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On positive eigenvectors of a linear eigenvalue problem

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Abstract. The linear eigenvalue problem in a real ordered Banach space is investigated. The purpose of this paper is to generalize some results of papers [4] and [6] concerning differential operators of the second order. In particular, the results obtained by use of the new method can be applied to some linear eigenvalue problems for differential operators of higher order (see Section 5).

Introduction. Kato and Hess [4] recently studied the elliptic eigenvalue problem

$$Lu = \lambda mu \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

in a bounded domain $\Omega \subset \boldsymbol{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$. Here L is a strongly uniformly elliptic differential expression of second order with real-valued Hölder continuous coefficients. The coefficient of u in L is nonnegative and m is a continuous real-valued function. Assuming only that $m(x_0) > 0$ for some $x_0 \in \Omega$, they prove that there is a unique positive eigenvalue λ_+ with corresponding positive eigenvector, whereas every other eigenvalue with nonnegative real part satisfies $\operatorname{Re}\lambda \geqslant \lambda_+$. If $m(x)$ is negative at some point in Ω, there is an analogous eigenvalue λ_-. Senn and Hess, continuing the work initiated in [4], treated in [6] the same eigenvalue problem with Neumann boundary conditions. Basing on the Kato and Hess idea, Bochenek [1] investigated the same eigenvalue problem with boundary conditions of a third type. The other method of generalization of Kato and Hess results was proposed in further papers by Bochenek [2] and [3]. The eigenvalue problem for a linear operator L in an ordered Hilbert space was considered. The method applied required the assumption that L is a selfadjoint operator.

The aim of the present paper is to generalize the Kato–Hess results [4]. The eigenvalue problem for a linear operator L in an ordered real Banach space is considered. The method is an adaptation of that used in [2], [3].

Let X be a real Banach space. Recall that a cone C (with vertex at 0) in X is a closed convex set such that (i) $x \in C$ and $t \geqslant 0$ imply $tx \in C$, and (ii) if $x \in C$

and $x \neq 0$ then $-x \notin C$. A cone C is called *reproducing* if $X = \{x-y:\ x,\ y \in C\}$. A cone C induces a partial ordering in X by $x \leqslant y$ if $y-x \in C$, and C is called *normal* if there exists a constant M such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$ for any elements $x,\ y \in C$. If $B\colon X \to X$ is a bounded linear operator such that $B(C) \subset C$, then B is called *positive* (with respect to C), in symbols $B \geqslant 0$.

It is known that if C is a normal reproducing cone in a real Banach space X and $B\colon X \to X$ is a bounded linear positive operator, then the spectral radius of B, i.e. the number $\operatorname{spr}(B) := \lim \|B^n\|^{1/n}$, is an element of $\sigma(B)$ ($\sigma(B)$ denotes the spectrum of B).

In the sequel we shall employ the standard notation of ordered Banach spaces: $x \geqslant 0$ if $x \in C$, $x > 0$ if $x \in C \setminus \{0\}$, and $x \gg 0$ if $x \in \dot{C} := \operatorname{Int} C$. Let L denote a linear operator defined in D_L dense in X. We impose the following assumptions upon the operator L:

(i) $L\colon D_L \to X$ is a closed operator,

(ii) for every $\alpha > 0$, $(L+\alpha)^{-1}$ exists and is a compact and strictly positive operator; i.e. $(L+\alpha)^{-1}$ maps $C \setminus \{0\}$ into $\dot{C}$.

Let $V\colon X \to X$ be a bounded linear operator such that $(V+\frac{1}{2})$ and $(\frac{1}{2}-V)$ are positive operators. We also assume that X is a real Banach space ordered by a normal reproducing cone C.

1. First eigenvalue problem with a parameter. In this section we consider the following eigenvalue problem with a parameter $t > t_0$:

$$(L+t)u = \varkappa(V+1)u, \tag{1}$$

where $\varkappa \in R$ is the eigenvalue parameter and

(2) $t_0 = \inf\{t \in R\colon (L+t)^{-1}$ exists and is a strictly positive operator on $X\}$.

