COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 1

MIXTURES OF NON-ATOMIC MEASURES. II

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0. Introduction. The purpose of this paper is to characterize all Borel spaces on which every measure is a mixture of non-atomic measures and also to characterize all spaces in which only non-atomic measures are mixtures of non-atomic measures. After fixing the notation and definitions in Section 1 we shall undertake these problems in Section 2. In the last section we shall study structure theoretic properties of Borel spaces introduced in Section 2.

1. Preliminaries. All measures are probability measures. Measure theoretic terminology is from [1], and Borel space terminology from [3]. Let $(X, \boldsymbol{A})$ be a Borel space. It is said to be *trivial* if $\boldsymbol{A} = \{\emptyset, X\}$. Let μ be a measure on $\boldsymbol{A}$. μ is a 0-1 *measure* if it takes only two values — 0 and 1. A measure μ is *singular* if there is an $A \in \boldsymbol{A}$ such that $\mu(A) = 1$ and, for every non-atomic measure P on $\boldsymbol{A}, P(A) = 0$; that is, if it is singular — in the customary sense of measure theory — with respect to the family of all non-atomic measures on $\boldsymbol{A}$. μ has *enough non-atomic measures around it* if it is not singular, that is, whenever $\mu(A) = 1$, there is a non-atomic measure P on $\boldsymbol{A}$ such that $P(A) > 0$ or, equivalently, such that $P(A) = 1$. (We would have liked to say μ is regular in this case, but the term "regular" has a different customary meaning). $(X, \boldsymbol{A})$ is *singular* if all 0-1 measures are singular. $(X, \boldsymbol{A})$ is *regular* if all 0-1 measures have enough non-atomic measures around them. Equivalently, for any non-empty $A \in \boldsymbol{A}$ there is a non-atomic measure on $\boldsymbol{A}$ supported by A. By standard normalization arguments, we can even say that $(X, \boldsymbol{A})$ is regular if for any non-empty $A \in \boldsymbol{A}$ there is a non-atomic measure μ on $\boldsymbol{A}$ with $\mu(A) > 0$. $(X, \boldsymbol{A})$ is *highly singular* if, for every 0-1 measure μ on $\boldsymbol{A}$, there is an atom $A \in \boldsymbol{A}$ such that $\mu(A) = 1$. Let now $(X, \boldsymbol{A})$ and $(Y, \boldsymbol{B})$ be two Borel spaces with measures μ and λ, respectively. If

$$\mu(A) = \int Q(y, A)\, d\lambda(y) \quad \text{for all } A \in \boldsymbol{A},$$

then Q is called *transition function* and we say that μ is a *mixture of* *measures* $Q(y, \cdot)$ with respect to a measure λ. If all $Q(y, \cdot)$ are non-atomic measures, then we say that μ is a *mixture of non-atomic measures.* μ is said to be *supported* by A if $A \in \boldsymbol{A}$ and $\mu(A) = 1$.

2. Mixture problem. In this section $(X, \boldsymbol{A})$ is a fixed Borel space. For our characterization problem we start with the following

LEMMA 1. *A 0-1 measure μ on $\boldsymbol{A}$ is a mixture of non-atomic measures iff it has enough non-atomic measures around it.*

Proof. The proof of the sufficiency is easy and we shall show only the necessity. Let Y be the set of all non-atomic measures on $\boldsymbol{A}$, and

$$B_A = \{P \in Y\colon P(A) < 1\} \quad \text{for } A \in \boldsymbol{A},$$

$$\boldsymbol{I} = \{B \subset Y\colon B \subset B_A \text{ for some } A \text{ with } \mu(A) = 1\},$$

$$\boldsymbol{B} = \{B \subset Y\colon B \in \boldsymbol{I} \text{ or } B^c \in \boldsymbol{I}\},$$

$$\lambda(B) = \begin{cases} 0 & \text{if } B \in \boldsymbol{I}, \\ 1 & \text{if } B^c \in \boldsymbol{I}, \end{cases}$$

$$Q(P, A) = P(A) \quad \text{for } (P, A) \in Y \times \boldsymbol{A}.$$

Then $\boldsymbol{I}$ is a σ-ideal of subsets of Y; $(Y, \boldsymbol{B})$ is a Borel space; $Y \notin \boldsymbol{I}$; λ is a measure on $\boldsymbol{B}$; Q is a transition function; and μ is a mixture of $Q(P, \cdot)$ with respect to λ.

