (a^s_{ij})_{i,j=1,2} = \begin{pmatrix} a_s & b_s \\ b_s & c_s \end{pmatrix}, \qquad s = 1, 2,$$

are constants, $d_{\vartheta_s} \subset \mathbf{R}^2$ is the angle between γ_s and γ_0, $s = 1, 2$. In the polar coordinates r, φ on $\mathbf{R}^2$, $\gamma_1, \gamma_0, \gamma_2$ are determined by $\varphi = -\vartheta_1$, $\varphi = 0$, $\varphi = \vartheta_2$, respectively; or equivalently, by $x_2 = -\tan\vartheta_1 x_1$, $x_2 = 0$, $x_2 = \tan\vartheta_2 x_1$, respectively. By $\bar n$ we denote the unit outward vector normal to the boundary; so

$$\bar n|_{\gamma_1} = (-\sin\vartheta_1, -\cos\vartheta_1), \qquad \bar n|_{\gamma_0} = (0, 1), \qquad \bar n|_{\gamma_2} = (-\sin\vartheta_2, \cos\vartheta_2).$$

We denote

$$\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right).$$

By ellipticity we have

$$p_s^2 \equiv \det A^s = a_s c_s - b_s^2 > 0, \qquad s = 1, 2.$$

Using the polar coordinates, we write (3.1) in the form

$$(3.2)\quad \begin{aligned} & L_s(r,\varphi)u_s \equiv \frac{\alpha_s}{r}\frac{\partial u_s}{\partial r}+\frac{\alpha_{s\varphi}}{r^2}\frac{\partial u_s}{\partial\varphi}-\frac{\alpha_{s\varphi}}{r}\frac{\partial^2 u_s}{\partial r\,\partial\varphi}+\beta_s\frac{\partial^2 u_s}{\partial r^2}+\frac{\alpha_s}{r^2}\frac{\partial^2 u_s}{\partial\varphi^2} = f_s, \\ & \qquad\qquad \text{in } d_{\vartheta_s},\ s=1,2, \\ & F_s(r,\varphi)u_s \equiv \frac{\alpha_{s\varphi}}{2}\frac{\partial u_s}{\partial r}-\frac{\alpha_s}{r}\frac{\partial u_s}{\partial\varphi} = \psi_s, \quad \text{on } \gamma_s,\ s=1,2, \\ & F_1(r,\varphi)u_1 = F_2(r,\varphi)u_2, \quad u_1=u_2, \quad \text{on } \gamma_0, \end{aligned}$$

where

$$(3.3)\quad \begin{aligned} \alpha_s &= a_s\sin^2\varphi-b_s\sin 2\varphi+c_s\cos^2\varphi, \\ \beta_s &= a_s\cos^2\varphi+b_s\sin 2\varphi+c_s\sin^2\varphi, \\ \alpha_{s\varphi\varphi} &= 2(\beta_s-\alpha_s), \qquad \beta_{s\varphi} = -\alpha_{s\varphi}, \end{aligned}$$

$s=1,2$, and $F_s(r,\varphi)u_s = A^s\cdot\nabla' u_s\bar{n}|_{\varphi=\varphi}$ where $\bar{n}=(-\sin\varphi,\cos\varphi)$.

Introducing the new variable $\tau=-\ln r$ and applying the Fourier transformation

$$\tilde{u}(\lambda,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{R^1} e^{-i\lambda\tau}u(\tau,\varphi)\,d\tau,$$

we replace problem (3.2) by

$$(3.4)\quad \begin{aligned} & L_s(\sigma,\varphi)\tilde{u}_s \equiv \alpha_s\tilde{u}_{s\varphi\varphi}+\alpha_{s\varphi}(1+\sigma)\tilde{u}_{s\varphi}+[(\beta_s-\alpha_s)\sigma+\beta_s\sigma^2]\tilde{u}_s \\ & \qquad = (-e^{-2\tau}f_s)^\sim \equiv -\tilde{F}_s \quad \text{in } \theta_s,\ s=1,2, \\ & F_s(\sigma,\varphi)\tilde{u}_s \equiv \alpha_s\tilde{u}_{s\varphi}+\tfrac{1}{2}\sigma\alpha_{s\varphi}\tilde{u}_s \\ & \qquad = (-e^{-\tau}\psi_s)^\sim \equiv \tilde{\Psi}_s \quad \text{for } \varphi=(-1)^s\vartheta_s,\ s=1,2, \\ & F_1(\sigma,\varphi)\tilde{u}_1 = F_2(\sigma,\varphi)\tilde{u}_2, \quad \tilde{u}_1=\tilde{u}_2 \quad \text{for } \varphi=0, \end{aligned}$$

where $\sigma=i\lambda$, $\theta_1=(-\vartheta_1,0)$, $\theta_2=(0,\vartheta_2)$.

We treat the homogeneous problem (3.4) as a nonlinear eigenvalue problem. According to considerations in [7], knowing that in the case of diagonal matrices $A^s=\mu_s\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $s=1,2$, the explicit form of eigenvalues and eigenvectors can be found (see [12]), we may assume the existence of eigenvalues and eigenvectors for (3.4) in the form

$$(3.5)\quad \sigma_k,\quad \varphi_{ski_k}(\varphi),\quad k=0,\mp1,\mp2,\ldots,\ s=1,2,$$

i_k is the multiplicity of the eigenvalue σ_k. For $\sigma=0$, instead of (3.4) we have

$$(3.6)\quad \begin{aligned} & \alpha_s\tilde{u}_{s\varphi\varphi}+\alpha_{s\varphi}\tilde{u}_{s\varphi}=0, && \text{in } \theta_s,\ s=1,2, \\ & \tilde{u}_{s\varphi}=0 && \text{for } \varphi=(-1)^s\vartheta_s,\ s=1,2, \\ & c_1u_{1\varphi}=c_2u_{2\varphi},\quad u_1=u_2 && \text{for } \varphi=0, \end{aligned}$$

which implies that

$$\tilde{u}_1 = \tilde{u}_2 = \text{const} \tag{3.7}$$

is the eigenvector.

To find solutions of the homogeneous problem (3.4) we introduce new variables (see [2], part 1, § 25.1)

$$\tilde{v}_s = \tilde{u}_s \left(\frac{\alpha_s}{c_s}\right)^{(1+\sigma)/2}, \quad s = 1, 2, \tag{3.8}$$

so that instead of the homogeneous problem (3.4) we have

$$\begin{aligned} &\tilde{v}_{s\varphi\varphi} + g_s(\sigma, \varphi)\tilde{v}_s = 0 && \text{in } \Theta_s,\ s = 1, 2, \\ &\alpha_s \tilde{v}_s - \tfrac{1}{2}\alpha_{s\varphi}\tilde{v}_s = 0 && \text{for } \varphi = (-1)^s \vartheta_s,\ s = 1, 2, \\ &\alpha_1 \tilde{v}_{1\varphi} - \tfrac{1}{2}\alpha_{1\varphi}\tilde{v}_1 = \alpha_2 \tilde{v}_2 - \tfrac{1}{2}\alpha_{2\varphi}\tilde{v}_2, && \tilde{v}_1 = \tilde{v}_2,\ \text{for } \varphi = 0, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} g_s(\sigma, \varphi) &= \frac{\frac{1}{2}\alpha_{s\varphi\varphi}\sigma + \beta_s\sigma^2}{\alpha_s} - \frac{(1+\sigma)^2}{4}\frac{\alpha_{s\varphi}^2}{\alpha_s^2} - \frac{1+\sigma}{2}\left(\frac{\alpha_{s\varphi}}{\alpha_s}\right)_\varphi \\ &= \frac{\sigma^2}{4\alpha_s^2}(4\alpha_s\beta_s - \alpha_{s\varphi}^2) - \frac{1}{4\alpha_s^2}(\alpha_{s\varphi}^2 - 4\alpha_s\beta_s + 4\alpha_s^2) \\ &= \frac{p_s}{\alpha_s^2}\sigma^2 + \frac{1}{\alpha_s^2}(p_s - \alpha_s^2), \end{aligned} \tag{3.10}$$

$$\alpha_s|_{\varphi=0} = c_s, \quad \alpha_{s\varphi}|_{\varphi=0} = -2b_s, \quad s = 1, 2. \tag{3.11}$$

To define weak solutions and to prove the existence of solutions to the problem (3.4) we need the following

LEMMA 3.1. *Let $\psi_s \in H_\mu^{k-1-1/2}(\gamma_s)$, $s = 0, 1, 2$, $2 \leqslant k \in \mathbf{Z}$, $\mu \in \mathbf{R}$. Then there exist functions $v_s \in H_\mu^k(d_{\vartheta_s})$, $s = 1, 2$, such that*

$$\begin{aligned} &A^s \cdot \nabla' v_s \cdot \bar{n} = \psi_s && \text{on } \gamma_s,\ s = 1, 2, \\ &A^1 \cdot \nabla' v_1 \cdot \bar{n} = A^2 \cdot \nabla' v_2 \cdot \bar{n} + \psi_0, \quad v_1 = v_2 && \text{on } \gamma_0; \end{aligned} \tag{3.12}$$

$$\sum_{s=1}^{2} \|v_s\|_{H_\mu^k(d_{\vartheta_s})} \leqslant c \sum_{s=0}^{2} \|\psi_s\|_{H_\mu^{k-1-1/2}(\gamma_s)}. \tag{3.13}$$

The converse is also valid. Let $v_s \in H_\mu^k(d_{\vartheta_s})$, $s = 1, 2$; then ψ_s determined by (3.12) belong to $H_\mu^{k-1-1/2}(\gamma_s)$, $s = 0, 1, 2$, and

$$\sum_{s=0}^{2} \|\psi_s\|_{H_\mu^{k-1-1/2}(\gamma_s)} \leqslant c \sum_{s=1}^{2} \|v_s\|_{H_\mu^k(d_{\vartheta_s})}. \tag{3.14}$$

The proof of this lemma is omitted because it is a simplified version of the proof of Lemma 4.1.

