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Characters of finitely generated C^*-algebras

by W. Szymański (Kraków)

The present paper, inspired by a result of Arveson [1], deals with some condition concerning the existence of non-trivial characters of finitely generated C^*-algebras (1).

Let $\mathscr{L}(H)$ be the algebra of all linear bounded operators on the Hilbert space H. Following Dash [3] we introduce the definition of the joint approximate point spectrum of a pair of operators T_1, T_2 from $\mathscr{L}(H)$. We say namely that $0 = (0, 0) \in C \times C$ (where C is the complex plane) belongs to the approximate — point spectrum $\sigma_\pi(T_1, T_2)$ of operators $T_1, T_2 \in \mathscr{L}(H)$ if $S_1 T_1 + S_2 T_2 \neq I$, for all $S_1, S_2 \in \mathscr{L}(H)$, where I is the identity in H. In general, $\lambda = (\lambda_1, \lambda_2) \in C \times C$ belongs to $\sigma_\pi(T_1, T_2)$ if $0 \in \sigma_\pi(T_1 - \lambda_1, T_2 - \lambda_2)$. A simple characterization of the set $\sigma_\pi(T_1, T_2)$ is given by the following lemma (see [4]):

Lemma 1. *0 is in $\sigma_\pi(T_1, T_2)$ if and only if there exists a sequence (f_n) of unit vectors in H such that:*

$$(*) \qquad \|T_1 f_n\| \to 0 \quad \text{and} \quad \|T_2 f_n\| \to 0 \quad \text{when } n \to \infty.$$

Proof. Let $0 \in \sigma_\pi(T_1, T_2)$. Then for all $S_1, S_2 \in \mathscr{L}(H)$ we have $S_1 T_1 + S_2 T_2 \neq I$. It follows that $0 \in \sigma_\pi(T_1^* T_1 + T_2^* T_2)$, where $\sigma_\pi(A)$ stands for the approximate point spectrum of an operator $A \in \mathscr{L}(H)$. Consequently there exists a sequence (f_n) of unit vectors in H such that $\|T_1^* T_1 f_n + T_2^* T_2 f_n\| \to 0$ when $n \to \infty$.

Since

$$\begin{aligned}\|T_1 f_n\|^2 + \|T_2 f_n\|^2 &= (T_1 f_n, T_1 f_n) + (T_2 f_n, T_2 f_n) \\ &= (T_1^* T_1 f_n, f_n) + (T_2^* T_2 f_n, f_n) \leqslant \|T_1^* T_1 f_n + T_2^* T_2 f_n\|,\end{aligned}$$

so $\|T_1 f_n\| \to 0$ and $\|T_2 f_n\| \to 0$ when $n \to \infty$.

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(1) The author presented the results of the present paper for $C^*(T)$-algebras at the seminar of professor W. Mlak in December 1970. After the major part of the present paper has been done the author found out that technically similar investigations have been performed independently by Istratescu [6] — see also the review of Suzuki in Math. Reviews vol. 41 (1971).

We now suppose that there exist operators $S_1, S_2 \in \mathscr{L}(H)$ such that $S_1T_1+S_2T_2 = I$. But then for every $f \in H$ we have $S_1T_1f+S_2T_2f = f$. If $\|f\| = 1$, then

$$1 = \|f\| = \|S_1T_1F+S_2T_2f\| \leqslant \|S_1\|\|T_1f\|+\|S_2\|\|T_2f\|$$

and consequently (*) fails.

Suppose H is a complex Hilbert space. It has been proved by Berberian [2] that there exists a Hilbert space $K \supset H$ and an isometric, involution preserving representation $\varphi\colon \mathscr{L}(H) \to \mathscr{L}(K)$ such that $\varphi(I_H) = I_K$ (where I_S stands for the identity operator in the space S) and for every T from $\mathscr{L}(H)$: $\sigma_\pi(T) = \sigma_\pi\big(\varphi(T)\big) = \sigma_p\big(\varphi(T)\big)$, where $\sigma_p(A)$ is the point spectrum of the operator A. In what follows we write $T^0 = \varphi(T)$ for $T \in \mathscr{L}(H)$.