THEOREM 1. *Every measure μ on $(X, \boldsymbol{A})$ is a mixture of non-atomic measures iff $(X, \boldsymbol{A})$ is regular.*

Proof. If $(X, \boldsymbol{A})$ is regular and μ is a 0-1 measure, then, by Lemma 1, μ is a mixture of non-atomic measures. Any measure μ is a convex combination of a non-atomic measure and of finite or countably many 0-1 measures. So μ itself, in its turn, is a mixture of non-atomic measures. The converse part follows directly from Lemma 1.

Remark 1. A 0-1 measure μ has enough non-atomic measures around it iff μ is a strong limit of the set of all non-atomic measures, that is, iff there is a net μ_a of non-atomic measures such that $\mu_a(A) \to \mu(A)$ for all $A \in \boldsymbol{A}$.

To see the necessity, note that, whenever $\mu(A) = 1$, there is an a_0 with $\mu_{a_0}(A) > 0$. For the converse part, observe that $\boldsymbol{D} = \{B\colon \mu(B) = 1\}$ is a directed set under set inclusion. For $B \in \boldsymbol{D}$ fix a non-atomic measure P_B supported by B. Then $\{P_B\colon B \in \boldsymbol{D}\}$ is a net strongly converging to μ. As a consequence, $(X, \boldsymbol{A})$ is regular iff the set of non-atomic measures is strongly dense in the set of all probability measures on $(X, \boldsymbol{A})$.

Remark 2. In Lemma 1, the mixand measures $Q(y, \cdot)$ are all non-atomic, but the mixing measure λ is 0-1 valued. One might ask whether

λ can be chosen to be non-atomic, so that μ would be a non-atomic mixture of non-atomic measures. We shall show that it is not possible. Let $(Y, \boldsymbol{B})$ be any Borel space and let $Q(y, A)$ be a transition function on $Y \times \boldsymbol{A}$ such that, for any y, $Q(y, \cdot)$ is non-atomic. Let λ be any measure on $\boldsymbol{B}$ such that μ is a mixture of Q with respect to λ. Let $\boldsymbol{B}_0 \subset \boldsymbol{B}$ be the σ-algebra generated by functions $Q(\cdot, A)$ for $A \in \boldsymbol{A}$. We show that λ takes on $\boldsymbol{B}_0$ only two values — 0 and 1. Indeed, by the integral representation of μ and by the fact that μ takes only two values, we have

$$\lambda\{y: Q(y, A) < r\} = 0 \text{ or } 1 \quad \text{for any } A \in \boldsymbol{A} \text{ and } 0 \leqslant r \leqslant 1.$$

So λ on $\boldsymbol{B}_0$ is two-valued.

THEOREM 2. *Every mixture of non-atomic measures on $(X, \boldsymbol{A})$ is non-atomic iff $(X, \boldsymbol{A})$ is singular.*

Proof. Necessity. By Lemma 1, no 0-1 measure is a mixture of non-atomic measures. If μ is a mixture of non-atomic measures and if μ is not non-atomic, then it is not difficult to verify that all atomic parts of μ are also mixtures of non-atomic measures. Thus, if μ is not non-atomic, then, by suitable normalizations, we can produce a 0-1 measure which is a mixture of non-atomic measures, contradicting the first sentence of our proof.

Sufficiency. If $(X, \boldsymbol{A})$ is not singular, then, by definition, there is a 0-1 measure with enough non-atomic measures around it, which will then be a mixture of non-atomic measures by Lemma 1.

Remark 3. If $(X, \boldsymbol{A})$ is singular, then for every $p \in X$ there is a set $A \in \boldsymbol{A}$ such that $p \in A$ and A does not support any non-atomic measure. Of course, the converse is not true. This can be seen by taking the unit square X and the σ-field $\boldsymbol{A}$ generated by Borel sets contained in countably many vertical lines. But $(X, \boldsymbol{A})$ is singular iff for any maximal σ-filter $\boldsymbol{F} \subset \boldsymbol{A}$ there is an $A \in \boldsymbol{F}$ such that A does not support any non-atomic measure. This is more or less a restatement of the definition of singularity.

