COLLOQUIUM MATHEMATICUM

VOL. XXIV 1972 FASC. 2

SOME REMARKS ON DVORETZKY'S THEOREM ON ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

BY

T. FIGIEL (WARSZAWA)

In the paper of Dvoretzky [1] very deep and meaningful results were formulated. They concerned the existence of nearly spherical k-dimensional central sections of an arbitrary symmetric convex body in a Euclidean space $\boldsymbol{R}^n$ of sufficiently high dimension. The proof is rather difficult, it depends on sophisticated estimations of various measures of some subsets of spheres and Grassmann manifolds. However, it is not complete; the proof of important Lemma 8 involves certain approximation procedure which is not quite obvious.

The purpose of the present paper is to fill this gap in Dvoretzky's proof and to make some simplifications. To achieve this we prove a result (Theorem 1) stating, roughly speaking, that the measure of the set of 2-dimensional linear subspaces of $\boldsymbol{R}^n$ intersecting the given Borel subset A of the unit sphere S^{n-1} is less than the lower Minkowski content of A.

In spite of its generality, we think that our proof is simpler than the direct justification of Dvoretzky's approximation which is known to us.

Lemma 8 of Dvoretzky is a particular case of Corollary 3 to Theorem 1. Instead of using Lemma 8, one can shorten and simplify the proof of Dvoretzky's Theorem 2 using our Corollary 1.

The other simplification we made is the deduction of Corollary 4 from Lemma 6, which gives the estimate of the mean value of the function $p(x) = \|x\|_\infty^{-1}$ on the unit sphere S^{n-1}. Dvoretzky's proof of this fact involved his Theorem 3B, whose proof is much more complicated.

We begin with some definitions and notations.

Let $\|\cdot\|$ be a Euclidean norm in $\boldsymbol{R}^n$, and let d be a geodesic distance on S^{n-1} given by the formula

$$d(x, y) \doteq 2 \arcsin(\tfrac{1}{2}\|x-y\|).$$

For any $A \subseteq S^{n-1}$, $x \in S^{n-1}$ and $r > 0$, we define

$$d(A, x) = \inf\{d(x, y) \colon y \in A\},$$

$$A_r = \{y \in S^{n-1}: \ d(A, y) \leqslant r\},$$
$$E_x = \{y \in S^{n-1}: \ d(x, y) = \tfrac{1}{2}\pi\},$$
$$(A, x) = \{y \in E_x: x\cos t + y\sin t \in A \ \text{ for some } \ t \in \boldsymbol{R}\}.$$

(This means that (A, x) is the image of a projection of A along the meridians through $\pm x$ on the equator E_x corresponding to the point x.)

Besides the usual Lebesgue measure $\mathscr{L}^n$ in $\boldsymbol{R}^n$ we shall use Hausdorff measures $\mathscr{H}^{n-1}$ and $\mathscr{H}^{n-2}$ (for definitions and basic properties see [2]). Let f be an $\mathscr{H}^{n-1}$-integrable function on S^{n-1}. We write for abbreviation

$$\int f(x)\,dx$$

instead of

$$\big(\mathscr{H}^{n-1}(S^{n-1})\big)^{-1} \int_{S^{n-1}} f(x)\,d\mathscr{H}^{n-1}x.$$

Let A be a Borel subset of S^{n-1}; we define

$$\mu(A) = \int \mathscr{H}^{n-2}(A, x)\,dx.$$

The significance of the functional μ is a consequence of the following remark (cf. [1]).

Let ν be the normalized invariant measure in the Grassmann manifold $G(n, 2)$ of all 2-dimensional linear subspaces of $\boldsymbol{R}^n$. Then, for every symmetric (i. e., $A = -A$) Borel subset A of S^{n-1},

$$\mu(A) = \mathscr{H}^{n-2}(S^{n-2})\nu\big(\{E \in G(n, 2): \ A \cap E \neq \emptyset\}\big).$$

Dvoretzky's proof needs a demonstration of the fact that the estimate

$$\mu(A) \leqslant \mathscr{H}^{n-2}(A) \tag{1}$$

holds for A in a sufficiently wide class of Borel subsets of S^{n-1} (containing for any subset X of sphere the boundaries of almost all sets X_r, $r > 0$).