We define the joint point spectrum $\sigma_p(T_1, T_2)$ of operators $T_1; T_2$ from $\mathscr{L}(H)$ as follows:

$$\sigma_p(T_1, T_2) = \{(\lambda_1, \lambda_2) \in C\times C\colon \text{there is an } f \in H,\ \|f\| = 1 \text{ such that } T_if = \lambda_if;\ i = 1, 2\}.$$

We will prove the following

THEOREM 1. *Let* $T_1, T_2 \in \mathscr{L}(H)$. *Then*

$$\sigma_\pi(T_1, T_2) = \sigma_\pi(T_1^0, T_2^0) = \sigma_p(T_1^0, T_2^0).$$

Proof. It is obvious that $\sigma_p(T_1^0, T_2^0) \subset \sigma_\pi(T_1^0, T_2^0)$. We refer to the notations of the proof of the theorem of Berberian from [2]. Let $0 \in \sigma_\pi(T_1, T_2)$. Then, by Lemma 1, there exists a sequence (f_n) of unit vectors in H such that (*) holds. But

$$\|T_1^0(f_n)'\|^2 = g\lim\|T_1f_n\|^2 = \lim\|T_1f_n\|^2 = 0,$$

$$\|T_2(f_n)'\|^2 = 0 \quad \text{and} \quad \|(f_n)'\| = 1.$$

Hence $0 \in \sigma_p(T_1^0, T_2^0)$, which proves that $\sigma_\pi(T_1, T_2) \subset \sigma_p(T_1^0, T_2^0)$. Since φ is an isometric mapping and I^0 is the identity in K, the inequality $S_1T_1+S_2T_2 \neq I$ is equivalent to the inequality $S_1^0T_1^0+S_2^0T_2^0 \neq I^0$, which proves that $0 \in \sigma_\pi(T_1, T_2)$ if and only if $0 \in \sigma_\pi(T_1^0, T_2^0)$. Hence $\sigma_\pi(T_1^0, T_2^0) = \sigma_\pi(T_1, T_2)$ which completes the proof.

Let $(\lambda_1, \lambda_2) = \lambda \in C\times C$, $(\bar\lambda_1, \bar\lambda_2) = \bar\lambda$ and $\lambda \in \sigma_\pi(T_1; T_2)$. We define the set

$$N_\pi^\lambda(T_1, T_2) = \{(f_n) \subset H\colon \|f_n\| = 1, \|T_1f_n-\lambda_1f_n\| \to 0, \|T_2f_n-\lambda_2f_n\| \to 0\}$$

and we say that λ, belonging to $\sigma_\pi(T_1, T_2)$, is in $C_\pi(T_1, T_2)$ if $\bar\lambda$ belongs to $\sigma_\pi(T_1^*, T_2^*)$ and $N_\pi^{\bar\lambda}(T_1^*, T_2^*) = N_\pi^\lambda(T_1, T_2)$. Now the following lemma can be proved:

LEMMA 2. *If* $\lambda = (\lambda_1, \lambda_2) \in C \times C$ *is in* $C_\pi(T_1, T_2)$, *then there exists a sequence* (f_n) *of unit vectors in* H *such that*

$$T_1^0(f_n)' = \lambda_1 (f_n)', \qquad T_2^0(f_n)' = \lambda_2 (f_n)',$$

$$T_1^{0*}(f_n)' = \bar{\lambda}_1 (f_n)', \qquad T_1^{0*}(f_n)' = \lambda_2 (f_n)'.$$

Proof. If $\lambda \in C_\pi(T_1, T_2)$, then $\lambda \in \sigma_\pi(T_1, T_2)$, $\bar{\lambda} \in \sigma_\pi(T_1^*, T_2^*)$ and $N_\pi^\lambda(T_1, T_2) = N_\pi^{\bar{\lambda}}(T_1^*, T_2^*)$. By Theorem 1 we get $\lambda \in \sigma_p(T_1^0, T_1^0)$ and $\bar{\lambda} \in \sigma_p(T_1^{0*}, T_2^{0*})$. Moreover, there exists a sequence (f_n) of unit vectors in H such that

$$\lim \|(T_1 - \lambda_1 I) f_n\|^2 = \lim \|(T_1^* - \bar{\lambda}_1 I) f_n\|^2 = \lim \|(T_2 - \lambda_2 I) f_n\|^2$$
$$= \lim \|(T_2^* - \bar{\lambda}_2 I) f_n\|^2 = 0.$$