Remark 4. As an analogue of Remark 1, it can be noted that $(X, \boldsymbol{A})$ is singular iff the set of non-atomic measures is strongly closed in the set of probability measures on $(X, \boldsymbol{A})$.

3. Structural properties of regular and singular spaces. Any regular space is atomless, but the converse is false in view of the following theorem:

THEOREM 3. *Any uncountable set X has an atomless Borel structure $\boldsymbol{A}$ such that $(X, \boldsymbol{A})$ has no non-atomic measures.*

Proof. Consider $Y = 2^X$, the X-fold product of the two-point space $\{0, 1\}$, and $\boldsymbol{B}$, the product σ-field, where each coordinate space is equipped with the σ-field of all its subsets. Let Z be the set of those points of Y which have zeros in all but finitely many coordinates, and let $\boldsymbol{A} = \{B \cap Z:$

$B \in \boldsymbol{B}\}$. Then $\boldsymbol{A}$ is atomless (as in [3]) and, since Z has the same cardinality as X, it suffices to show that there are no non-atomic measures on $\boldsymbol{A}$. Let P be any non-atomic measure on $\boldsymbol{B}$. It suffices to exhibit a set W such that $Z \subset W \in \boldsymbol{B}$ and $P(W) = 0$. To this end, fix a countably generated σ-algebra $\boldsymbol{B}' \subset \boldsymbol{B}$ such that the restriction of P to $\boldsymbol{B}'$ is non-atomic [2]. We may assume that $\boldsymbol{B}'$ is generated by coordinate maps on Y determined by points of a countable set $D \subset X$. For each finite $F \subset D$, let p_F be the point of Y with all coordinates zero except those in F, let A_F be the $\boldsymbol{B}'$-atom containing p_F, and let W be the union of all A_F's so obtained as F varies through finite subsets of D. It is easy to see that this W is what we are looking for, thus completing the proof.

Remark 5. Actually, the proof of Theorem 3 yields the following assertion: Any uncountable set X has an atomless Borel structure $\boldsymbol{A}$ such that any countably generated $\boldsymbol{A}_0 \subset \boldsymbol{A}$ has only countably many atoms. Clearly, such a space cannot support a non-atomic measure.

We now give some examples of regular spaces.

Example 1. *If $(X_1, \boldsymbol{A}_1)$ is regular and $(X_2, \boldsymbol{A}_2)$ is any Borel space, then their product is again regular.*

Example 2. *If $(X_i, \boldsymbol{A}_i)$, $i \geqslant 1$, is a sequence of atomless Borel structures, then their product $(X, \boldsymbol{A})$ is regular.*

To this end, we first prove that if $p \in A \in \boldsymbol{A}$, then there are sets $A_i \in \boldsymbol{A}_i$ such that

$$p \in \mathop{\times}_{i \geqslant 1} A_i \subset A.$$

For this, fix a countably generated $\boldsymbol{B}_i \subset \boldsymbol{A}_i$ such that $A \in \mathop{\times}_{i \geqslant 1} \boldsymbol{B}_i$, the product σ-field, and then let A_i be the $\boldsymbol{B}_i$-atom containing the i-th coordinate of p. Clearly, $p \in \mathop{\times}_{i \geqslant 1} A_i \subset A$. Now, if $A \in \boldsymbol{A}$ is a non-empty set, then take non-empty sets $A_i \in \boldsymbol{A}_i$ such that $\mathop{\times}_{i \geqslant 1} A_i \subset A$. Each $(X_i, \boldsymbol{A}_i)$ being atomless, one can fix a measure μ_i on $\boldsymbol{A}_i$ supported by A_i and taking values $0, \frac{1}{2}, 1$. The product measure μ is supported by A and is non-atomic.

Example 3. *Any uncountable product $(X, \boldsymbol{A})$ of non-trivial Borel spaces $(X_i, \boldsymbol{A}_i)$ is regular.*

First note that $\boldsymbol{A}$ has a non-atomic measure. For if μ_i is any measure on $\boldsymbol{A}_i$ taking values $0, \frac{1}{2}, 1$, then the product measure μ is non-atomic on $\boldsymbol{A}$. But, given any set $A \in \boldsymbol{A}$, it depends only on countably many coordinates, and so the same kind of argument can be repeated to produce a non--atomic measure supported by A.