We shall prove the following

LEMMA 1. *For every $t > t_0$ the problem* (1) *has a first eigenvalue $\varkappa_1 = \varkappa_1(t)$ such that $\varkappa_1(t) > 0$, with a corresponding eigenvector $u_t > 0$. Moreover, the mapping $t \to \varkappa_1(t)$ is a continuous function for $t > t_0$.*

Proof. Let $t > t_0$ be a fixed number. By the assumptions concerning the operators L and V, we see that

$$K_t := (L+t)^{-1}(V+1)$$

is a strictly positive compact operator in X. Thus it has a positive spectral radius. The Krein–Rutman theorem [5] guarantees that $\nu_1(t) := \operatorname{spr}(K_t)$ is the only eigenvalue of K_t whose associated eigenspace contains a positive vector. Moreover, the geometric and algebraic multiplicities of $\nu_1(t)$ are equal to one. If $\varkappa$ is any eigenvalue of (1), we have $\varkappa \geqslant \varkappa_1(t)$, where $\varkappa_1(t) := 1/\nu_1(t)$. Let us remark that equation (1), for $t > t_0$, is equivalent to the equation

$$u = \varkappa K_t u \tag{3}$$

in X. Since K_t is a strictly positive operator, the first eigenvector u_t of (1) corresponding to $\varkappa_1(t)$ for $t > t_0$, belongs to $\dot{C}$.

Taking into account that $L\colon D_L \to X$ is a closed densely defined operator, we can define the (Banach) adjoint operator $L^*\colon X^* \supset D_{L^*} \to X^*$ and we have $[(L+t)^{-1}]^* = (L^*+t)^{-1}$. By the Krein–Rutman theorem, $\nu_1(t)$ is an eigenvalue of the operator K_t^* with a corresponding positive eigenvector $u_t^* \in X^*$; i.e.

$$\nu_1(t) u_t^* = K_t^* u_t^*. \tag{4}$$

Since u_t^* is a positive linear functional on X^*, the inequality $\langle u, u_t^* \rangle > 0$ is satisfied for $u \in X$ and $u \gg 0$.

Assuming that $t > t_0$ is a fixed number and $h \in R$ satisfies the condition $t+h > t_0$, we obtain

$$(L+t+h)u_{t+h} = \varkappa_1(t+h)(V+1)u_{t+h},$$

$$(L+t)u_t = \varkappa_1(t)(V+1)u_t.$$

In what follows,

$$\begin{aligned}\langle L(u_{t+h}-u_t)+t(u_{t+h}-u_t)-\varkappa_1(t)(V+1)(u_{t+h}-u_t),\, u_t^*\rangle \\ = -h\langle u_{t+h}, u_t^*\rangle + [\varkappa_1(t+h)-\varkappa_1(t)]\ \langle (V+1)u_{t+h}, u_t^*\rangle,\end{aligned}$$

or

$$\begin{aligned}\langle u_{t+h}-u_t,\, L^*u_t^* + tu_t^* - \varkappa_1(t)(V^*+1)u_t^*\rangle \\ = -h\langle u_{t+h}, u_t^*\rangle + [\varkappa_1(t+h)-\varkappa_1(t)]\ \langle (V+1)u_{t+h}, u_t^*\rangle.\end{aligned}$$

From the last equality we get

$$[\varkappa_1(t+h)-\varkappa_1(t)]\langle (V+1)u_{t+h}, u_t^*\rangle = h\langle u_{t+h}, u_t^*\rangle. \tag{5}$$

Equality (5) with the assumptions concerning the operator V lead to

$$0 \leqslant \frac{\varkappa_1(t+h)-\varkappa_1(t)}{h} = \frac{\langle u_{t+h}, u_t^*\rangle}{\langle Vu_{t+h}, u_t^*\rangle + \langle u_{t+h}, u_t^*\rangle} \leqslant 2.$$

Therefore, $|\varkappa_1(t+h)-\varkappa_1(t)| \leqslant 2|h|$, and consequently

$$\lim \varkappa_1(t+h) = \varkappa_1(t), \tag{6}$$

which completes the proof.

Lemma 1 and the results of paper [7] (see Theorem 2 of [7]) lead to the following.

LEMMA 2. *For every $t > t_0$ the mappings $t \to u_t$ and $t \to u_t^*$ are continuous.*

THEOREM 1. *For every $t > t_0$ the function $\varkappa_1 = \varkappa_1(t)$ is differentiable and*

$$\varkappa_1'(t) = \frac{\langle u_t, u_t^*\rangle}{\langle (V+1)u_t, u_t^*\rangle}. \tag{7}$$

Proof. From equality (5) we get

$$\frac{\varkappa_1(t+h)-\varkappa_1(t)}{h} = \frac{\langle u_{t+h}, u_t^*\rangle}{\langle (V+1)u_{t+h}, u_t^*\rangle}.$$

Applying Lemma 2 to the last equality, we obtain equality (7).