Now, using this lemma we can consider the problem (3.4) with $\tilde{\Psi}_s = 0$, $s = 1, 2$. This problem will be denoted by (3.4)′.

To define a weak solution to the problem (3.4)′ we multiply (3.4)′ by $\bar{v}_s$ and integrate over θ_s. Summing over s we get

$$(3.15)\qquad -\sum_{s=1}^{2}\int_{\theta_s}\left(\alpha_s\tilde{u}_{s\varphi\varphi}+\alpha_{s\varphi}(1+\sigma)\tilde{u}_{s\varphi}+[(\beta_s-\alpha_s)\sigma+\beta_s\sigma^2]\tilde{u}_s\right)\bar{v}_s = \sum_{s=1}^{2}\int_{\theta_s}\tilde{F}_s\bar{v}_s,$$

where $\bar{v}$ denotes the conjugate transposed element to v. Integrating by parts in (3.15) and using $(3.4)'_{2,3}$ we get

$$(3.16)\qquad \sum_{s=1}^{2}\int_{\theta_s}\left(\alpha_s\tilde{u}_{s\varphi}+\tfrac{1}{2}\sigma\alpha_{s\varphi}\tilde{u}_s\right)\bar{v}_{s\varphi}-\sum_{s=1}^{2}\int_{\theta_s}\left(\tfrac{1}{2}\sigma\alpha_{s\varphi}\tilde{u}_{s\varphi}+\beta_s\sigma^2\tilde{u}_s\right)\bar{v}_s = \sum_{s=1}^{2}\int_{\theta_s}\tilde{F}_s\bar{v}_s.$$

Therefore, by a generalized solution to the problem (3.4)′ we mean functions $\tilde{u}_s\in H^1(\theta_s)$, $a = 1, 2$, which satisfy the identity (3.16) for every $v_s\in H^1(\theta_s)$, $s = 1, 2$.

To find the adjoint operator to the operator of problem (3.4)′ we integrate by parts in (3.16), thus obtaining

$$(3.17)\qquad \begin{aligned} &L_s^*(\sigma,\varphi)v_s \equiv \alpha_s v_{s\varphi\varphi}+(1-\bar{\sigma})\alpha_{s\varphi}v_{s\varphi}+[-(\beta_a-\alpha_s)\bar{\sigma}+\beta_s\bar{\sigma}^2]v_s = -\tilde{F}_s \\ &\qquad\qquad\qquad\qquad \text{in } \theta_s,\ s = 1, 2,\\ &F_s^*(\sigma,\varphi)v_s \equiv \alpha_s v_{s\varphi}-\tfrac{1}{2}\bar{\sigma}\alpha_{s\varphi}v_s = 0 \qquad \text{for } \varphi = (-1)^s\vartheta_s,\ s = 1, 2,\\ &F_1^*(\sigma,\varphi)v_1 = F_2^*(\sigma,\varphi)v_2,\quad v_1 = v_2 \qquad \text{for } \varphi = 0. \end{aligned}$$

Using the notion of weak solution (3.16) we have

THEOREM 3.2. *Let $\tilde{F}_s\in H^k(\theta_s)$, $s = 1, 2$, and let σ be different from σ_k, $k = 0, \mp 1, \mp 2, \ldots$, which occur in (3.5). Then there exists a unique weak solution $\tilde{u}_s$ of (3.16) which belongs to $H^{k+2}(\theta_s)$, $s = 1, 2$, and satisfies the estimate*

$$(3.18)\qquad \sum_{s=1}^{2}\left(\|\tilde{u}_s\|^2_{H^{k+2}(\theta_s)}+|\lambda|^{2(k+2)}\|\tilde{u}_s\|^2_{L_2(\theta_s)}\right) \leqslant c\sum_{s=1}^{2}\left(\|\tilde{F}_s\|^2_{H^k(\theta_s)}+|\lambda|^{2k}\|\tilde{F}_s\|^2_{L_2(\theta_s)}\right).$$

where $|\sigma| = |\lambda|$.

Proof. We introduce the space $H^1(\theta) = H^1(\theta_1)\oplus H^1(\theta_2)$ with the scalar product

$$(3.19)\qquad (U, V)_{H^1(\theta)} = \sum_{s=1}^{2}\int_{\theta_s}\left(\alpha_s u_{s\varphi}\bar{v}_{s\varphi}+u_s\bar{v}_s\right),$$

where $U = (u_1, u_2)$, $V = (v_1, v_2)$, and by ellipticity (see (1.2)) we have $a_0^s\leqslant\alpha_s\leqslant b_0^s$, $a_0^s\leqslant\beta_s\leqslant b_0^s$, $s = 1, 2$. Adding and subtracting the term $(t-1)\sum_{s=1}^{2}\int_{\theta_s}\tilde{u}_s\bar{v}_s$ to (3.16) we get

(3.20) $(\tilde{U}, V)_{H^1(\theta)} + (A_t \tilde{U}, V)_{H^1(\theta)} - (\sigma^2 + t)(B\tilde{U}, V)_{H^1(\theta)} = (\tilde{\mathscr{F}}, V)_{H^1(\theta)}$,

where $\mathscr{F} = (F_1, F_2)$, and by the Riesz theorem we have

(3.21)
$$(A_t \tilde{U}, V)_{H^1(\theta)} = \sum_{s=1}^{2} \int_{\theta_s} [\tfrac{1}{2}\sigma\alpha_{s\varphi}(\tilde{u}_s \bar{v}_{s\varphi} - \tilde{u}_{s\varphi}\bar{v}_s) + (t-1)\tilde{u}_s v_s],$$
$$(B\tilde{U}, V)_{H^1(\theta)} = \sum_{s=1}^{2} \int_{\theta_s} \tilde{u}_s \bar{v}_s,$$
$$(\tilde{\mathscr{F}}, V)_{H^1(\theta)} = \sum_{s=1}^{2} \int_{\theta_s} \tilde{F}_s \bar{v}_s.$$

By embedding theorems the operators A_t, B are compact and continuous. Moreover, (3.20) can be written in the operator form

(3.22) $$\tilde{U} + A_t \tilde{U} - (\sigma^2 + t) B\tilde{U} = \tilde{\mathscr{F}}.$$

Assuming that t is such that

(3.23) $$\|\tilde{U}\|^2_{H^1(\theta)} + (A_t \tilde{U}, \tilde{U})_{H^1(\theta)} \geqslant \tfrac{1}{2}\|\tilde{U}\|^2_{H^1(\theta)},$$

we have the existence of the operator $(I + A_t)^{-1}$. To satisfy (3.23) we write

(3.24) $$\int_{\theta_a} \tfrac{1}{2}a_0^s |\tilde{u}_{s\varphi}|^2 + (t - \tfrac{1}{2})|\tilde{u}_s|^2 \geqslant \tfrac{1}{2}\varepsilon|\sigma|^2 (\sup_{\theta_s} |\alpha_{s\varphi}|)^2 \int_{\theta_s} |\tilde{u}_{s\varphi}|^2 + (2\varepsilon)^{-1} \int_{\theta_s} |\tilde{u}_s|^2.$$

Putting

$$\varepsilon = \frac{a_0^s}{|\sigma|^2 (\sup_{\theta_s} |\alpha_{s\varphi}|)^2}$$

and

(3.25) $$t \geqslant \frac{|\sigma|^2 (\sup_{\theta_s} |\alpha_{s\varphi}|)^2}{2a_0^s} + \frac{1}{2}$$

we obtain (3.23). For t satisfying (3.25) we write (3.22) in the form (see [4, 9])

(3.26) $$\tilde{U} - (\sigma^2 + t)(I + A_t)^{-1} B\tilde{U} = (I + A_t)^{-1}\tilde{\mathscr{F}}.$$

For $\sigma \neq \sigma_k$ (see (3.5)) the homogeneous equation (3.26) (with $\mathscr{F} = 0$) has no solutions, so that $\|(I + A_t)^{-1}\| \leqslant c/(\sigma^2 + 1)$, and by the Fredholm theorem we get the existence of a unique solution to the equation (3.26) and the estimate

(3.27) $$\|\tilde{U}\|_{H^1(\theta)} \leqslant \frac{c}{\sigma^2 + 1}\|\tilde{\mathscr{F}}\|_{H^1(\theta)} = \frac{c}{\sigma^2 + 1}\|\tilde{F}\|_{L_2(\theta)}.$$

Returning to (3.4), we have the existence of solutions of the problem (3.4) such that $\tilde{u}_s \in H^2(\theta_s)$, $s = 1, 2$, and

$$\sum_{s=1}^{2} (\|\tilde{u}_s\|^2_{H^2(\theta_s)} + |\lambda|^2 \|\tilde{u}_s\|^2_{H^1(\theta_s)} + |\lambda|^4 \|\tilde{u}_s\|^2_{L_2(\theta_s)}) \leqslant c \sum_{s=1}^{2} \|\tilde{F}_s\|^2_{L_2(\theta_s)}. \tag{3.28}$$

Continuing our considerations we are able to show that $\tilde{u}_s \in H^{k+2}(\theta_s)$, $s = 1, 2$, and (3.18) is satisfied. ■

Integrating (3.18) with respect to λ along the line $\operatorname{Im}\lambda = h$, where $h = -\mu + k + 1$, $h \neq \sigma_i$, $i = 0, \mp 1, \ldots$, and σ_i are the eigenvalues in (3.5), and repeating the considerations from [3] we derive

THEOREM 3.3. *Let σ_i be as in* (3.5) *and suppose that*

$$\sigma_i \neq k + 1 - \mu, \tag{3.29}$$

where $0 \leqslant k \in \mathbf{Z}$, $\mu \in \mathbf{R}$. Let $f_s \in H^k_\mu(d_{\vartheta_s})$, $\psi_{s'} \in H^{k+1/2}_\mu(\gamma_{s'})$, $s = 1, 2$, $s' = 0, 1, 2$.