Using some clever method, similar to that of Ohmann [3], Dvoretzky concludes that it suffices to consider the sets A of the form ∂X_r, where X is finite (∂Y denote the boundary of Y with respect to S^{n-1}). His Lemma 8 contains merely the sketch of the proof of (1) in this simple case.

Our method avoids a difficult direct comparison of μ and $\mathscr{H}^{n-2}$. Namely we prove

THEOREM 1. *If X is a Borel subset of S^{n-1}, then*

$$\mu(X) \leqslant \sigma(X) \overset{\mathrm{df}}{=} \liminf_{r \to 0+} \frac{1}{2r}\mathscr{H}^{n-1}(X_r).$$

This result reduces in many important cases the proof of (1) to the verification of the inequality $\sigma(A) \leqslant \mathscr{H}^{n-2}(A)$, which frequently is a direct consequence of known theorems relating Hausdorff measure to Minkowski content (see, e. g., [2]). We shall give such examples after the proof of the theorem.

In the final part of this proof, we use some modification of Ohmann-Dvoretzky method, for which we shall need some simple geometric estimations.

To gain this we fix a Borel subset $X \subset S^{n-1}$ and real numbers

$$0 < q \leqslant r \leqslant s < t < \tfrac{1}{2}\pi.$$

Let $S = \{z_1, \ldots, z_k\} \subseteq X$ be a minimal finite q-net for X (i. e., $X \subseteq S_q$). Let χ be the characteristic function of the set $V = X_t \cup (-X_t)$. We write

$$a = \mathscr{H}^{n-2}(S^{n-2}) \int \chi(x)\,dx. \tag{2}$$

Putting $C = 2\mathscr{H}^{n-2}(S^{n-2})\big(\mathscr{H}^{n-1}(S^{n-1})\big)^{-1}$, we have

$$\begin{aligned} a &= \tfrac{1}{2}C\mathscr{H}^{n-1}(V) = \tfrac{1}{2}C\mathscr{H}^{n-1}\big(X_t \cup (-X_t)\big) \\ &\leqslant \tfrac{1}{2}C\big(\mathscr{H}^{n-1}(X_t) + \mathscr{H}^{n-1}(-X_t)\big) = C\mathscr{H}^{n-1}(X_t). \end{aligned} \tag{3}$$

LEMMA 1. *Under the assumptions above, the following inequality holds:*

$$\mu(X) \leqslant \mu(\partial S_r) + a.$$

Proof. Let $x \in S^{n-1} \setminus V$ and $y \in (X, x)$. Consider the function

$$f(u) = d(S, x\cos u + y\sin u), \qquad 0 \leqslant u \leqslant \pi.$$

Since f is continuous and

$$\inf_{u \in [0,\pi]} f(u) \leqslant q \leqslant r < t < f(0).$$

there exists a $u_0 \in (0, \pi)$ such that $f(u_0) = r$. It is clear that $(x\cos u_0 + y\sin u_0) \in \partial S_r$. Hence we have proved that $y \in (\partial S_r, x)$. Since y was an arbitrary point of (X, x), we infer that

$$\mathscr{H}^{n-2}(X, x) \leqslant \mathscr{H}^{n-2}(\partial S_r, x). \tag{4}$$

For $x \in V$ we have

$$\mathscr{H}^{n-2}(X, x) \leqslant \mathscr{H}^{n-2}(E_x) \leqslant \mathscr{H}^{n-2}(S^{n-2}) + \mathscr{H}^{n-2}(\partial S_r, x). \tag{5}$$

It follows from (4) and (5) that, for any $x \in S^{n-1}$,

$$\mathscr{H}^{n-2}(X, x) \leqslant \mathscr{H}^{n-2}(\partial S_r, x) + \mathscr{H}^{n-2}(S^{n-2})\chi(x).$$

We conclude the proof of the lemma integrating this inequality with respect to $\mathscr{H}^{n-1}$ and recalling (2).