Since $\langle (V+1)u_t, u_t^*\rangle > 0$ and $\langle u_t, u_t^*\rangle > 0$, we derive from Theorem 1:

COROLLARY 1. *The function $t \to \varkappa_1(t)$ is increasing for $t > t_0$.*

The first eigenvalue $\varkappa_1(t)$ of problem (1) has been so far defined for $t > t_0$. Since $t \to \varkappa_1(t)$ is an increasing function, there exists $0 \leqslant \varkappa_0 := \lim \varkappa_1(t_n)$ for any sequence $(t_n) \subset (t_0, +\infty)$ with $t_n \searrow t_0$. Let u_{t_n}, with $\|u_{t_n}\| = 1$, be an eigenfunction associated with $\varkappa_1(t_n)$. We have

$$(L+t_n)u_{t_n} = \varkappa_1(t_n)(V+1)u_{t_n}. \tag{8}$$

Equality (8) may be rewritten in the form

$$u_{t_n} = \varkappa_1(t_n)(L+1+t_n)^{-1}(V+1)u_{t_n} + (L+1+t_n)^{-1}u_{t_n}. \tag{9}$$

Since $(L+1+t_n)^{-1}(V+1)$ and $(L+1+t_n)^{-1}$ are compact operators on X such that $(L+1+t_n)^{-1}(V+1) \to (L+1+t_0)^{-1}(V+1)$ and $(L+1+t_n)^{-1} \to (L+1+t_0)^{-1}$ in $B(X, X)$ as $n \to \infty$, the sequence (u_{t_n}) is relatively compact in X. Hence (for a subsequence) $u_{t_n} \to u_{t_0}$ in X, $u_{t_0} > 0$, and

$$u_{t_0} = \varkappa_0(L+1+t_0)^{-1}(V+1)u_{t_0} + (L+1+t_0)^{-1}u_{t_0} \tag{10}$$

or

$$(L+t_0)u_{t_0} = \varkappa_0(V+1)u_{t_0}. \tag{11}$$

Two cases can occur.

(A) $\varkappa_0 = 0$. In this case equation (10) leads to the equation $u_{t_0} = (L+1+t_0)^{-1}u_{t_0}$ with $u_{t_0} > 0$. The Krein–Rutman theorem applied to the compact and strictly positive operator $(L+1+t_0)^{-1}$, implies that $1 = \operatorname{spr}(L+1+t_0)^{-1}$. Further $\dim(\operatorname{Ker}[1-(L+1+t_0)^{-1}]) = 1$ and $\dim \operatorname{Ker}(L+t_0) = 1$. From equation (11) we get that 0 is a simple eigenvalue of the problem (1), for $t = t_0$, with positive eigenvector u_{t_0}, i.e. $\operatorname{Ker}(L+t_0) = \operatorname{span}[u_{t_0}]$.

(B) $\varkappa_0 > 0$. From equation (11) we obtain that there exists the operator $(L+t_0)^{-1}$ and $(L+t_n)^{-1} \to (L+t_0)^{-1}$ in $B(X, X)$ for $t_n \searrow t_0$. Hence $(L+t_0)^{-1}$ is a compact and positive operator on X. We now prove that $(L+t_0)^{-1}$ is a strictly positive operator on X. Indeed, let $u \in C$ be any vector and let $v := (L+t_0)^{-1}u$. We have $v \in C$. Since $u = (L+t_0)v$, we have $(u+v) = (L+1+t_0)v$. Thus $v = (L+1+t_0)^{-1}(u+v)$ and so $v \in \dot{C}$. This proves that $(L+t_0)^{-1}$ is a strictly positive operator on X. This contradicts the definition of t_0, showing that case (B) is impossible.

Summarizing, we have

COROLLARY 2. *The problem* (1) *has the first eigenvalue* $\varkappa_1(t)$ *for* $t \geq t_0$, *where* t_0 *is defined by* (2), *and* $\varkappa_1(t) > 0$ *for* $t > t_0$, $\varkappa_1(t_0) := \varkappa_0 = 0$. *The mapping* $t \to \varkappa_1(t)$ *is continuous and strictly increasing for* $t \geq t_0$. *The eigenvalue* $\varkappa_1(t)$ *is simple with a positive eigenvector* $u_t > 0$; *i.e.* $\operatorname{Ker}(L+t) = \operatorname{span}[u_t]$ *for* $t \geq t_0$.