Then there exists a unique solution of the problem (3.1) *such that $u_s \in H^{k+2}_\mu(d_{\vartheta_s})$, $s = 1, 2$, and*

$$\sum_{s=1}^{2} \|u_s\|_{H^{k+2}_\mu(d_{\vartheta_s})} \leqslant c \Big(\sum_{s=1}^{2} \|f_s\|_{H^k_\mu(d_{\vartheta_s})} + \sum_{s'=0}^{2} \|\psi_{s'}\|_{H^{k+1/2}_\mu(\gamma_{s'})} \Big). \tag{3.30}$$

Let $f_s \in H^k_\mu(d_{\vartheta_s}) \cap H^{k'}_{\mu'}(d_{\vartheta_s})$, $\psi_{s'} \in H^{k+1/2}_\mu(\gamma_{s'}) \cap H^{k'+1/2}_{\mu'}(\gamma_{s'})$, $s = 1, 2$, $s' = 0, 1, 2$, and

$$k' + 1 - \mu' < \sigma_i < k + 1 - \mu, \tag{3.31}$$

where i belongs to some subset of $\mathbf{Z}$.

The solutions $u_s \in H^{k+2}_\mu(d_{\vartheta_s})$, $u'_s \in H^{k'+2}_{\mu'}(d_{\vartheta_s})$, $s = 1, 2$, satisfy

$$u_s = u'_s + \sum_i \sum_{j=0}^{k_i} c_{sij} r^{-i\lambda_j} \ln^j r\, \varphi_{sij}(\varphi), \tag{3.32}$$

where λ_i are poles of multiplicities k_i and summation is spread over all i such that $\sigma_i = \operatorname{Im}\lambda_i$ satisfy (3.31).

To show the existence of solutions of the problem (3.1) in $W^{k+2}_\mu(d_{\vartheta_s})$, $s = 1, 2$, we need the following facts.

LEMMA 3.4. *Let $f_s(x)$, $\psi_s(x)$ be homogeneous polynomials of degree $l-2$, $l-1$, respectively,*

$$f_s = \sum_{i_1+i_2=l-2} f_{si_1i_2} x_1^{i_1} x_2^{i_2}, \qquad \psi_s = \psi_{sl-1} |x'|^{l-1}, \tag{3.33}$$

where $f_{si_1i_2}$, ψ_{sl-1}, $s = 1, 2$, are constants.

If $l \neq \sigma_k$, $k = 0, \mp 1, \ldots$, where σ_k are as in (3.5), *then the problem* (3.1) *has a unique solution of the form of a homogeneous polynomial of degree l:*

$$u_s = \sum_{i_1+i_2=l} u_{si_1i_2} x_1^{i_1} x_2^{i_2}; \tag{3.34}$$

$u_{si_1i_2}$ are constants, $s = 1, 2$.

Proof. We are looking for solutions of the problem (3.2) of the form $u_s = r^l P_s(\varphi)$, $s = 1, 2$, with right-hand sides (3.33). Therefore we get the problem (3.4), where σ is replaced by l. For $l \neq \sigma_i$ (see (3.5), by Theorem 3.2 we have the existence of a unique solution $P_s(\varphi)$, $s = 1, 2$. Assuming that $u_s = r^l P_s(\varphi)$, $s = 1, 2$, are not homogeneous polynomials and inserting them in to (3.1) we get a contradiction with (3.33). The polynomials of the first degree must be calculated explicitly. They are determined uniquely if $a_1 + b_1 \neq a_2 + b_2$. ∎

From [3], § 4 and also from [10], [14] we have

LEMMA 3.5. *Let* $v_{s\varphi} \in H_\mu^{l-j}(d_{\vartheta_s})$, $0 \leqslant j < l$, $j, l \in \mathbf{Z}$, $0 \leqslant \mu \in \mathbf{Z}$, $\alpha = (\alpha_1, \alpha_2)$, $j = |\alpha| = \alpha_1 + \alpha_2$, $0 \leqslant \alpha_i \in \mathbf{Z}$, $i = 1, 2$, $s = 1, 2$, $\sigma = \mu + 1$, $j + \sigma \leqslant l$. *Then there exist functions* $v_s \in W_\mu^l(d_{\vartheta_s}) \cap H^{j+\sigma-1}(d_{\vartheta_s})$ *such that* $D^\alpha v_s - v_{s\alpha} \in H_\mu^\sigma(d_{\vartheta_s}) \cap W_\mu^{l-j}(d_{\vartheta_s})$ *and*

$$(3.35) \qquad \|v_s\|_{W_\mu^l(d_{\vartheta_s})} + \|v_s\|_{H_\mu^{j+\sigma-1}(d_{\vartheta_s})} + \|D^\alpha v_s - v_{s\alpha}\|_{H_\mu^\sigma(d_{\vartheta_s})} \leqslant c \sum_{|\alpha| = j} \|v_{s\alpha}\|_{W_\mu^{l-j}(d_{\vartheta_s})}, \qquad s = 1, 2.$$

Now we prove

LEMMA 3.6. *Let* $f_s \in W_\mu^k(d_{\vartheta_s})$, $\psi_s \in W_\mu^{k+1/2}(\gamma_s)$, $s = 1, 2$, $0 \leqslant k \in \mathbf{Z}$, $\mu \in \mathbf{R}^+$. *Then there exist functions* v_s *such that* $D^2 v_s \in W_\mu^k(d_{\vartheta_s})$, $s = 1, 2$,

$$(3.36) \qquad \begin{aligned} & f_s + \nabla' A^s \nabla' v_s = g_s \in H_\mu^k(d_{\vartheta_s}), \\ & \psi_s - A^s \nabla' v_s \cdot \bar{n}|_{\gamma_s} = \varphi_s \in H_\mu^{k+1/2}(\gamma_s), \qquad s = 1, 2; \end{aligned}$$

$$(3.37) \qquad \sum_{s=1}^{2} (\|D^2 v_s\|_{W_\mu^k(d_{\vartheta_s})} + \|g_s\|_{H_\mu^k(d_{\vartheta_s})} + \|\varphi_s\|_{H_\mu^{k+1/2}(\gamma_s)}) \leqslant c \sum_{s=1}^{2} (\|f_s\|_{W_\mu^k(d_{\vartheta_s})} + \|\psi_s\|_{W_\mu^{k+1/2}(\gamma_s)}) \equiv cX.$$

Proof. (a) Let μ be a noninteger. We introduce homogeneous polynomials

$$(3.38) \qquad \begin{aligned} & \psi_{sq} = \left(\frac{\partial}{\partial r}\right)^q \psi_s \bigg|_{r=0} \frac{r^q}{q!}, \qquad q < k - \mu, \\ & f_{sq} = \sum_{|\alpha| = q} D_{x'}^\alpha f_s \bigg|_{x'=0} \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1! \alpha_2!}, \qquad q < k - 1 - \mu, \ |\alpha| = \alpha_1 + \alpha_2, \ s = 1, 2. \end{aligned}$$

Taking the functions (3.38) as the right-hand sides of (3.1), we get by Lemma 3.4 the existence of homogeneous polynomials v_{sq}, $q < k + 1 - \mu$, $s = 1, 2$, which are solutions of the problem

$$(3.39) \qquad \begin{aligned} & -A^s \nabla' \nabla' v_{sq} = f_{sq}, \qquad A^s \nabla' v_{sq}|_{\gamma_s} = \psi_{sq}, \\ & A^1 \nabla' v_{1q} \bar{n}|_{\gamma_0} = A^2 \nabla' v_{2q} \bar{n}|_{\gamma_0}, \qquad v_{1q}|_{\gamma_0} = v_{2q}|_{\gamma_0}, \qquad s = 1, 2. \end{aligned}$$

Let $v_s = \sum_{q<k+1-\mu} v_{sq}\,\xi(x')$, $s = 1, 2$. Then, by Lemma 2.2 (a) and its version for traces of functions, the properties (3.36) and the estimates for the last two terms in the left-hand side of (3.37) are satisfied. To estimate the first term in (3.37) we use the fact that v_s, $s = 1, 2$, depend on the derivatives

$$\left(\frac{\partial}{\partial r}\right)^q \psi_s\Big|_{r=0}, \quad q < k-\mu,$$

$$D^\alpha_{x'} f_s|_{x'=0}, \quad |\alpha| < k-1-\mu,\ s = 1, 2.$$

Hence, using the embeddings (see [13])

$$\text{(3.40)}\qquad \begin{aligned} \|D^\alpha u\|_{L_\infty(d_\vartheta)} &\leqslant c\,\|u\|_{W^k_\mu(d_\vartheta)}, & |\alpha| < k-1-\mu,\\ \|D^q u\|_{L_\infty(\gamma)} &\leqslant c\,\|u\|_{W^{k+1/2}_\mu(\gamma)}, & q < k-\mu, \end{aligned}$$

we get (3.37)

(b) Let $\mu+1 = \sigma$ be an integer. In this case the construction of v is divided into two steps. At first we construct the polynomials v_{sq} (see (3.39)) for $q \leqslant k+1-\sigma = k-\mu$.