Let $z = z_i$ be a point of S and let $m > (t-r)^{-1}$ be an integer. For $\xi \in (\boldsymbol{R}^n)^* \setminus \{0\}$ we define $U_\xi = \xi^{-1}\big((-\infty, 0)\big)$. Consider the set

$$F_{i,m} = \{\xi \in (\boldsymbol{R}^n)^* \setminus \{0\} : U_\xi \cap \{z_i\}_{r+1/2m} = \emptyset\}.$$

The family $\{U_\xi\}_{\xi \in F_{i,m}}$ is an open covering of the compact set $\{x \in S^{n-1}: d(x, z) \geqslant r + 1/m\}$. Choose a finite subcovering $\{U_{\xi_1}, \ldots, U_{\xi_l}\}$ and define

$$G_i^m = S^{n-1} \setminus \bigcup_{j=1}^{l} U_{\xi_j}, \qquad H_i^m = \partial G_i^m, \qquad W^m = \bigcup_{i=1}^{k} G_i^m.$$

It follows from the construction that

$$S_{r+1/2m} \subseteq W^m \subseteq S_t. \tag{6}$$

Let p_i^m denote the projection of $\partial\{z_i\}_r$ onto H_i^m along the meridians through $\pm z_i$. An elementary computation shows that p_i^m is Lipschitzian with a Lipschitz constant

$$L_m \leqslant (\sin r)^{-2} \sin^2\left(r + \frac{1}{m}\right) \leqslant \left(1 + \frac{1}{rm}\right)^2.$$

Hence for every Borel subset $A \subseteq \partial\{z_i\}_r$

$$\mathscr{H}^{n-2}\big(p_i^m(A)\big) \leqslant \left(1 + \frac{1}{rm}\right)^{2(n-2)} \mathscr{H}^{n-2}(A). \tag{7}$$

LEMMA 2. $\liminf\limits_{m\to\infty} \mathscr{H}^{n-2}(\partial W^m) \leqslant \mathscr{H}^{n-2}(\partial S_r)$.

Proof. We define $A_i = \{z_i\}_r \cap \partial S_r$ for $1 \leqslant i \leqslant k$. Then we have

$$\mathscr{H}^{n-2}(\partial S_r) = \sum_{i=1}^{k} \mathscr{H}^{n-2}(A_i), \tag{8}$$

$$\partial W^m \subseteq \bigcup_{i=1}^{k} p_i^m(A_i) \quad \text{for any } m. \tag{9}$$

Equality (8) is a simple consequence of the minimality of the set S, since for $1 \leqslant i < j \leqslant k$ we have $z_i \neq z_j$ and, therefore,

$$\mathscr{H}^{n-2}\big((\partial\{z_i\}_r) \cap (\partial\{z_j\}_r)\big) = 0.$$

In order to prove (9), we take a $y \in \partial W^m$ and choose a j, $1 \leqslant j \leqslant k$, such that $d(y, z_j) = \min\limits_{1\leqslant i\leqslant k} d(y, z_i) = d_0$. Let $\bar{y}$ be the unique point of S^{n-1} such that $d(\bar{y}, z_j) = r$ and $d(\bar{y}, y) = d_0 - r$. For any $j' \neq j$ we have $d(\bar{y}, z_{j'}) \geqslant d(y, z_{j'}) - d(y, \bar{y}) \geqslant d_0 - (d_0 - r) = r$, hence $\bar{y} \in \partial S_r$ and, since $\bar{y} \in \{z_j\}_r$, we obtain $\bar{y} \in A_j$ and $y = p_j^m(\bar{y}) \in p_j^m(A_j) \subseteq \bigcup\limits_{i=1}^{k} p_i^m(A_i)$, which, because $y \in \partial W^m$ was arbitrary, concludes the proof of (9).