2. Second eigenvalue problem with a parameter. In this section we consider the following eigenvalue problem:

$$(L-\tau V)v = \mu v. \tag{12}$$

In (12) L and V are the operators defined in Introduction, τ is a parameter and $\mu \in R$ is the eigenvalue parameter.

We shall prove

THEOREM 2. *The problem* (12) *has a simple eigenvalue* $\mu_1(\tau)$ *for each* $\tau \in R$ *with a positive eigenvector* $v_\tau > 0$; *i.e.* $\operatorname{Ker}(L-\tau v) = \operatorname{span}[v_\tau]$ *for* $\tau \in R$.

Proof. It is known (see Corollary 2) that the problem

$$(L+t)u = \varkappa(V+1)u \tag{13}$$

has the first eigenvalue $\varkappa_1(t)$, for $t \geq t_0$, where t_0 is defined by (2) with $u_t > 0$. This means that

$$(L+t)u_t = \varkappa_1(t)(V+1)u_t.$$

This equality may be rewritten in the form

$$[L-\varkappa_1(t)V]u_t = [\varkappa_1(t)-t]u_t, \qquad t \geq t_0. \tag{14}$$

Since $t \to \varkappa_1(t)$ is continuous and strictly increasing for $t \geq t_0$, the function $\varkappa_1^{-1}$ exists and $\varkappa_1^{-1}$: $[0, +\infty) \to R$. If we write $\tau := \varkappa_1(t)$, for $t \geq t_0$, we get $t = \varkappa_1^{-1}(\tau)$. Equality (14) takes on the form

$$(L-\tau V)u_t = \mu_1(\tau)u_t, \tag{15}$$

where $\mu_1(\tau) := \tau - \varkappa_1^{-1}(\tau)$ for $\tau \geq 0$.

This shows that $\mu_1(\tau) = \tau - \varkappa_1^{-1}(\tau)$ is an eigenvalue of the problem (12), with positive eigenvector $v_\tau := u_t > 0$ for $\tau := \varkappa_1(t)$. Since $\varkappa_1(t)$ is a simple eigenvalue of (1), this leads to the conclusion that the problem (12) has $\mu_1(\tau)$ as a simple eigenvalue with positive eigenvector v_τ for each $\tau \geq 0$.

Suppose that $\tau < 0$ in (12). Then equation (12) may be written in the form

$$[L-(-\tau)(-V)]v = \mu v.$$

On account of the first part of the proof, we get the assertion of Theorem 2.

THEOREM 3. *For each* $\tau \in R$ *the mapping* $\tau \to \mu_1(\tau)$ *is a continuous, differentiable function and*

$$\mu_1'(\tau) = -\frac{\langle V v_\tau, v_\tau^* \rangle}{\langle v_\tau, v_\tau^* \rangle}. \tag{16}$$

Proof. The claim follows directly from Corollary 2 and Theorems 1 and 2.

3. Eigenvalue problem with an indefinite weight operator. In this section we investigate the eigenvalue problem

$$Lu = \lambda Vu, \tag{17}$$

where L and V are the operators defined in Introduction. The considerations this and subsequent sections need the following

HYPOTHESIS Z. There exists $\alpha_0 \in R$, $\alpha_0 > t_0$, and $w_0 \in D_L$, $w_0 > 0$ such that

$$Lw_0 - \alpha_0 Vw_0 \leqslant 0. \tag{18}$$

LEMMA 3. *Under assumption* (18), *there exist* λ: $0 \leqslant \lambda \leqslant \alpha_0$, *and* $u \in D_L$, $u > 0$, *such that* $Lu = \lambda Vu$.

Proof. From (18) we obtain

$$(L+\alpha_0)w_0 - \alpha_0(V+1)w_0 \leqslant 0. \tag{19}$$

The positivity of the operator $(L+\alpha_0)^{-1}$ leads to the inequality

$$w_0 \leqslant \alpha_0 K_{\alpha_0} w_0, \tag{20}$$

where $K_t := (L+t)^{-1}(V+1)$ for $t > t_0$.

Now the assertion of Lemma 3 is a direct consequence of Lemma 3, Lemma 1 (in [4]) and Lemma 4 (in [6]).