Introducing

$$\text{(3.41)}\qquad v^1_s = \sum_{q\leqslant k-\mu} v_{sq}\,\xi(x'), \quad s = 1, 2.$$

by embedding theorems (3.40) (see [13]) we have

$$f^1_s = f_s + \nabla' A^s \nabla' v^1_s \in W^k_\mu(d_{\vartheta_s}), \quad \psi^1_s = \psi_s - A^s \nabla' v^1_s\,\bar n|_{\gamma_s} \in W^{k+1/2}_\mu(\gamma_2),$$
$$D^2 v^1_s \in W^k_\mu(d_{\vartheta_s}), \quad s = 1, 2,$$

and

$$\text{(3.42)}\qquad \sum_{s=1}^2 \left(\|D^2 v^1_s\|_{W^k_\mu(d_{\vartheta_s})} + \|f^1_s\|_{W^k_\mu(d_{\vartheta_s})} + \|\psi^1_s\|_{W^{k+1/2}_\mu(\gamma_s)}\right) \leqslant cX.$$

Moreover

$$\text{(3.43)}\qquad \begin{aligned} D^\alpha_{x'} f^1_s|_{x'=0} &= 0, & |\alpha| \leqslant k-2-\mu,\\ D^j_r \psi^1_r|_{r=0} &= 0, & j \leqslant k-1-\mu. \end{aligned}$$

Now we define functions $v_{s(\alpha)}$, $(\alpha) = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1+\alpha_2 = k+1-\mu \equiv l$, by the relations

$$-\partial^\varrho_{x_1}\partial^{l-2-\varrho}_{x_2}(A^s\nabla'\nabla' v_{s(0,0)}) = \partial^\varrho_{x_1}\partial^{l-2-\varrho}_{x_2} f^1_s, \quad s = 1, 2,\ \varrho = 0, \ldots, l-2,$$

$$(\cos\varphi\partial_{x_1} + \sin\varphi\partial_{x_2})^{l-1} A^s \nabla' v_{s(0,0)}\,\bar n = (\cos\varphi\partial_{x_1} + \sin\varphi\partial_{x_2})^{l-1}\tilde\psi_s,$$

$$\varphi = (-1)^s\vartheta_s, \quad s = 1, 2,$$

$$\text{(3.44)}\qquad \bar n|_{\varphi=-\vartheta_1} = (-\sin\vartheta_1,\ -\cos\vartheta_1),$$

$$\vec{n}|_{\varphi=\vartheta_2} = (-\sin\vartheta_2, \cos\vartheta_2),$$
$$\partial_{x_1}^{l-1} A^1 \nabla' v_{1(0,0)}\vec{n} = \partial_{x_1}^{l-1} A^2 \nabla' v_{2(0,0)}\vec{n},$$
$$\partial_{x_1}^{l} v_{1(0,0)} = \partial_{x_1}^{l} v_{2(0,0)},$$

where $\tilde{\psi}_s$ is an extension of ψ_s to d_{ϑ_s}, $s = 1, 2$,

$$\partial_{x_1} v_{s(\alpha_1,\alpha_2)} = v_{s(\alpha_1+1,\alpha_2)}, \qquad \partial_{x_2} v_{s(\alpha_1,\alpha_2)} = v_{s(\alpha_1,\alpha_2+1)}.$$

To show the existence and uniqueness of solutions of (3.44), we consider the problem

$$\begin{aligned} -\nabla' A^s \nabla' P_s &= h_s, \quad x\in d_{\vartheta_s},\ s = 1,2,\\ A^s \nabla' P_s \vec{n}|_{\gamma_s} &= \varphi_s, \quad s = 1, 2,\\ A^1 \nabla' P_1 \vec{n}|_{\gamma_0} &= A^2 \nabla' P_2 \vec{n}|_{\gamma_0},\\ P_1|_{\gamma_0} &= P_2|_{\gamma_0}, \end{aligned} \tag{3.45}$$

where

$$P_s(x) = \sum_{q=0}^{l} p_{sq}\frac{x_1^q x_2^{l-q}}{q!(l-q)!}, \qquad h_s = \sum_{\varrho=0}^{l-2} h_{s\varrho}\frac{x_1^\varrho x_2^{l-2-\varrho}}{\varrho!(l-2-\varrho)!}, \qquad p_{sq} = v_{s(q,l-q)},$$

$$h_{s\varrho} = \partial_{x_1}\partial_{x_2}^{l-2-\varrho} f_s^1, \qquad \varphi_s = \psi_{sl-1}\frac{(\cos(-1)^s\vartheta_s x_1 + \sin(-1)^s\vartheta_s x_2)^{l-1}}{(l-1)!},$$

$$\psi_{sl-1} = (\cos(-1)^s\vartheta_s\partial_{x_1} + \sin(-1)^s\vartheta_s\partial_{x_2})^{l-1}\tilde{\psi}_s, \qquad s = 1, 2.$$

By Lemma 3.4 for $l \neq \sigma_k$, where σ_k are as in (3.5), we have the existence of a unique solution to (3.45), hence also for (3.44).

Now, putting $l = k+2, j = k+1-\mu$, $\sigma = 1+\mu$, from Lemma 3.5 we derive the existence of functions v_s^2, $s = 1, 2$, such that $D^2 v_s^2 \in W_\mu^k(d_{\vartheta_s})$, $v_s^2 \in H_\mu^{k+1}(d_{\vartheta_s})$ and

$$\|D^2 v_s^2\|_{W_\mu^k(d_{\vartheta_s})} + \|v_s^2\|_{H_\mu^{k+1}(d_{\vartheta_s})} + \|D^{k+1-\mu} v_s^2 - v_{s(\alpha)}\|_{H_\mu^\sigma(d_{\vartheta_s})} \leqslant c\sum_{|\alpha|=k+1-\mu}\|v_{s(\alpha)}\|_{W_\mu^{1+\mu}(d_{\vartheta_s})} \leqslant cX, \qquad s = 1, 2, \tag{3.46}$$

where the last inequality follows from the construction of $v_{s(\alpha)}$, $s = 1, 2$. By the definition of $v_{s(\alpha)}$, $|\alpha| = k+1-\mu$, and (3.46) we get

$$\|D^{k-1-\mu}(A^s\nabla'\nabla' v_s^2 + f_s^1)\|_{H_\mu^\sigma(d_{\vartheta_s})} + \|D^{k-\mu}(\psi_s^1 - A^s\nabla' v_s^2\cdot\vec{n}|_{\gamma_s})\|_{H_\mu^{\sigma-1/2}(\gamma_s)} \leqslant cX, \qquad s = 1, 2. \tag{3.47}$$

Hence by Lemma 2.2 and (3.43) we get (3.36) and (3.37) for $v_s = v_s^1 + v_s^2$, $s = 1, 2$. ∎

Finally we have

THEOREM 3.7. *Let* $0 \leqslant \mu \in \mathbf{R}$, $0 \leqslant k \in \mathbf{Z}$, $\sigma_1 > k+1-\mu \geqslant 0$, $f_s \in W_\mu^k(d_{\vartheta_s})$, $\psi_s \in W_\mu^{k+1/2}(\gamma_s)$, $s = 1, 2$ *(where* σ_1 *is the first positive eigenvalue in* (3.5)).

Then the problem (3.1) *has a solution such that* $D_{x'}^{\alpha} u_s \in W_\mu^k(d_{\vartheta_s})$, $s = 1, 2$, $|\alpha| = 2$, *and*

$$(3.48)\qquad \sum_{s=1}^{2} \sum_{|\alpha|=2} \|D_{x'}^{\alpha} u_s\|_{W_\mu^k(d_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} (\|f_s\|_{W_\mu^k(d_{\vartheta_s})} + \|\psi_s\|_{W_\mu^{k+1/2}(\gamma_s)}).$$

Proof. We use Lemma 3.6 and then Theorem 3.3. ■

Remark 3.8 (see [10], § 3). Let $u_s(x')$, $s = 1, 2$, be solutions of the problem (3.1). Then $\lambda^2 u_s(\lambda^{-1} x')$, $s = 1, 2$, $\lambda > 0$, are solutions of the same problem with $f_s(\lambda^{-1} x')$. $\lambda\psi_s(\lambda^{-1} x')$, $s = 1, 2$, instead of $f_s(x')$, $\psi_s(x')$, $s = 1, 2$. Then we obtain the estimate (3.48) for $\lambda^2 u_s(\lambda^{-1} x')$, $s = 1, 2$. Passing with λ to 0 we get

$$(3.49)\qquad \sum_{s=1}^{2} \|u_s\|_{L_\mu^{k+2}(d_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} (\|f_s\|_{L_\mu^k(d_{\vartheta_s})} + \|\psi_s\|_{L_\mu^{k+1/2}(\gamma_s)}).$$

4. Existence of solutions to the problem (1.1) with constant coefficients in a dihedral angle in R^n

At first we extend Lemma 3.1 to the case of dihedral angles. This result is necessary to define weak solutions and to prove the existence of solutions of the following problem:

$$(4.1)\qquad \begin{aligned} -\sum_{i,j=1}^{n} a_{ij}^s \frac{\partial^2 u_s}{\partial x_i \partial x_j} &= f_s && \text{in } \mathcal{D}_{\vartheta_s},\ s = 1, 2,\\ \sum_{i,j=1}^{n} a_{ij}^s \frac{\partial u_s}{\partial x_j} n_i\Big|_{\varphi=(-1)^s\vartheta_s} &= \psi_s && \text{on } \Gamma_s,\ s = 1, 2,\\ \sum_{i,j=1}^{n} a_{ij}^1 \frac{\partial u_1}{\partial x_j} n_i\Big|_{\varphi=0} &= \sum_{i,j=1}^{n} a_{ij}^2 \frac{\partial u_2}{\partial x_j} n_i\Big|_{\varphi=0} + \psi_0 && \text{on } \Gamma_0,\\ u_1|_{\varphi=0} &= u_2|_{\varphi=0} && \text{on } \Gamma_0, \end{aligned}$$

where $\bar{n}|_{\varphi=0} = (0, 1, 0, \ldots, 0)$, $\bar{n}|_{\varphi=-\vartheta_1} = (-\sin\vartheta_1, -\cos\vartheta_1, 0, \ldots, 0)$, $\bar{n}|_{\varphi=\vartheta_2} = (-\sin\vartheta_2, \cos\vartheta_2, 0, \ldots, 0)$.

LEMMA 4.1. *Let* $\psi_s \in H_\mu^{k-1-1/2}(\Gamma_s)$, $s = 0, 1, 2$, $k \in \boldsymbol{Z}$, $k \geqslant 2$, $\mu \in \boldsymbol{R}$. *Then there exist functions* $v_s \in H_\mu^k(\mathcal{D}_{\vartheta_2})$, $s = 1, 2$, *such that*

$$(4.2)\qquad \begin{aligned} &A^s \nabla v_s \cdot \bar{n} = \psi_s && \textit{on } \Gamma_s,\ s = 1, 2,\\ &A^1 \nabla v_1 \bar{n} = A^2 \nabla v_2 \cdot \bar{n} + \psi_0, \quad v_1 = v_2 && \textit{on } \Gamma_0, \end{aligned}$$

$$(4.3)\qquad \sum_{s=1}^{2} \|v_s\|_{H_\mu^k(\mathcal{D}_{\vartheta_s})} \leqslant c \sum_{s=0}^{2} \|\psi_s\|_{H_\mu^{k-1-1/2}(\Gamma_s)},$$

where $A^s = (a_{ij}^s)_{i,j=1,\ldots,n}$.