It follows from (9), (7) and (8) that

$$\liminf_{m\to\infty}\mathscr{H}^{n-2}(\partial W^m) \leqslant \liminf \mathscr{H}^{n-2}\Big(\bigcup_{i=1}^{k} p_i^m(A_i)\Big)$$

$$\leqslant \liminf_{m\to\infty}\sum_{i=1}^{k}\mathscr{H}^{n-2}\big(p_i^m(A_i)\big) \leqslant \sum_{i=1}^{k}\limsup_{m\to\infty}\mathscr{H}^{n-2}\big(p_i^m(A_i)\big)$$

$$\leqslant \sum_{i=1}^{k}\mathscr{H}^{n-2}(A_i) = \mathscr{H}^{n-2}(\partial S_r).$$

The next lemma was proved by Dvoretzky [1], for the sake of completeness we briefly sketch his proof.

LEMMA 3. *If A is a Borel subset of some equator E_y, then*

$$\mu(A) = \mathscr{H}^{n-2}(A).$$

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of disjoint Borel subsets of E_y. Let $x \in S^{n-1}\setminus E_y$. Since the projection along the meridians of E_y onto E_x is a homeomorphism, we have

$$\mathscr{H}^{n-2}\Big(\bigcup_{i=1}^{\infty} A_i, x\Big) = \sum_{i=1}^{\infty}\mathscr{H}^{n-2}(A_i, x).$$

Integrating this relation over S^{n-1}, we infer that μ restricted to the σ-algebra of Borel subsets of E_y is a σ-additive measure.

Observe that $\mu(E_y) = \mathscr{H}^{n-2}(E_y)$ and both μ and $\mathscr{H}^{n-2}$ are invariant with respect to orthogonal transformations of E_y. Hence the lemma is a direct consequence of the uniqueness of a normalized invariant measure on a homogeneous space of locally compact group (see, e. g., [2], Theorem 2.7.11).

LEMMA 4. $\mathscr{H}^{n-2}(\partial S_r) \geqslant 2\mu(X) - 4a$.

Proof. Observe that each of the sets H_i^m is a union of a finite number of subsets of equators. Since $\partial W^m \subseteq \bigcup_{i=1}^{k} H_i^m$, there exists a decomposition $\partial W^m = \bigcup_{j=1}^{p} C_j$, where C_j are Borel subsets of some equators such that $\mathscr{H}^{n-2}(C_j \cap C_{j'}) = 0$ for $1 \leqslant j < j' \leqslant p$. Clearly we may assume that every C_j is contained in a unique equator, say E_j. Using Lemma 3, we obtain

$$\mathscr{H}^{n-2}(\partial W^m) = \mathscr{H}^{n-2}\Big(\bigcup_{j=1}^{p} C_j\Big) = \sum_{j=1}^{p}\mathscr{H}^{n-2}(C_j) = \sum_{j=1}^{p}\mu(C_j) \tag{10}$$

$$= \sum_{j=1}^{p}\int \mathscr{H}^{n-2}(C_j, x)\,dx = \int \sum_{j=1}^{p}\mathscr{H}^{n-2}(C_j, x)\,dx$$

$$= \int\Big(\int_{E_x} N(v, x)\,d\mathscr{H}^{n-2}v\Big)dx,$$

where $N(v, x)$ denotes the number of j's, $1 \leqslant j \leqslant p$, such that the meridian joining x and $-x$ through v intersects the set C_j.

We estimate from below the internal integral.

Let $x \in S^{n-1} \setminus (V \cup \bigcup_{j=1}^{p} E_j)$, $v \in (\partial S_r, x)$ and let

$$f(u) = d(W^m, x\cos u + v\sin u), \qquad 0 \leqslant u \leqslant \pi,$$

$$u_1 = \inf\{u: f(u) = 0\}, \qquad u_2 = \sup\{u: f(u) = 0\}.$$

If $m > (t-r)^{-1}$, then it follows from (6) that $u_1 \neq u_2$. Hence $v^i = x\cos u_i + v\sin u_i$, $i = 1, 2$, are two different points of ∂W^m. Since $x \notin \bigcup_{j=1}^{p} E_j$, they must belong to different sets C_j. We have thus proved that

$$N(v, x) \geqslant 2\psi(v),$$

where ψ is the characteristic function of a set $(\partial S_r, x)$. Integrating this inequality over E_x, we obtain inequality

$$\int_{E_x} N(v, x)\, d\mathscr{H}^{n-2} v \geqslant 2\mathscr{H}^{n-2}(\partial S_r, x) \qquad \text{for } x \in S^{n-1} \setminus (V \cup \bigcup_{j=1}^{p} E_j).$$