Comparison of the eigenvalue problems (12) and (17) implies

COROLLARY 3. *A number* λ_0 *is an eigenvalue of* (17) *with positive eigenvector* u_0 *if and only if* $\mu_1(\lambda_0) = 0$, *where* $\mu_1(\tau)$ *is the first eigenvalue of problem* (12) *with positive eigenvector* v_τ, $\tau \in R$. *The eigenvector of problem* (17), *corresponding to* λ_0, *is* $u_0 = \sigma v_{\lambda_0}$, $\sigma > 0$.

COROLLARY 4. *If inequality* (18) *holds and if* $\mu_1(0) > 0$, *then the function* $\tau \to \mu_1(\tau)$ *has at least one positive zero-point* τ_0, *i.e.* $\mu_1(\tau_0) = 0$, $\tau_0 > 0$.

Remark 1. If $\tilde{V}$: $X \to X$ is a bounded linear operator such that $\tilde{V} - V \geqslant 0$ on X and the operators L and V satisfy Hypothesis Z, then also the operators L and $\tilde{V}$ satisfy this hypothesis (L and V are defined in Introduction).

For $t \in I := (-\infty, \bar{t})$, we consider the equation (cf. [4])

$$Lu = \lambda(V-t)u, \tag{21}$$

where

$$\bar{t} := \sup\{t \in R\colon L \text{ and } (V-t) \text{ satisfy Hypothesis Z}\}. \tag{22}$$

Therefore the eigenvalue problem (21) has the first eigenvalue $\lambda_1 = \lambda_1(t) \geqslant 0$, for $t \in I$, with corresponding eigenvector $u_t > 0$. By Proposition 1 (in [4]), $t \to \lambda_1(t)$ is a strictly increasing function in $t \in I$, and by use of Lemma 5 (in [4]) we get $\lambda_1(t) \to +\infty$ as $t \to \bar{t}$.

For $t \in I$ and $\tau \in R$, let us consider the following problem:

$$Lv - \tau(V-t)v = \mu v, \tag{23}$$

where μ is an eigenvalue parameter. From Theorem 2 it follows that the problem (23) has a simple eigenvalue $\mu(\tau, t)$ for each $\tau \in R$ and $t \in I$ with a positive eigenvector $v_{\tau,t} > 0$. One can observe that if

$$Lv_{\tau,t} - \tau(V-t)v_{\tau,t} = \mu(\tau, t)v_{\tau,t},$$

then

$$Lv_{\tau,t} - \tau V v_{\tau,t} = [\mu(\tau, t) - \tau t]\, v_{\tau,t}. \tag{24}$$

Equality (24) means that

$$\mu(\tau, t) = \mu_1(\tau) + \tau t, \qquad \tau \in R,\ t \in I, \tag{25}$$

where $\mu_1(\tau)$ is the first eigenvalue of the problem (12) for $\tau \in R$.

LEMMA 4. *Suppose that* $\bar{t} > 0$, *where* $\bar{t}$ is defined by (22). *Then* $\mu_1(\tau_0) = 0$ *and* $\tau_0 > 0$ *lead to*

$$\mu_1'(\tau) < 0 \quad \textit{for } \tau > \tau_0. \tag{26}$$

Proof. From (25) it follows that $R \times I \ni (\tau, t) \to \mu(\tau, t)$ is a continuous and differentiable function. Since $0 \in I$, we have. $\mu(\tau, 0) = \mu_1(\tau)$ for $\tau \in R$. From Lemma 3 we conclude, in view of Corollaries 3 and 4, that for every $t \in I$ the equation

$$\mu_1(\tau) + \tau t = 0 \tag{27}$$

has a solution $\tau = \lambda_1(t)$. This solution $\lambda_1(t) \geqslant 0$ is the first eigenvalue of the problem (21), with a positive eigenvector $u_t \in D_L$. We know that $I \ni t \to \lambda_1(t)$ is a strictly increasing function; hence there exists λ_1^{-1}: $(0, +\infty) \ni \tau \to \lambda_1^{-1}(\tau) \in R$, which is also strictly increasing. For $\tau > 0$ we get from (27)

$$t = -\frac{\mu_1(\tau)}{\tau} = \lambda_1^{-1}(\tau). \tag{28}$$

Differentiability of $\tau \to \mu_1(\tau)$ together with (28) imply the differentiability of the function $\tau \to \lambda_1^{-1}(\tau)$ and

$$\frac{d\lambda_1^{-1}}{d\tau} = -\frac{\mu_1'(\tau) + t}{\tau} \quad \text{for } \tau \neq 0. \tag{29}$$