The converse is also valid. Let $v_s \in H^k_\mu(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$; *then* ψ_s *determined by* (4.2) *belong to* $H^{k-1-1/2}_\mu(\Gamma_s)$, $s = 1, 2$, *and*

$$\sum_{s=0}^{2} \|\psi_s\|_{H^{k-1-1/2}_\mu(\Gamma_s)} \leqslant c \sum_{s=1}^{2} \|v_s\|_{H^k_\mu(\mathcal{D}_{\vartheta_s})}. \tag{4.4}$$

A proof can be obtained by means of a suitable partition of unity in $\mathcal{D}_{\vartheta_s}$, $s = 1, 2$ (see [6], Lemma 1.2).

Using the lemma we can consider the problem (4.1) with $\psi_s = 0$, $s = 0, 1, 2$. This problem will be denoted by (4.1)′.

By a generalized solution to the problem (4.1)′ we mean functions $u_s \in \mathcal{H}(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$, which satisfy the integral identity

$$\sum_{s=1}^{2} \sum_{i,j=1}^{n} \int_{\mathcal{D}_{\vartheta_s}} a^s_{ij} \frac{\partial u_s}{\partial x_i} \frac{\partial \eta_s}{\partial x_j} = \sum_{s=1}^{2} \int_{\mathcal{D}_{\vartheta_s}} f_s \eta_s \tag{4.5}$$

for arbitrary, $\eta_s \in \mathcal{H}(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$.

If

$$N = \sum_{s=1}^{2} \Big(\int_{\mathcal{D}_{\vartheta_s}} |f_s|^2 |x - x_0|^2 \, dx \Big)^{1/2} < \infty, \tag{4.6}$$

where $x_0 \in M$, then the right-hand side of (4.5) is bounded above as follows:

$$\sum_{s=1}^{2} \int_{\mathcal{D}_{\vartheta_s}} f_s \eta_s \leqslant \sum_{s=1}^{2} \Big(\int_{\mathcal{D}_{\vartheta_s}} |f_s|^2 |x-x_0|^2 \, dx \Big)^{1/2} \Big(\int_{\mathcal{D}_{\vartheta_s}} |\eta_s|^2 |x-x_0|^{-2} \, dx \Big)^{1/2}$$
$$\leqslant cN \sum_{s=1}^{2} \|\eta_s\|_{\mathcal{H}(\mathcal{D}_{\vartheta_s})}.$$

Hence the right-hand side of (4.5) is a linear continuous functional on $\mathcal{H}(\mathcal{D}_{\vartheta_1}) \oplus \mathcal{H}(\mathcal{D}_{\vartheta_2})$. Therefore the Riesz theorem implies

THEOREM 4.2. *Let* f_s, $s = 1, 2$, *be such that* (4.6) *is satisfied. Then there exists a generalized solution of* (4.1)′ *such that* $u_s \in \mathcal{H}(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$, *which satisfies the integral identity* (4.5) *and*

$$\sum_{s=1}^{2} \|u_s\|_{\mathcal{H}(\mathcal{D}_{\vartheta_s})} \leqslant cN. \tag{4.7}$$

Now we prove the main result of this section.

THEOREM 4.3. *Let* $k \in \mathbf{Z}_+$, $\mu \in \mathbf{R}_+$ *be such that*

$$\sigma_1 > 1 + k - \mu \geqslant 0, \tag{4.8}$$

where σ_1 *is the first positive eigenvalue in* (3.5). *Let* $f_s \in L^k_\mu(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$.

*Then there exists a uniqu*e *solution of the problem* (4.1) *such that* $u_s \in L^{k+2}_\mu(\mathcal{D}_{\vartheta_s}) \cap \mathcal{H}(\mathcal{D}_{\vartheta_s})$, $s = 1, 2$, *and*

$$\sum_{s=1}^{2} \|u_s\|_{L^{k+2}_\mu(\mathcal{D}_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} \|f_s\|_{L^k_\mu(\mathcal{D}_{\vartheta_s})}. \tag{4.9}$$

Proof. Let $\tilde{u}_s(x', \xi)$ be the Fourier transform of the function u_s with respect to variables z, defined by the formula

$$\tilde{f}_s(x', \xi) = (2\pi)^{-(n-2)/2} \int_{R^{n-2}} f_s(x) e^{-iz\cdot\xi} dz,$$

where $x = (x', z)$, $\xi = (\xi_1, \ldots, \xi_{n-2}) \in R^{n-2}$, $z\cdot\xi = x_3\xi + \ldots + x_n\xi_{n-2}$. Problem (4.1) converts via the Fourier transformation to the following one:

$$(4.10)\quad \begin{aligned} &-\sum_{i,j=1}^{2} a_{ij}^s \tilde{u}_{sx_ix_j} - i\sum_{i=1}^{2}\sum_{j=3}^{n}(a_{ij}^s + a_{ji}^s)\tilde{u}_{sx_i}\xi_{j-2} \\ &\qquad + \sum_{i,j=3}^{n} a_{ij}^s \xi_{i-2}\xi_{j-2}\tilde{u}_s = \tilde{f}_s, \quad s = 1, 2, \\ &\sum_{i=1}^{2}\Big(\sum_{j=1}^{2} a_{ij}^s \frac{\partial \tilde{u}_s}{\partial x_j} n_i + i\sum_{j=3}^{n} a_{ij}^s \tilde{u}_s \xi_{j-2} n_i\Big)\Big|_{\varphi=(-1)^s\vartheta_s} = 0, \quad s = 1, 2, \\ &\sum_{i=1}^{2}\Big(\sum_{j=1}^{2} a_{ij}^1 \frac{\partial \tilde{u}_1}{\partial x_j} n_i + i\sum_{j=3}^{n} a_{ij}^1 \tilde{u}_1 \xi_{j-2} n_i\Big)\Big|_{\varphi=0} \\ &\qquad = \sum_{i=1}^{2}\Big(\sum_{j=1}^{2} a_{ij}^2 \frac{\partial \tilde{u}_2}{\partial x_j} n_i + i\sum_{j=3}^{n} a_{ij}^2 \tilde{u}_2 \xi_{j-2} n_i\Big)\Big|_{\varphi=0}, \\ &\tilde{u}_1 = \tilde{u}_2 \quad \text{for } \varphi = 0. \end{aligned}$$

Multiplying $(4.10)_1$ by $\bar{\eta}_s$, integrating over d_{ϑ_s}, summing over s, integrating by parts and using $(4.10)_{2,3,4}$ we get

$$(4.11)\quad \sum_{s=1}^{2}\int_{d_{\vartheta_s}}\Big[\sum_{i,j=1}^{2} a_{ij}^s \tilde{u}_{sx_i}\bar{\eta}_{sx_j} + i\sum_{i=1}^{2}\sum_{j=3}^{n} a_{ij}^s(\tilde{u}_s\bar{\eta}_{sx_i} - \tilde{u}_{sx_i}\bar{\eta}_s)\xi_{j-2} + \sum_{i,j=3}^{n} a_{ij}^s \xi_{i-2}\xi_{j-2}\tilde{u}_s\bar{\eta}_s\Big] = \sum_{s=1}^{2}\int_{d_{\vartheta_s}} \tilde{f}_s\bar{\eta}_s.$$

Now, by a generalized solution to the problem (4.10) we mean functions $\tilde{u}_s$, $s = 1, 2$, which satisfy the integral identity (4.11) for arbitrary functions $\eta_s \in H^1(d_{\vartheta_s})$, $s = 1, 2$. The left-hand side of (4.11) is equal to $\sum_{s=1}^{2}\sum_{i,j=1}^{n} a_{ij}^s \eta_{si}\bar{x}_{sj}$, where $\eta = (\nabla'\tilde{u}_s, i\tilde{u}_s\xi)$, $x = (\nabla'\eta_s, i\eta_s\xi)$; so by ellipticity (see (1.2)), the left-hand side of (4.11) generates a scalar product in $H^1(d_{\vartheta_1}) \oplus H^1(d_{\vartheta_2})$ for almost all ξ. The right-hand side of (4.11) is a linear functional in $H^1(d_{\vartheta_1}) \oplus H^1(d_{\vartheta_2})$; hence by the Riesz theorem we get the existence of weak solutions to the problem (4.10) in $H^1(d_{\vartheta_1}) \oplus H^1(d_{\vartheta_2})$.