Since for $x \in V$ we have

$$\int_{E_x} N(v, x)\, d\mathscr{H}^{n-2} v \geqslant 0 \geqslant 2\left(\mathscr{H}^{n-2}(\partial S_r, x) - \mathscr{H}^{n-2}(S^{n-2})\right)$$

and $\mathscr{H}^{n-1}(\bigcup_{i=1}^{p} E_j) = 0$, there is, by (10) and (2),

$$\begin{aligned} \mathscr{H}^{n-2}(\partial W^m) &= \int\Big(\int_{E_x} N(v, x)\, d\mathscr{H}^{n-2}(v)\Big) dx \\ &\geqslant 2\int \left(\mathscr{H}^{n-2}(\partial S_r, x) - \mathscr{H}^{n-2}(S^{n-2})\chi(x)\right) dx \\ &= 2\left(\mu(\partial S_r) - a\right) \qquad \text{for } m > (t-r)^{-1}. \end{aligned}$$

Lemma 4 follows immediately from this estimate and Lemmas 1 and 2.

LEMMA 5. $\int_{q}^{s} \mathscr{H}^{n-2}(\partial X_r)\, dr = \mathscr{H}^{n-1}(X_s \setminus X_q)$.

Proof. We use the following fact ([2], Theorem 3.2.11).

If A is a Borel subset of $\boldsymbol{R}^n$, and $f: A \to \boldsymbol{R}$ is a Lipschitzian map, then

$$\int_A \|Df(x)\|\, d\mathscr{L}^n x = \int_{\boldsymbol{R}} \mathscr{H}^{n-1}\left(A \cap f^{-1}(y)\right) dy. \tag{11}$$

Fix an $\varepsilon > 0$, and let $A = \{tx\colon t\in[1, 1+\varepsilon], x\in X_s\setminus X_q\}$. Define $f: A \to \boldsymbol{R}$ by the formula

$$f(x) = d\left(X, \frac{x}{\|x\|}\right).$$

It is easy to verify that f is a contraction and

$$(1+\varepsilon)^{-1} \leqslant \operatorname*{inf\,ess}_{x\in A} \|Df(x)\| \leqslant \operatorname*{sup\,ess}_{x\in A} \|Df(x)\| \leqslant 1. \tag{12}$$

It follows from (11) and (12) that

$$(1+\varepsilon)^{-1}\mathscr{L}^n(A) \leqslant \int_{\boldsymbol{R}} \mathscr{H}^{n-1}\big(A\cap f^{-1}(y)\big)\,dy \leqslant \mathscr{L}^n(A). \tag{13}$$

Observe that

$$\mathscr{L}^n(A) = \frac{1}{n}\,[(1+\varepsilon)^n - 1]\mathscr{H}^{n-1}(X_s\setminus X_q),$$

$$\mathscr{H}^{n-1}\big(A\cap f^{-1}(y)\big) = \begin{cases} \dfrac{1}{n-1}\,[(1+\varepsilon)^{n-1}-1]\mathscr{H}^{n-2}(\partial X_y) & \text{for } q < y \leqslant s,\\ 0 & \text{otherwise.}\end{cases}$$

Substituting these values in (13), dividing by ε, and letting ε approach 0, we obtain the desired formula.

Proof of the theorem. Without loss of generality we may assume that $\sigma(X) < \infty$ and therefore

$$\lim_{t\to 0+} \mathscr{H}^{n-1}(X_t) = 0. \tag{14}$$

It follows from Lemma 4 that for $r\in[q, s]$ we have

$$\mathscr{H}^{n-1}(\partial S_r) \geqslant 2\mu(X) - 4a.$$

Integrating this inequality over the segment $[q, s]$ and using Lemma 5, we obtain

$$(s-q)\big(2\mu(X)-4a\big) \leqslant \int_q^s \mathscr{H}^{n-2}(\partial S_r)\,dr = \mathscr{H}^{n-1}(S_s\setminus S_q) < \mathscr{H}^{n-1}(X_s).$$

Recalling (3) and letting q approach 0 (s, t — fixed), we infer that

$$s\big(2\mu(X) - 4a\big) \leqslant \mathscr{H}^{n-1}(X_s),$$

$$\mu(X) \leqslant \frac{1}{2s}\mathscr{H}^{n-1}(X_s) + 2C\mathscr{H}^{n-1}(X_t).$$

We conclude the proof letting s approach 0 (t fixed) and then using (14).