The above-given examples are the only ways of manufacturing product spaces which are regular, in view of the following theorem:

THEOREM 4. *A product of Borel spaces is regular iff either one of them is regular, or infinitely many are atomless, or uncountably many are non-trivial.*

Proof. The necessity is proved by Examples 1-3. For the converse part, suppose none of the conditions holds. Since only countably many spaces are non-trivial, we can assume that the space $(X, \boldsymbol{A})$ is a product of $(X_i, \boldsymbol{A}_i)$, $i \geqslant 1$. Since finitely many spaces are atomless, let us say that the first k spaces are atomless and let A_i, $i \geqslant k+1$, be any atoms of the spaces X_i, $i \geqslant k+1$, respectively. Of course, if $k = 0$, then, clearly,

$$A = \mathop{\times}_{i\geqslant 1} A_i$$

is an atom of $\boldsymbol{A}$, and so $(X, \boldsymbol{A})$ is not regular. Thus $k \geqslant 1$. Fix 0-1 measures μ_i on $\boldsymbol{A}_i$, $i \leqslant k$, so that μ_i does not have enough non-atomic measures around it and μ_i, $i > k$, is the point mass at A_i in $(X_i, \boldsymbol{A}_i)$. Let μ be the product measure. We claim that μ does not have enough non-atomic measures around it. More specifically, let $B_i \in \boldsymbol{A}_i$, $i \leqslant k$, be such that $\mu_i(B_i) = 1$ and such that there is no non-atomic measure on $(X_i, \boldsymbol{A}_i)$ supported by B_i. We prove that the set

$$B = \mathop{\times}_{i\geqslant 1} B_i$$

does not support any non-atomic measure, where we have taken $B_i = A_i$ for $i > k$. This is immediate, since if P is any such non-atomic measure, then each marginal P_i is supported by B_i and, moreover, one of the marginals P_i, $i \leqslant k$, has a non-atomic part.

COROLLARY 1. *A finite product of Borel spaces is regular iff one of them is regular.*

COROLLARY 2. *A countable product of Borel spaces is regular iff either one of them is regular or infinitely many of them are atomless.*

We now turn our attention to singular spaces.

Example 4. *A finite product of singular spaces is singular.*

For, let $(X, \boldsymbol{A})$ be the product of singular spaces $(X_i, \boldsymbol{A}_i)$, $i \leqslant n$. Let μ be a 0-1 measure on $\boldsymbol{A}$ with marginals μ_i, $i \leqslant n$. It is not difficult to see that μ is the product of μ_i, $i \leqslant n$. Let $A_i \in \boldsymbol{A}_i$ be such that $\mu_i(A_i) = 1$ and A_i does not support any non-atomic measure. Clearly,

$$A = \mathop{\times}_{i\leqslant n} A_i$$

has μ-measure 1, and if it supports a non-atomic measure P, then each marginal P_i is supported by A_i and, moreover, one of the marginals P_i would have a non-atomic part.

Remark 6. A countable product of singular spaces need not be singular, because the spaces exhibited by Theorem 3 are singular, but their countable product is regular in view of Theorem 4.

Example 5. *A countable product of highly singular spaces is highly singular, and hence singular.*

For, if μ is a 0-1 measure on a countable product, then it is a product of its marginals, and if, moreover, each marginal is concentrated on an atom, then μ is concentrated on the product of these atoms which is an atom of the product.

Example 6. *An uncountable product of trivial spaces is highly singular, and hence singular.*

The above-given examples take care of product spaces which are singular because of the following

THEOREM 5. *A product of Borel spaces is singular iff finitely many of the spaces are singular, countably many are highly singular and the remaining are trivial.*

Proof. The proof of the necessity is essentially contained in the above-given examples. For the sufficiency, let the product space $(X, \boldsymbol{A})$ be singular. If uncountably many of the coordinate spaces are non-trivial, then, by Theorem 4, the product space is, in fact, regular. So we can safely assume that all but countably many spaces are trivial and, to simplify notation, we might even consider $(X, \boldsymbol{A})$ to be the product of $(X_i, \boldsymbol{A}_i)$, $i \geqslant 1$. As the first step, suppose all spaces are not highly singular. So fix μ_i on $\boldsymbol{A}_i$, a 0-1 measure not concentrated on any atom, and let μ on $\boldsymbol{A}$ be the product measure. We claim μ to have enough non-atomic measures around it. For, if $\mu(B) = 1$, first get countably generated σ-algebras $\boldsymbol{B}_i \subset \boldsymbol{A}_i$ such that