Now we show the regularity of the weak solution. Putting $\eta_s = \tilde{u}_s\xi^{2-2\mu}$ $(\xi^2 = \xi_1^2 + \ldots + \xi_{n-2}^2)$ into (4.11) we get

$$(4.12)\quad \begin{aligned} \xi^{2-2\mu}\sum_{s=1}^{2}\sum_{i,j=1}^{n}\int_{d_{\vartheta_s}} a_{ij}^s \eta_{si}\bar{\eta}_{sj} &= \xi^{2-2\mu}\sum_{s=1}^{2}\int_{d_{\vartheta_s}} \tilde{f}_s\bar{\tilde{u}}_s \\ &\leqslant \xi^{2-2\mu}\sum_{s=1}^{2}\Big(\int_{d_{\vartheta_s}} |\tilde{f}_s|^2 |x'|^{2\mu} dx'\Big)^{1/2}\Big(\int_{d_{\vartheta_s}} |\tilde{u}_s|^2 |x'|^{-2\mu} dx'\Big)^{1/2}, \end{aligned}$$

where $\eta_s = (\nabla' \tilde{u}_s, i\tilde{u}_s \xi)$. Observe that

$$\xi^{2-2\mu} \int_{d\vartheta_s} |\tilde{u}_s|^2 |x'|^{-2\mu} dx' \leqslant c \int_{d\vartheta_s} (|\nabla' \tilde{u}_s|^2 + \xi^2 |\tilde{u}_s|^2) dx', \qquad \mu \in [0, 1) \tag{4.13}$$

(this follows from the interpolation inequality (2.4)). By the ellipticity condition (1.2),

$$\int_{d\vartheta_s} a_{ij}^s \eta_{si} \bar{\eta}_{sj} \geqslant a_0^s \int_{d\vartheta_s} |\eta_s|^2 = a_0^s \int_{d\vartheta_s} (|\nabla' \tilde{u}_s| + \xi^2 |\tilde{u}_s|^2) dx'. \tag{4.14}$$

Using (4.13) and (4.14) in (4.12) we obtain

$$\xi^{2-2\mu} \sum_{s=1}^{2} \int_{d\vartheta_s} (|\nabla' \tilde{u}_s|^2 + \xi^2 |\tilde{u}_s|^2) dx' \leqslant c \sum_{s=1}^{2} \int_{d\vartheta_s} |\tilde{f}_s|^2 |x'|^{2\mu} dx'. \tag{4.15}$$

Now we show the boundedness of the integral $\sum_{s=1}^{2} \xi^2 \int_{d\vartheta_s} (|\nabla' \tilde{u}_s|^2 + \xi^2 |\tilde{u}_s|^2) \times |x'|^{2\mu} dx'$.

To do this we put $\eta_s = \tilde{u}_s V_\varrho(x', \xi)$ in (4.11) where $\varrho \gg 1$ and $V_\varrho = \min(\varrho |\xi|^{-2\mu}, \max(|x'|^{2\mu}, |\xi|^{-2\mu})) \xi^2$ is a bounded continuous function of x'. Hence

$$\sum_{s=1}^{2} \Big(\sum_{i,j=1}^{n} \int_{d\vartheta_s} a_{ij}^s \eta_{si} \bar{\eta}_{sj} V_\varrho + \sum_{i,j=1}^{2} \int_{d\vartheta_s} a_{ij}^s \tilde{u}_{sx_i} \bar{\tilde{u}}_s V_{\varrho x_j} + i \sum_{i=1}^{2} \sum_{j=3}^{n} a_{ij}^s |\tilde{u}_s|^2 V_{\varrho x_i} \xi_{j-2} \Big) = \sum_{s=1}^{2} \int_{d\vartheta_s} \tilde{f}_s \bar{\tilde{u}}_s V_\varrho. \tag{4.16}$$

By the ellipticity condition (1.2) we get from (4.16)

$$\begin{aligned} a_0 J_\varrho \equiv a_0 \sum_{s=1}^{2} \int_{d\vartheta_s} (|\nabla' \tilde{u}_s|^2 + \xi^2 |\tilde{u}_s|^2) V_\varrho & \\ \leqslant \sum_{s=1}^{2} \Big(\int_{d\vartheta_s} |\tilde{f}_s|^2 |x'|^{2\mu} dx' \Big)^{1/2} \Big(\int_{d\vartheta_s} |\tilde{u}_s|^2 V_\varrho^2 |x'|^{-2\mu} dx' \Big)^{1/2} & \\ + c_1 \sum_{s=1}^{2} \Big(\int_{d\vartheta_s} |\nabla' \tilde{u}_s|^2 V_\varrho dx' \Big)^{1/2} \Big(\int_{d\vartheta_s} |\tilde{u}_s|^2 |\nabla' V_\varrho|^2 V_\varrho^{-2} dx' \Big)^{1/2} & \\ + c_2 \sum_{s=1}^{2} \Big(\int_{d\vartheta_s} |\tilde{u}_s|^2 \xi^2 V_\varrho dx' \Big)^{1/2} \Big(\int_{d\vartheta_s} |\tilde{u}_s|^2 |\nabla' V_\varrho|^2 V_\varrho^{-1} dx' \Big)^{1/2}. & \end{aligned} \tag{4.17}$$

Using the Young inequality in (4.17) we get

$$J_\varrho \leqslant c \sum_{s=1}^{2} \int_{d\vartheta_s} (|\tilde{f}_s|^2 |x'|^{2\mu} + |\tilde{u}_s|^2 |\nabla' V_\varrho|^2 V_\varrho^{-1} + \varepsilon |\tilde{u}_s|^2 V_\varrho^2 |x'|^{-2\mu}) dx'. \tag{4.18}$$

From $|\nabla' V_\varrho|^2 V_\varrho^{-1} \leqslant (2\mu)^2 \xi^{4-2\mu}$ and (4.15) it follows that

$$\begin{aligned} \int_{d\vartheta_s} |\tilde{u}_s|^2 |\nabla' V_\varrho|^2 V_\varrho^{-1} dx' &\leqslant (2\mu)^2 \xi^{4-2\mu} \int_{d\vartheta_s} |\tilde{u}_s|^2 dx' \\ &\leqslant c \sum_{s=1}^{2} \int_{d\vartheta_s} |\tilde{f}_s|^2 |x'|^{2\mu} dx'. \end{aligned} \tag{4.19}$$

Moreover,

$$(4.20)\quad \int_{d_{\vartheta_s}} |\tilde{u}_s|^2 V_\varrho^2 |x'|^{-2\mu} dx' \leqslant \xi^2 \int_{d'_{\vartheta_s}} |\tilde{u}_s|^2 V_\varrho dx' + \int_{d_{\vartheta_s}\setminus d'_{\vartheta_s}} |\tilde{u}_s|^2 |x'|^{-2\mu} dx' \xi^{4-4\mu},$$

where $d'_{\vartheta_s} = \{x' \in d_{\vartheta_s}\colon |x'| \geqslant |\xi|^{-1}\}$. From (4.13) and (4.15) it follows that the last term is estimated by the right-hand side of (4.15). Therefore, using (4.19) and (4.20) in (4.18), for sufficiently small ε we get

$$(4.21)\quad J_\varrho \leqslant c \sum_{s=1}^{2} \int_{d_{\vartheta_s}} |\tilde{f}_s|^2 |x'|^{2\mu} dx'.$$

Passing with ϱ to infinity we get

$$(4.22)\quad \sum_{s=1}^{2} \xi^2 \int_{d_{\vartheta_s}} (|\nabla' \tilde{u}_s|^2 + \xi^2 |\tilde{u}_s|^2) |x'|^{2\mu} dx' \leqslant c \sum_{s=1}^{2} \int_{d_{\vartheta_s}} |\tilde{f}_s|^2 |x'|^{2\mu} dx'.$$

Now we are in a position to prove the existence of solutions to the problem (4.10) in $L^2_\mu(\mathscr{D}_{\vartheta_1}) \oplus L^2_\mu(\mathscr{D}_{\vartheta_2})$. By the Hardy inequality (2.13) and by (4.22) we have

$$(4.23)\quad \sum_{s=1}^{2} \|\xi \tilde{u}_s\|^2_{H^1_\mu(d_{\vartheta_s})} \leqslant c\xi^2 \sum_{s=1}^{2} \int_{d_{\vartheta_s}} |\nabla' \tilde{u}_s|^2 |x'|^{2\mu} dx' \leqslant c \sum_{s=1}^{2} \int_{d_{\vartheta_s}} |\tilde{f}_s|^2 |x'|^{2\mu} dx',$$

where $0 < \mu \leqslant 1$.

By (4.22), the second and the third terms in the left-hand side od $(4.10)_1$ belong to $L_\mu(d_{\vartheta_1}) \oplus L_\mu(d_{\vartheta_2})$. By Theorem 2.1 we see that the terms with parameter ξ in $(4.10)_{2,3}$ belong to $H^{1/2}_\mu(d_{\vartheta_1}) \oplus H^{1/2}_\mu(d_{\vartheta_2})$ (since in view of (4.23), $\xi\tilde{u}_s$, $s = 1, 2$, belong to $H^1_\mu(d_{\vartheta_1}) \oplus H^1_\mu(d_{\vartheta_2})$). Then by Theorem 3.7 and Remark 3.8 we infer that $\tilde{u}_s \in L^2_\mu(d_{\vartheta_s})$, $s = 1, 2$, and

$$(4.24)\quad \sum_{s=1}^{2} \|\tilde{u}_s\|^2_{L^2_\mu(d_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} \|\tilde{f}_s\|^2_{L_\mu(d_{\vartheta_s})}.$$

Now, using the estimates (4.22), (4.24) and the Parseval identity we get (4.9) for $k = 0$ and $\mu \in (0, 1)$.