The following corollary contains exactly the fact which is needed for the proof of Dvoretzky's results:

COROLLARY 1. *If X be a subset of S^{n-1}, then, for almost all $r > 0$,*

$$\mu(\partial X_r) \leqslant \mathscr{H}^{n-2}(\partial X_r).$$

Proof. Inasmuch as $(\partial X_r)_s \subset X_{r+s} \setminus X_{r-s'}$ for $0 < s < s' < r$, we have

$$\mathscr{H}^{n-1}\big((\partial X_r)_s\big) \leqslant \mathscr{H}^{n-1}(X_{r+s}) - \mathscr{H}^{n-1}(X_{r-s}).$$

Hence, for every $r > 0$,

$$\mu(\partial X_r) \leqslant \sigma(\partial X_r) = \liminf_{s\to 0_+} \frac{1}{2s} \mathscr{H}^{n-1}\big((\partial X_r)_s\big)$$

$$\leqslant \liminf_{s\to 0_+} \frac{1}{2s} \big(\mathscr{H}^{n-1}(X_{r+s}) - \mathscr{H}^{n-1}(X_{r-s})\big).$$

Since the right-hand side is, by Lemma 5, equal to $\mathscr{H}^{n-2}(\partial X_r)$ for almost all r, the proof is complete.

In order to formulate the next corollary, we denote by $K(X, r)$ the set

$$\{y \in \boldsymbol{R}^n \colon \|x-y\| \leqslant r \text{ for some } x \in X\},$$

where $X \subset \boldsymbol{R}^n$ and $r > 0$.

COROLLARY 2. *If A is a Borel subset of S^{n-1} such that*

$$\liminf_{r\to 0_+} \frac{1}{\pi r^2} \mathscr{L}^n\big(K(A, r)\big) \leqslant \mathscr{H}^{n-2}(A),$$

then (1) *holds.*

Proof. It is enough to prove that, for every subset X of the sphere and every $\varepsilon > 0$,

$$\liminf_{r\to 0_+} \frac{1}{\pi r^2} \mathscr{L}^n\big(K(X, r)\big) \geqslant \sigma(X) - \varepsilon. \tag{14}$$

Consider the function $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$, $f(x) = \|x\|$. Since $\|Df(x)\| = 1$ for any $x \neq 0$, we obtain, using the theorem cited in the proof of Lemma 5, that

$$\mathscr{L}^n\big(K(X, r)\big) = \int_{\boldsymbol{R}} \mathscr{H}^{n-1}\{x \in K(X, r) \colon \|x\| = t\}\, dt. \tag{15}$$

Observe that

$$\{x \in K(X, r) \colon \|x\| = t\} = t X_{a(r,t)}, \tag{16}$$

where the function a does not depend on the set X and satisfies the following condition:

$$a(r, t) < r \quad \text{for } t \in (1-r, 1+r). \tag{17}$$

Since there exists a $\delta > 0$ such that for $s < \delta$

$$\text{(18)} \qquad \frac{1}{2s}\mathscr{H}^{n-1}(X_s) \geqslant \sigma(X) - \varepsilon,$$

we infer, using (15)-(18), that for $r < \delta$ there is

$$\text{(19)} \qquad \mathscr{L}^n\big(K(X, r)\big) = \int_{1-r}^{1+r} t^{n-1}\mathscr{H}^{n-1}(X_{a(r,t)})\,dt$$

$$\geqslant 2\big(\sigma(X) - \varepsilon\big) \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt.$$

Inequality (14) is an easy consequence of (19) and the relation

$$\text{(20)} \qquad \lim_{r\to 0_+} \frac{1}{\pi r^2} \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt = \frac{1}{2}.$$

Indeed, it suffices to divide (19) by πr^2 and let r approach 0.