Since $\tau \to \lambda_1^{-1}(\tau)$ is an increasing function, it follows from (29) that $\mu_1'(\tau) \leqslant -t$ for $\tau > 0$. Hence, if $t > 0$ then

$$\mu_1'(\tau) < 0. \tag{30}$$

By assumption, $\mu_1(\tau_0) = 0$ and $\tau_0 > 0$. Hence, by Corollary 4 we infer that τ_0 is the first eigenvalue of problem (17) with a positive eigenvector. From (27) it follows that $\tau_0 = \lambda_1(0)$. On the other hand, the monotonicity of the function $\tau = \lambda_1(t)$ in $t > 0$ results for $\tau > \tau_0$. Therefore, from (30) we obtain (26). This completes the proof of Lemma 4.

THEOREM 4. *Suppose, under the assumption of Lemma 4, that the inverse L^{-1} exists and is a positive operator on X. Then the eigenvalue problem (17) admits an eigenvalue $\lambda_1 > 0$ which is the unique positive eigenvalue having a positive eigenvector. Moreover, λ_1 has the following properties:*

(i) *if $\lambda \in R$ is an eigenvalue of (17) and $\lambda \geq 0$, then $\lambda \geq \lambda_1$;*

(ii) *$\nu_1^* := 1/\lambda_1$ is an eigenvalue of L^{-1}: $X \to X$ with geometric and algebraic multiplicities 1.*

Proof. The assumption that L^{-1} exists leads to the conclusion that, in particular, $\lambda = 0$ is not an eigenvalue of the problem (17). Hence, it follows from Lemma 3 that the problem (17) admits an eigenvalue $\lambda_1 > 0$ with a positive eigenvector u_1. By virtue of Corollary 3, $\mu_1(\lambda_1) = 0$, where $\mu_1(\tau)$ is the first eigenvalue of (12) with a positive eigenvector v_τ, $\tau \in R$. By positivity of L^{-1} we have $\mu_1(0) > 0$. On the other hand, from (26) we obtain that $\tau \to \mu_1(\tau)$ is a strictly decreasing function for $\tau > \lambda_1$. This implies that $\lambda_1 > 0$ is the unique positive solution of the equation $\mu_1(\tau) = 0$. Consequently, from Corollary 3 we obtain that $\lambda_1 > 0$ is the unique positive eigenvalue of (17) with positive eigenvector.

In order to prove (i), let us observe that $\lambda_1 > 0$ is the eigenvalue of (17) with a positive eigenvector, if and only if λ_1 is a solution of the equation

$$t = \varkappa_1(t). \tag{31}$$

Here $\varkappa_1(t)$ is the first eigenvalue of problem (1) for $t \geq 0$. We know that if $\varkappa(t)$ is an eigenvalue of (1), then $\varkappa(t) \geq \varkappa_1(t)$ for every $t \geq 0$. Suppose that $\lambda \geq 0$ is an eigenvalue of (17). Then λ satisfies the equation $\lambda = \varkappa(\lambda)$, where $\varkappa(t)$ is an eigenvalue of (1) for $t \geq 0$. From the equalities $\lambda_1 = \varkappa_1(\lambda_1)$, $\lambda = \varkappa(\lambda)$ and inequality $\varkappa(t) \geq \varkappa_1(t)$ (for every $t \geq 0$) we get the inequality $\lambda \geq \lambda_1$.

Assertion (ii) is a direct consequence of the Krein–Rutman theorem and the fact that λ_1 is the eigenvalue of problem (1) for $t = \lambda_1$ (see the proof of Lemma 1).

4. Eigenvalue problem (17) in the exceptional case. Theorem 4 concerns the regular case of the problem (17), i.e. such that there exists the operator L^{-1}, positive on X. In this section we consider the exceptional case of (17), i.e. the case of noninvertible L. From the assumptions concerning the operator L (see Introduction) and from Theorem 2 we conclude that in this case problem (12) has the simple eigenvalue $\mu_1(0) = 0$ with an eigenvector $v_0 > 0$ and $\mathrm{Ker}(L) = \mathrm{span}[v_0]$. By Theorem 3, there exists $\mu_1'(0)$ and

$$\mu_1'(0) = -\frac{\langle Vv_0, v_0^* \rangle}{\langle v_0, v_0^* \rangle}, \tag{32}$$

where $v_0^* > 0$, $v_0^* \in X^*$ and $\mathrm{Ker}(L^*) = \mathrm{span}[v_0^*]$.