For $k > 0$ the proof goes by induction. Let us assume that (4.9) holds for $j \leqslant l-1$. To show (4.9) for $j = l$, assume $\tilde{f}_s \in \mathscr{E}^l_\mu(d_{\vartheta_s})$, $s = 1, 2$. Then

$$(4.25)\quad \xi\nabla' \tilde{u}_s \in \mathscr{E}^l_\mu(d_{\vartheta_s}), \qquad \xi^2 \tilde{u}_s \in \mathscr{E}^l_\mu(d_{\vartheta_s}).$$

Hence by inductive assumption,

$$(4.26)\quad \|\nabla' \tilde{u}_s\|_{\mathscr{E}^l_\mu(d_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} \|\tilde{f}_s\|_{\mathscr{E}^{l-1}_\mu(d_{\vartheta_s})},$$

$$(4.27)\quad \|\tilde{u}_s\|^2_{\mathscr{E}^l_\mu(d_{\vartheta_s})} \leqslant c \sum_{s=1}^{2} \|\tilde{f}_s\|^2_{\mathscr{E}^{l-2}_\mu(d_{\vartheta_s})},$$

and so

$$(4.28)\qquad \xi^2 \|\nabla' \tilde{u}_s\|^2_{\mathscr{E}^l_\mu(d\vartheta_s)} + \xi^4 \|\tilde{u}_s\|^2_{\mathscr{E}^l_\mu(d\vartheta_s)} \leqslant c \sum_{s=1}^{2} (\xi^2 \|\tilde{f}_s\|^2_{\mathscr{E}^{l-1}_\mu(d\vartheta_s)} + \xi^4 \|\tilde{f}_s\|^2_{\mathscr{E}^{l-2}_\mu(d\vartheta_s)}) \leqslant c \sum_{s=1}^{2} \|\tilde{f}_s\|^2_{\mathscr{E}^l_\mu(d\vartheta_s)}.$$

From (4.26), (4.27) it follows that $\tilde{u}_s \xi|_{\gamma_\sigma} \in \mathscr{E}^{l+1/2}_\mu(\gamma_\sigma)$, $\sigma = 0, 1, 2$, $s = 1, 2$. Therefore by Theorem 3.3 and Remark 3.8 we have

$$(4.29)\qquad \sum_{s=1}^{2} \|\tilde{u}_s\|^2_{\mathscr{E}^{l+2}_\mu(d\vartheta_s)} \leqslant c \sum_{s=1}^{2} \|\tilde{f}_s\|^2_{\mathscr{E}^l_\mu(d\vartheta_s)},$$

and hence, by the Parseval identity, $u_s \in L^{l+2}_\mu(\mathscr{D}_{\vartheta_2})$, $s = 1, 2$, and we get estimate (4.9) for $k = l$. Hence the theorem is proved for $\mu \in (0, 1)$ and $k > 0$ such that (4.8) is satisfied.

For a proof in the case $\mu = 0$, $\mu \geqslant 1$, see Theorem 4.2 in [10]. ∎

Theorems 4.2 and 4.3 imply

COROLLARY 4.4. *Let k, μ be as in Theorem 4.2. Let $f_s \in W^k_\mu(\mathscr{D}_{\vartheta_s})$ have compact supports, $s = 1, 2$. Then there exist solutions of the problem (4.3) such that $D^2 u_s \in W^k_\mu(\mathscr{D}_{\vartheta_s})$, $u_s \in \mathscr{H}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, and*

$$(4.23)\qquad \sum_{s=1}^{2} (\|D^2 u_s\|_{W^k_\mu(\mathscr{D}_{\vartheta_s})} + \|u_s\|_{\mathscr{H}(\mathscr{D}_{\vartheta_s})}) \leqslant c \sum_{s=1}^{2} \|f_s\|_{W^k_\mu(\mathscr{D}_{\vartheta_s})}.$$

Therefore the space $\Lambda^{k+2}_\mu(\mathscr{D}_\vartheta)$ can be used.

5. The problem (1.1) with variable coefficients in dihedral angles

In this section we consider the following problem:

$$(5.1)\qquad \begin{aligned} -\nabla A^s \nabla u_s + a^s \cdot \nabla u_s + b^s u_s &= f_s && \text{in } \mathscr{D}_{\vartheta_s},\ s = 1, 2,\\ A^s \cdot \nabla u_s \bar{n}|_{\varphi=(-1)^s \vartheta_s} &= \psi_s && \text{on } \Gamma_s,\ s = 1, 2,\\ A^1 \cdot \nabla u_1 \bar{n}|_{\varphi=0} &= A^2 \cdot \nabla u_2 \bar{n}|_{\varphi=0}, && \text{on } \Gamma_0,\\ u_1|_{\varphi=0} &= u_2|_{\varphi=0} && \text{on } \Gamma_0, \end{aligned}$$

where $A^s = (a^s_{ij} + b^s_{ij}(x))_{i,j=1,\dots,n}$, a^s_{ij} are constant matrices, $(b^s_{ij}(x))_{i,j=1,\dots,n}$, $a^s = (a^s_1(x), \dots, a^s_n(x))$, $b^s = b^s(x)$ have supports in $K^s_1 = \{x \in \mathscr{D}_{\vartheta_s}\colon |x| \leqslant 1\}$, and $b^s_{ij}(x) \in C^{k+1}(\mathscr{D}_{\vartheta_s})$, $a^s_i(x)$, $b^s(x) \in C^k(\mathscr{D}_{\vartheta_s})$, $i, j = 1, \dots, n$, $s = 1, 2$. Assume

$$(5.2)\qquad \sum_{s=1}^{2} \Big(\sum_{i,j=1}^{n} |b^s_{ij}|_{C^{k+1}(\mathscr{D}_{\vartheta_s})} + \sum_{i=1}^{n} |a^s_i|_{C^k(\mathscr{D}_{\vartheta_s})} + |b^s|_{C^k(\mathscr{D}_{\vartheta_s})} \Big) \leqslant \delta.$$

By a generalized solution of the problem (5.1) we mean functions $u_s \in \mathscr{H}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, satisfying the integral identity

(5.3) $$\sum_{s=1}^{2} \mathscr{A}^s(u_s, \eta_s) = \sum_{s=1}^{2} \Big(\int_{\mathscr{D}_{\vartheta_s}} f_s \eta_s + \int_{\Gamma_s} \psi_s \eta_s \Big),$$

where

$$\mathscr{A}^s(u_s, \eta_s) = \int_{\mathscr{D}_{\vartheta_s}} A^s \nabla u_s \nabla \eta_s + a^s \nabla u_s \eta_s + b^s u_s \eta_s,$$

valid for arbitrary functions $\eta_s \in \mathscr{H}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$.

By the Hardy inequality we have

$$\|u_s\|^2_{L_2(K_1^s)} \leqslant \int_{\mathscr{D}_{\vartheta_s}} \frac{|u_s|^2}{|x|^2} dx \leqslant c \|u_s\|^2_{\mathscr{H}(\mathscr{D}_{\vartheta_s})},$$

so

(5.4) $$\mathscr{A}^s(u_s, u_s) \geqslant a_0^s \|u_s\|^2_{\mathscr{H}(\mathscr{D}_{\vartheta_s})} - c\delta \big(\|u_s\|_{\mathscr{H}(\mathscr{D}_{\vartheta_s})} + \|u_s\|_{L_2(K_1^s)} \big) \|u_s\|_{L_2(K_1^s)}$$

$$\geqslant \tfrac{1}{2} a_0^s \|u_s\|^2_{\mathscr{H}(\mathscr{D}_{\vartheta_s})} \quad \text{for } \delta \leqslant \frac{a_0^s}{2c}.$$

THEOREM 5.1. *Let f_s, ψ_s, $s = 1, 2$, have compact supports and satisfy*

(5.5) $$N^2 = \sum_{s=1}^{2} \Big(\int_{\mathscr{D}_{\vartheta_s}} |f_s|^2 |x - x_0|^2 dx + \int_{\Gamma_s} |\psi_s|^2 |x - x_0|^2 ds \Big) < \infty,$$

where $x_0 \in M$. Let δ be sufficiently small. Then there exists a generalized solution of (5.1) *such that the integral identity* (5.3) *is satisfied and*

(5.6) $$\sum_{s=1}^{2} \|u_s\|_{\mathscr{H}(\mathscr{D}_{\vartheta_s})} \leqslant cN.$$

The existence follows from the Lax–Milgram Theorem. Inequality (5.6) follows from (5.3) applied to $\eta_s = u_s$, $s = 1, 2$, and (5.4).

Theorem 5.2. *Let k, μ be as in Theorem* 4.3. *Let $f_s \in W_\mu^k(\mathscr{D}_{\vartheta_s})$, $\psi_s \in W_\mu^{k+1/2}(\Gamma_s)$, $s = 1, 2$, have compact supports. Let δ be sufficiently small. Then the problem* (5.1) *has a unique solution such that $D^2 u_s \in W_\mu^k(\mathscr{D}_{\vartheta_s})$, $u_s \in \mathscr{H}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, and*

(5.7) $$\sum_{s=1}^{2} \big(\|D^2 u_s\|_{W_\mu^k(\mathscr{D}_{\vartheta_s})} + \|u_s\|_{\mathscr{H}(\mathscr{D}_{\vartheta_s})} \big) \leqslant c \sum_{s=1}^{2} \big(\|f_s\|_{W_\mu^k(\mathscr{D}_{\vartheta_s})} + \|\psi_s\|_{W_\mu^{k+1/2}(\Gamma_s)} \big).$$

Proof. We construct a solution of the problem (5.1) by the following method of successive approximations:

$$-\sum_{i,j=1}^{n} a_{ij}^s u_{sx_ix_j}^{(m+1)} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} b_{ij}^s \frac{\partial u_s^{(m)}}{\partial x_j} - a^s \cdot \nabla u_s^{(m)} - b^s u_s^{(m)} + f_s$$

$$\equiv l_1 u_s^{(m)} + f_s, \quad s = 1, 2,$$

$$\sum_{j=1}^{2}\sum_{i=1}^{n} a_{ij}^{s} u_{sx_i}^{(m+1)} n_j|_{\varphi=(-1)^s\vartheta_s} = -\sum_{i=1}^{2}\sum_{j=1}^{n} b_{ij}^{s}(x)\frac{\partial u_s^{(m)}}{\partial x_j} n_i\Big|_{\varphi=(-1)^s\vartheta_s} + \psi_s$$
$$\equiv l_2 u_s^{(m)} + \psi_s, \qquad s = 1, 2,$$

(5.8)
$$\sum_{j=1}^{2}\sum_{i=1}^{n} (a_{ij}^{1} u_{1x_i}^{(m+1)} n_j - a_{ij}^{2} u_{2x_i}^{(m+1)} n_j)|_{\varphi=0}$$
$$= \sum_{i=1}^{2}\sum_{j=1}^{n}\left(b_{ij}^{2}\frac{\partial u_2^{(m)}}{\partial x_j} - b_{ij}^{1}\frac{\partial u_1^{(m)}}{\partial x_j}\right) n_i\Big|_{\varphi=0} \equiv \sum_{s=1}^{2} l_3^s u_s^{(m)}.$$
$$u_1^{(m+1)}|_{\varphi=0} = u_2^{(m+1)}|_{\varphi=0},$$

for $m \geqslant 0$ and $u^{(0)} = 0$. Assuming that $u_s^{(m)} \in \Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, and using

$$\sum_{s=1}^{2} (\|l_1 u_s^{(m)}\|_{W_\mu^k(\mathscr{D}_{\vartheta_s})} + \|l_2 u_s^{(m)}\|_{W_\mu^{k+1/2}(\Gamma_s)} + \|l_3^s u_s^{(m)}\|_{W_\mu^{k+1/2}(\Gamma_0)})$$
$$\leqslant c\delta \sum_{s=1}^{2} \|u_s^{(m)}\|_{\Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})},$$