To prove (20) observe that for any equator $E = E_x$ we have

$$\text{(21)} \quad \lim_{r\to 0_+} \frac{1}{\pi r^2} \mathscr{L}^n\big(K(E, r)\big) = \mathscr{H}^{n-2}(E) = \lim_{r\to 0_+} \frac{1}{2r}\mathscr{H}^{n-1}(E_r) = \sigma(E) \neq 0.$$

Using (21), we obtain easily from (19) the inequality

$$\sigma(E) \geqslant 2\big(\sigma(E) - \varepsilon\big) \limsup_{r\to 0_+} \frac{1}{\pi r^2} \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt.$$

Since ε was arbitrary and $\sigma(E) \neq 0$, we obtain

$$\limsup_{r\to 0_+} \frac{1}{\pi r^2} \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt \leqslant \frac{1}{2}.$$

Now observe that by (21) there exists a $\delta_1 > 0$ such that

$$\frac{1}{2s}\mathscr{H}^{n-1}(E_s) \leqslant \sigma(E) + \varepsilon \qquad \text{for } s \in (0, \delta_1).$$

This inequality implies, as before,

$$\text{(22)} \qquad \mathscr{L}^n\big(K(E, r)\big) \leqslant 2\big(\sigma(E) + \varepsilon\big) \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt.$$

Using (22) instead of (19) and reasoning as in the preceding proof, we obtain

$$\liminf_{r\to 0_+} \frac{1}{\pi r^2} \int_{1-r}^{1+r} t^{n-1} a(r, t)\,dt \geqslant \frac{1}{2}$$

which completes the proof of Corollary 2.

COROLLARY 3. *If $A \subset S^{n-1}$ is the image of a compact subset of $\boldsymbol{R}^{n-2}$ under a Lipschitzian map, then*

$$\mu(A) \leqslant \mathscr{H}^{n-2}(A).$$

Proof. Since, by the theorem of M. Kneser (cf. [2], Theorem 3.2.39), the assumption satisfied by A implies

$$\lim_{r\to 0_+} \frac{1}{\pi r^2} \mathscr{L}^n\big(K(A, r)\big) = \mathscr{H}^{n-2}(A),$$

Corollary 3 is a special case of Corollary 2.

Let us pass to the problem of simplification of Dvoretzky's Theorem 3.

Let $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n \setminus \{0\}$ and $p_n(x) = \big(\max(|x_1|, \ldots, |x_n|)\big)^{-1}$. We prove the following

LEMMA 6. *If M_n is the mean value of the function p_n on the unit sphere S^{n-1}, then $M_n = O(\sqrt{n/\log n})$.*

Proof. We may assume that $n \geqslant 2$. Define the function $f_n \in L_1(\boldsymbol{R}^n, \mathscr{L}^n)$ by the formula

$$f_n(x) = \pi^{-n/2} \exp(-\|x\|^2) p_n(x).$$

Let $I_n = \int_{\boldsymbol{R}^n} f_n(x)\, d\mathscr{L}^n x$. Passing to spherical coordinates and using the well-known relation $\mathscr{H}^{n-1}(S^{n-1}) = 2\pi^{n/2}\Gamma(n/2)^{-1}$, we obtain

$$\begin{aligned} I_n &= \int_0^\infty dr \int_{rS^{n-1}} f_n(x)\, d\mathscr{H}^{n-1}x \\ &= \int_0^\infty \mathscr{H}^{n-1}(S^{n-1}) r^{n-1} e^{-r^2} r^{-1} \pi^{-n/2} M_n\, dr \\ &= 2M_n \Gamma\left(\frac{n}{2}\right)^{-1} \int_0^\infty r^{n-2} e^{-r^2} dr = M_n \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n}{2}\right)^{-1}. \end{aligned}$$