THEOREM 5. *Suppose, under the assumption of Lemma 4, that the operator V is such that $\langle Vv_0, v_0^* \rangle \neq 0$. Then the problem* (17) *admits a unique eigenvalue $\lambda_1 \neq 0$ which has a positive eigenvector. More precisely, $\lambda_1 > 0$ if $\langle Vv_0, v_0^* \rangle < 0$, and $\lambda_1 < 0$ if $\langle Vv_0, v_0^* \rangle > 0$.*

Proof. By assumptions of the theorem, and by (32), we have $\mu_1(0) = 0$ and $\mu_1'(0) \neq 0$. Suppose that $\langle Vv_0, v_0^* \rangle < 0$, i.e. $\mu_1'(0) > 0$. Continuity of the function $\tau \to \mu_1(\tau)$ leads to $\mu_1(\tau) > 0$ in a right neighbourhood of 0. Reasoning as in the proof of Theorem 4 gives the statement of Theorem 5 for the case $\langle Vv_0, v_0^* \rangle < 0$.

For $\langle Vv_0, v_0^* \rangle > 0$, it suffices to write (17) in the form $Lu = (-\lambda)(-V)u$ and use the first part of this proof.

THEOREM 6. *Suppose, under the assumption of Lemma 4, that the operator V fulfils $\langle Vv_0, v_0^* \rangle = 0$. Then 0 is the only eigenvalue of the problem* (17) *having a positive eigenvector.*

Proof. According to the assumptions of this section, the eigenvalue problem (21) has the eigenvalue $\lambda_0 = 0$ with a positive eigenvector v_0 for every $t \in \boldsymbol{R}$. For every $t \neq 0$ we have $\langle (V-t)v_0, v_0^* \rangle \neq 0$. Hence, by Theorem 5, there exists an eigenvalue $\lambda_1(t) \neq 0$ of the problem (21) with a positive eigenvector v_t. Since $t \to \lambda_1(t)$ is a strictly increasing function, there exists λ_1^{-1}, given by (28). From (28) we get

$$\frac{d\lambda_1^{-1}}{d\tau} = -\frac{\mu_1'(\tau)\tau - \mu_1(\tau)}{\tau^2} \quad \text{for } \tau \neq 0. \tag{33}$$

Since $d\lambda_1^{-1}/d\tau \geqslant 0$ for $\tau \neq 0$, we have by (33)

$$\mu_1(\tau) \geqslant \mu_1'(\tau)\tau \quad \text{for } \tau \neq 0. \tag{34}$$

Inequality (34) with conditions $\mu_1(0) = 0$ and $\mu_1'(0) = 0$ shows that $\tau \to \mu_1(\tau)$ is a concave function in a neighbourhood of 0. Hence $\mu_1(\tau) < 0$ in this neighbourhood for $\tau \neq 0$. On the other hand, if there exists $\tau_0 \neq 0$ such that $\mu_1'(\tau_0) = 0$, then $\mu_1(\tau_0) \geqslant 0$. These properties of the function $\tau \to \mu_1(\tau)$ imply $\mu_1(\tau) \neq 0$ for $\tau \neq 0$. Consequently the problem (17) has no eigenvalue $\lambda \neq 0$ with a positive eigenvector. Theorem 6 is thus established.

5. Application to the elliptic eigenvalue problems of higher order. In this section we show that the results of Sections 1–4 may be applied to certain linear eigenvalue problems for differential operators of higher order.

Let Ω be an open bounded subset of $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$. We consider the eigenvalue problem

$$Lu = \lambda Vu \quad \text{in } \Omega, \tag{35}$$

where L is a differential operator defined in the Banach space $X := C(\bar{\Omega})$ by a differential expression of the form

(36) $$\mathscr{L} := \mathscr{L}_1 \mathscr{L}_2 \ldots \mathscr{L}_p.$$

Here

(37) $$\mathscr{L}_k \varphi := \sum_{i,j=1}^{n} a_{ij}^k \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k \frac{\partial \varphi}{\partial x_i} + c^k \varphi, \qquad k = 1, \ldots, p,$$

is a strongly uniformly elliptic differential expression of the second order with real-valued coefficient functions $a_{ij}^k = a_{ji}^k$, b_i^k, $c^k \geqslant 0$, belonging to $C^{2k-2+\theta}(\bar{\Omega})$ $(0 < \theta < 1)$. We consider equation (35) with the boundary conditions