Lemma 4.1 and Corollary 4.4 we see that $u_s^{(m+1)} \in \Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, and

(5.9)
$$\sum_{s=1}^{2} \|u_s^{(m+1)} - u_s^{(m)}\|_{\Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})} \leqslant c\delta \sum_{s=1}^{2} \|u_s^{(m)} - u_s^{(m-1)}\|_{\Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})}.$$

Therefore, for sufficiently small δ, the inequality (5.9) guarantees that the sequence $\{u_s^{(m)}\}$, $s = 1, 2$, converges to a solution $u_s \in \Lambda_\mu^{k+2}(\mathscr{D}_{\vartheta_s})$, $s = 1, 2$, of the problem (5.1). ∎

6. Existence in a bounded domain

In a bounded domain $\Omega = \Omega_1 \cup \Omega_2$ we consider the following problem (more general than the problem (1.1)):

(6.1)
$$-\nabla A^s(x)\nabla u_s + a^s(x)\cdot\nabla u_s + b^s(x) u_s = f_s \qquad \text{in } \Omega_s,\ s = 1, 2,$$
$$A^s(x)\nabla u_s\cdot\bar{n} = \psi_s \qquad \text{on } S_s,\ s = 1, 2,$$
$$A^1\nabla u_1\cdot\bar{n} = A^2\nabla u_2\cdot\bar{n}, \qquad u_1 = u_2, \text{ on } \Gamma,$$

where $A^s = \{a_{ij}^s(x)\}_{i,j=1,\dots,n}$ are symmetric matrices (see Section 1), $a_{ij}^s \in C^{k+1}(\Omega_s)$, a^s, $b^s \in C^k(\Omega_s)$, $s = 1, 2$.

By a weak solution for (6.1) we mean functions $u_s \in H^1(\Omega_s)$, $s = 1, 2$, which satisfy the integral identity

(6.2)
$$\sum_{s=1}^{2} \mathscr{A}^s(u_s, \eta_s) = \sum_{s=1}^{2} \left(\int_{\Omega_s} f_s\eta_s + \int_{S_s} \psi_s\eta_s\right) \equiv l(\eta),$$

for arbitrary $\eta_s \in H^1(\Omega_s)$, where

$$\mathscr{A}^s(u_s, \eta_s) = \int_{\Omega} A^s \nabla u_s \nabla \eta_s + a^s \cdot \nabla u_s \eta_s + b^s u_s \eta_s, \qquad s = 1, 2.$$

If $f_s \in L_2(\Omega_s)$, $\psi_s \in L_2(S_s)$, $s = 1, 2$, the right-hand side of (6.2) is a linear functional on $H^1(\Omega) = H^1(\Omega_1) \oplus H^1(\Omega_2)$ and

$$\|l\| = \sum_{s=1}^{2} (\|f_s\|_{L_2(\Omega_s)} + \|\psi_s\|_{L_2(S_s)}). \tag{6.3}$$

Then the problem of existence of a weak solution is formulated in the form of the operator equation in $H^1(\Omega)$

$$U + VU = F, \tag{6.4}$$

where $U = (u_1, u_2)$, V is a compact continuous operator and

$$\|F\|_{H^1(\Omega)} = \|l\|. \tag{6.5}$$

Having the existence of a weak solution and knowing that $S_s, \Gamma \in C^{k+2}$, $s = 1, 2$, we infer that it belongs to $H^{k+2}(\Omega_s')$, $s = 1, 2$, where Ω_s' is an arbitrary subdomain Ω_s such that $\operatorname{dist}(L, \Omega_s') > 0$.

Assume that $L \in C^{k+2}$. Then we can establish the regularity of solutions in a neighbourhood of the edge L. Let $z \in L$ and let U_z be a neighbourhood of z. Let $\mathscr{D}_{\vartheta_s(z)}$ be the dihedral angle between tangent spaces $\Gamma_s = T_z S_s$ and $\Gamma_0 = T_z \Gamma$, $s = 1, 2$, (where $T_z \Gamma$ is the tangent space to Γ at $z \in \Gamma$), whose edge is an $(n-2)$-dimensional linear space tangent to L at z. Therefore there exists a diffeomorphism $T_z^s \in C^{k+2}$, such that $\mathscr{D}_{\vartheta_s(z)} \supset V_z^{s'} = T_z^s V_z^s$, where $V_z^s = \Omega_s \cap U_z$ and

$$V_z^{s'} \ni \xi = T_z^s x, \qquad x \in V_z^s. \tag{6.6}$$

Let $\xi \in K_\varrho(z) = \{\xi \in \mathscr{D}_{\vartheta_1} \cup \mathscr{D}_{\vartheta_2} : |\xi - z| < \varrho\}$. We introduce new coordinates $y = z + (\xi - z)\varrho^{-1}$ in which the problem (6.1) can be written in the form

$$\begin{aligned} -\nabla A^{s\varrho} \nabla u_s + a^{s\varrho} \nabla u_s + b^{s\varrho} u_s &= \varrho^2 f_s, && \text{in } \Omega_s,\ s = 1, 2, \\ A^{s\varrho} \nabla u_s \bar{n} &= \varrho \psi_s && \text{on } \Gamma_s,\ s = 1, 2, \\ A^{1\varrho} \nabla u_1 \bar{n} = A^{2\varrho} \nabla u_2 \bar{n}, &\quad u_1 = u_2, && \text{on } \Gamma_0, \end{aligned} \tag{6.7}$$

where $a_{ij}^{s\varrho}(y) = a_{ij}^{s'}(z + \varrho(y - z))$, $a^{s\varrho}(y) = \varrho a^{s'}(z + \varrho(y - z))$, $b^{s\varrho}(y) = \varrho^2 b^{s'} \times (z + \varrho(y - z))$ and the dash denotes the coefficients of equation $(6.1)_1$ after transformation (6.6). Therefore, for sufficiently small δ, the condition (5.2) is satisfied. Hence we have

THEOREM 6.1. *Let $S_s, \Gamma, L \in C^{k+2}$, $A^s \in C^{k+1}$, $a^s, b^s \in C^k$, and assume that $f_s(\xi)$, $\psi_s(\xi)$, have bounded norms $\|f_s\|_{W^k_{\mu(z)}(K_\varrho(z))}$, $\|\psi_s\|_{W^{k+1/2}_{\mu(z)}(K_\varrho^{(s)}(z))}$, $s = 1, 2$, where $K_\varrho^{(s)}(z) = K_\varrho(z) \cap \Gamma_s$, $s = 1, 2$, and*

$$\sigma_1(z) > k + 1 - \mu(z) > 0, \qquad \mu(z) \geqslant 0. \tag{6.8}$$

Then the weak solution of the problem (6.1) *has a bounded norm* $\|u_s\|_{W^{k+2}_{\mu(z)}(\tilde{V}^s_z)}$, *where* $\operatorname{dist}(\tilde{V}^s_z, \Omega_s \setminus V^s_z) > 0$, $s = 1, 2$, *and*

$$(6.9)\qquad \sum_{s=1}^{2} \|u_s\|_{W^{k+2}_{\mu(z)}(\tilde{V}^s_z)} \leqslant c \sum_{s=1}^{2} \left(\|u_s\|_{H^1(V^s_z)} + \|f_s(\xi)\|_{W^k_{\mu(z)}(K_0)} + \|\psi_s(\xi)\|_{W^{k+1/2}_{\mu(z)}(K_0)}\right).$$

The proof is almost the same as that of Theorem 6.1 in [10].

Summing over all neighbourhoods of L we obtain

THEOREM 6.2. *Let* S_s, Γ, $L \in C^{k+2}$, $A^s \in C^{k+1}$, a^s, $b^s \in C^k$, $f_s \in W^k_{\mu(z)}(\Omega_s)$, $\psi_s \in W^{k+1/2}_{\mu(z)}(S_s)$, $s = 1, 2$, *and assume that* (6.8) *is satisfied for all* $z \in L$. *Then the weak solution of* (6.1) *belongs to* $W^{k+2}_{\mu(z)}(\Omega_s)$, $s = 1, 2$, *and*

$$(6.10)\qquad \sum_{s=1}^{2} \|u_s\|_{W^{k+2}_{\mu(z)}(\Omega_s)} \leqslant c \sum_{s=1}^{2} \left(\|u_s\|_{H^1(\Omega_s)} + \|f_s\|_{W^k_{\mu(z)}(\Omega_s)} + \|\psi_s\|_{W^{k+1/2}_{\mu(z)}(S_s)}\right).$$

Now we consider the problem (1.1). The solutions of this problem are determined up to an arbitrary constant. For definiteness we assume

$$(6.11)\qquad \int_{\Omega_s} u_s = 0, \qquad s = 1, 2.$$

Then we have

THEOREM 6.3. *Let the assumptions of Theorem* 6.2 *and* (6.11) *be satisfied. Then there exists a unique solution of the problem* (1.1) *such that* $u_s \in W^{k+2}_{\mu(z)}(\Omega_s)$, $s = 1, 2$, *and*

$$(6.12)\qquad \sum_{s=1}^{2} \|u_s\|_{W^{k+2}_{\mu(z)}(\Omega_s)} \leqslant c \sum_{s=1}^{2} \left(\|f_s\|_{W^k_{\mu(z)}(\Omega_s)} + \|\psi_s\|_{W^{k+1/2}_{\mu(z)}(S_s)}\right).$$

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