$$B \in \mathop{\times}_{i \geqslant 1} \boldsymbol{B}_i.$$

Let B_i be the $\boldsymbol{B}_i$-atom with μ_i-measure 1 and, since B_i is not an $\boldsymbol{A}_i$-atom, fix a $0, \frac{1}{2}, 1$ measure P_i on $\boldsymbol{A}_i$ supported by B_i. If P is the product of these measure P_i, then it is not difficult to see that P is non-atomic and $P(B) = 1$. Next observe that the same kind of contradiction can be arrived at by assuming only that infinitely many coordinate spaces are not highly singular. For, in this case we imitate the above-given proof by fixing point masses in the highly singular coordinate spaces. Thus we can assume that all but finitely many of the spaces are highly singular. Suppose X_i, $i \geqslant k$, are highly singular. We want to show that X_i, $i < k$, are singular. To see, for instance, that $(X_1, \boldsymbol{A}_1)$ is singular, let μ_1 be any 0-1 measure on $\boldsymbol{A}_1$ and, for $i \geqslant 2$, let μ_i be a point mass concentrated at a pre-fixed point $x_i \in X_i$. If μ is the product measure, then μ is a 0-1 measure. So fix $B \in \boldsymbol{A}$ such that $\mu(B) = 1$ and such that B does not support any non-atomic measure. If A is the $(x_2 x_3 \ldots)$-section of B, then $\mu_1(A) = 1$ and A does not support any non-atomic measure. This completes the proof.

We state, without proof, a similar theorem for highly singular spaces:

THEOREM 6. *A product of Borel spaces is highly singular iff countably many of the spaces are highly singular and the remaining are trivial.*

Remark 7. Every Borel space $(X, \boldsymbol{A})$, where $\boldsymbol{A}$ is countably generated, is singular. In fact, it is highly singular. Of course, there are singular spaces that are not highly singular, like the atomless structures of Theorem 3. Observe that a highly singular space is atomic, but not conversely. The countable co-countable structure on an uncountable set is such an example. One can get even rich examples of such spaces (rich in the sense that it supports many non-atomic measures).

Remark 8. Let $\boldsymbol{A}$ be the power set of X. If there is no measurable cardinal, then $(X, \boldsymbol{A})$ is always highly singular. Let a measurable cardinal exist, the first one being $\varkappa$. Then $(X, \boldsymbol{A})$ is highly singular iff $\operatorname{card} X < \varkappa$. Further, if the Lebesgue measure on $[0, 1]$ cannot be extended to all its subsets as a measure, then $(X, \boldsymbol{A})$ is always singular. If the Lebesgue measure can be extended, then $(X, \boldsymbol{A})$ is singular iff $\operatorname{card} X < \varkappa$.

Remark 9. A measurable subspace of a regular space is regular. That is, if $(X, \boldsymbol{B})$ is regular and $Y \in \boldsymbol{B}$, then $(Y, \boldsymbol{B} \cap Y)$ is regular. But a non-measurable subspace need not be regular. In fact, it can be singular as constructed in Theorem 3. On the other hand, any subspace of a singular space is singular and that of a highly singular space is highly singular.

Remark 10. In Remark 9 we considered the case where the σ-field is kept fixed but the underlying set is reduced. Let us now keep the set fixed and reduce the Borel structure. To this end we observe that, in general, the power set being a highly singular structure, it is possible to have a regular structure contained in a singular structure. On the other hand, since the product of a regular structure with any structure is again regular, it is also possible to have a singular structure contained in a regular structure.

Remark 11. Since many of our statements involve 0-1 measures, it is possible to formulate them using σ-ideals. Such a formulation did neither give a better understanding nor simplify the proofs. So we have refrained from such reformulations.

REFERENCES

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Mixtures of nonatomic measures*, Proceedings of the American Mathematical Society 33 (1972), p. 507-510.

[2] — *A remark on nonatomic measures*, Annals of Mathematical Statistics 43 (1972), p. 369-370.
[3] B. V. Rao, *On Borel structures*, Colloquium Mathematicum 21 (1970), p. 199-204.

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Reçu par la Rédaction le 15. 3. 1974