Since $\Gamma(n/2)\Gamma\big((n-1)/2\big)^{-1} \sim \sqrt{n/2}$, it is enough to show that $I_n = 0(1/\sqrt{\log n})$. Let $C(t) = p_n^{-1}\big([t^{-1}, \infty)\big)$ and

$$\psi(t) = \int_{C(t)} f_n(x)\, d\mathscr{L}^n x,$$

$$\eta(t) = \pi^{-n/2} \int_{C(t)} \exp(-\|x\|^2)\, d\mathscr{L}^n x = \left(\frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du\right)^n.$$

Then

$$\psi'(t) = t^{-1}\eta'(t) \quad \text{for } t > 0.$$

Integrating by parts, we obtain

$$I_n = \int_0^\infty d\psi(t) = \int_0^\infty t^{-1} d\eta(t) = \int_0^\infty t^{-2}\eta(t)\,dt \tag{23}$$

$$= \int_0^\infty t^{-2}\left(\frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du\right)^n dt.$$

Using a simple inequality

$$\int_0^t e^{-u^2}du \leqslant \min\left(\frac{\sqrt{\pi}}{2}, \frac{2t}{t+1}\right) \qquad \text{for } t \geqslant 0, \tag{24}$$

we easily infer that for any positive A the following estimate holds:

$$I_n = \int_0^\infty t^{-2}\left(\frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du\right)^n dt = \int_0^A + \int_A^\infty$$

$$\leqslant \frac{16}{\pi}\int_0^A (t+1)^{-2}\left(\frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du\right)^{n-2} dt + \int_A^\infty t^{-2}dt$$

$$= \frac{16}{\pi}\,\frac{\sqrt{\pi}}{2(n-1)}\int_0^A (t+1)^{-2}e^{t^2}\frac{d}{dt}\left[\left(\frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du\right)^{n-1}\right]dt + A^{-1}$$

$$< \frac{5}{n-1}\sup_{0\leqslant t\leqslant A}(t+1)^{-2}e^{t^2} + A^{-1}.$$

It is easy to check that

$$\sup_{0\leqslant t\leqslant A}(t+1)^{-2}e^{t^2} = \max\left(1, (A+1)^{-2}e^{A^2}\right),$$

hence substituting $A = \sqrt{\log n}$ we have for n large enough (greater than 5) the estimate

$$I_n \leqslant \frac{1}{\sqrt{\log n}} + \frac{5}{n-1}\,\frac{n}{\log n} \leqslant \frac{1}{\sqrt{\log n}} + \frac{6}{\log n},$$

which completes the proof of the lemma.

Remark 1. Estimating more carefully, one can obtain

$$I_n = \frac{1}{\sqrt{\log n}} + O\left(\frac{\log\log n}{(\log n)^{3/2}}\right).$$

In a similar way, one can prove that, for any $a > 0$,

$$\mathscr{H}^{n-1}(S^{n-1})^{-1} \int\limits_{S^{n-1}} p_n(x)^a d\mathscr{H}^{n-1} x$$

$$= \left(\frac{n}{2\log n}\right)^{a/2} + O\left(n^{a/2}(\log n)^{-(a+2)/2} \log\log n\right).$$

COROLLARY 4. *If R_n is a positive number such that*

$$\mathscr{H}^{n-1}\{x \in S^{n-1}\colon\ p_n(x) \geqslant R_n\} = \tfrac{1}{2}\mathscr{H}^{n-1}(S^{n-1}),$$

then $R_n = O(\sqrt{n/\log n})$.

Proof follows from Lemma 6 and an obvious estimation

$$M_n = \int p_n(x)\,dx \geqslant \frac{1}{2} R_n.$$

Remark 2. The estimate $R_n = o(\sqrt{n})$ from Theorem 3A of Dvoretzky, being in many cases sufficient, follows directly from (23), (24) and Lebesgue's theorem.

REFERENCES

[1] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proceedings of the Symposium on Linear Spaces, Jerusalem 1961, p. 123-160.

[2] H. Federer, *Geometric measure theory*, Berlin 1969.

[3] D. Ohmann, *Eine Verallgemeinerung der Steinerschen Formel*, Mathematische Annalen 129 (1955), p. 209-212.

Reçu par la Rédaction le 15. 9. 1970