(38) $$\varphi_k = 0 \quad \text{on } \partial\Omega,\ k = 1, \ldots, p,$$

or

(39) $$\frac{d\varphi_k}{d\nu} - h^k \varphi_k = 0 \quad \text{on } \partial\Omega,\ k = 1, \ldots, p,$$

where $\varphi_1 := u$, $\varphi_k := \mathscr{L}_{p-k+2} \ldots \mathscr{L}_p u$, $k = 2, \ldots, p$. In (39), h^k $(k = 1, \ldots, p)$ are real-valued, nonnegative and continuous functions on $\partial\Omega$; ν is an outward pointing, nowhere tangent smooth vector field on $\partial\Omega$.

Equipped with the positive cone $C_X := \{v \in X\colon v(x) \geqslant 0 \ \forall x \in \bar{\Omega}\}$, X is an ordered Banach space. It is known that for each $k = 1, \ldots, p$ the differential expression $\mathscr{L}_k$ with the boundary condition (38) or (39) defines an operator L_k in the space X. By the maximum principle, $(L_k + \varepsilon)^{-1}$ is a strictly positive operator for each $\varepsilon > 0$; i.e. it maps $C_X \setminus \{0\}$ into $\dot{C}_X$ (see [4] and [6]).

Let us define the operator L by the formula

(40) $$L := L_1 L_2 \ldots L_p$$

with domain D_L defined by

(41) $$D_L := \{u \in C^{2p}(\bar{\Omega})\colon u \text{ satisfies (38) or (39) on } \partial\Omega\}.$$

One can prove, using the Krein–Rutman theorem, that the operator L defined above satisfies assumptions (i) and (ii) from Introduction. Let us define V as the multiplication operator induced by a function $m \in X$. Without loss of generality we assume that $|m| \leqslant \frac{1}{2}$ on $\bar{\Omega}$. Then $(V + \frac{1}{2})$ and $(\frac{1}{2} - V)$ are positive operators on X. We are now interested in the situation where m changes sign in Ω.

We shall prove the following

THEOREM 7. *Suppose $m(x_0) > 0$ for some $x_0 \in \Omega$. Then the operators L and V defined above satisfy Hypothesis* Z; *i.e., there exist $\alpha_0 \in R$, $\alpha_0 > 0$ and a function $w_0 \in D_L \cap C_X$ such that*

(42) $$L w_0 - \alpha_0 V w_0 \leqslant 0.$$

Proof. The continuity of m ensures that there exists $\varrho > 0$ such that $\bar{B}_\varrho \subset \Omega$ and $m(x) > 0$ for all $x \in B_\varrho$, where $B_\varrho := \{x \in \mathbf{R}^n: \|x - x_0\| < \varrho\}$. Let $w_0 \in C^\infty(\bar{\Omega})$ be a function such that $\operatorname{supp} w_0 \subset \bar{B}_\varrho$ and $w_0(x) > 0$ for all $x \in B_\varrho$. Obviously, $w_0 \in D_L$, $w_0 > 0$, i.e. $w_0 \in C_X$. It what follows, $Lw_0 = \mathscr{L} w_0$, and so

$$\mathscr{L} w_0 = 0 \quad \text{in } \Omega \setminus \bar{B}_\varrho .$$

Since $Vw_0 = mw_0$, we obtain

$$mw_0 > 0 \quad \text{in } B_\varrho \quad \text{and} \quad mw_0 = 0 \quad \text{in } \bar{\Omega} \setminus \bar{B}_\varrho .$$

This implies $\mathscr{L} w_0 = mw_0$ in $\bar{\Omega} \setminus \bar{B}_\varrho$, and we get for sufficiently large $\alpha_0 > 0$

$$\mathscr{L} w_0 - \alpha_0 m w_0 \leqslant 0 \quad \text{in } B_\varrho .$$

Therefore, we obtain for all $x \in \bar{\Omega}$

$$(\mathscr{L} w_0)(x) - \alpha_0 m(x) w_0(x) \leqslant 0. \tag{43}$$

Inequality (43) is equivalent to (42). This proves Theorem 7.

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