Hence

$$\|(T_1^0 - \lambda_1 I^0)(f_n)'\|^2 = g\lim \|(T_1 - \lambda_1 I) f_n\|^2 = \lim \|(T_1 - \lambda_1 I) f_n\|^2 = 0,$$

i.e. $T_1^0(f_n)' = \lambda_1(f_n)'$. Using similar arguments we have

$$T_1^{0*}(f_n)' = \bar{\lambda}_1 (f)', \qquad T_2^0(f_n)' = \lambda_2 (f_n)', \qquad T_2^{0*}(f_n)' = \bar{\lambda}_2 (f_n)',$$

and the proof is complete.

Every homomorphic mapping form a Banach algebra A onto the complex plane C is called a *character on* A.

Now we apply the results form the preceding section to a construction of non-trivial characters of finitely generated C^*-algebras. Suppose $T_1, T_2 \in \mathscr{L}(H)$. Let $Q^{(i_1, \dots, i_n)}$ be a product $Q_1^{i_1} \dots Q_n^{i_n}$, where each Q_s $(s = 1, \dots, n)$ equals to one of the operators T_1, T_1^*, T_2, T_2^*. For $w(z_1, \dots$ $\dots, z_n) = z_1^{i_1} \dots z_1^{i_n}$ $(z_i \in C)$ we put $w(T_1, T_2) = Q^{(i_1, \dots, i_n)}$. For $w(z_1, \dots, z_n) = \sum a_{i_1, \dots, i_n} z_1^{i_1} \dots z_n^{i_n}$ we define $w(T_1, T_2) = \sum a_{i_1 \dots i_n} Q^{(i_1 \dots i_n)}$. The algebra $C^*(T_1, T_2)$ is, by definition, generated by operators of the form $w(T_1, T_2)$. The properties of the Berberian isomorphism $T \to T^0$ imply that $C^*(T_1, T_2)$ and $C^*(T_1^0, T_2^0)$ are isometrically isomorphic.

THEOREM 2. *Assume that* $\lambda = (\lambda_1, \lambda_2) \in C_\pi(T_1, T_2)$. *Then there exists a character* χ: $C^*(T_1, T_2) \to C$ *such that* $\chi(T_1) = \lambda_1$ *and* $\chi(T_2) = \lambda_2$.

Proof. Let $\tilde{f} = (f_n)$ be as in Lemma 2. Let $w(z_1, \dots, z_n) = \sum a_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n}$. Then $w(\lambda_1, \dots, \lambda_n) \tilde{f} = w(T_1^0, T_2^0) \tilde{f}$. Hence

$$|w(\lambda_1, \dots, \lambda_n)| = \|w(\lambda_1, \dots, \lambda_n) \tilde{f}\| = \|w(T_1^0, T_2^0) \tilde{f}\| \leqslant \|w(T_1^0, T_2^0)\|.$$

It follows that there is a well defined character $\bar{\chi}$ on $C^*(T_1^0, T_2^0)$ such that $\bar{\chi}(T_1^0) = \lambda_1$, $\bar{\chi}(T_2^0) = \lambda_2$. Since $C^*(T_1, T_2)$ and $C^*(T_1^0, T_2^0)$ are isometrically isomorphic, the proof is complete.

Now we define the sets:

$$\sigma_\varrho(T_1, T_2) = \{(\lambda_1, \lambda_2) \in C\times C\colon \text{ for all } S_1, S_2 \in \mathscr{L}(H)\colon\ S_1(T_1^*-\bar{\lambda}_1)+ +S_2(T_2^*-\bar{\lambda}_2) \neq I\},$$

$$\sigma_{\pi\varrho}(T_1, T_2) = \{(\lambda_1, \lambda_2) \in C\times C\colon \text{ for all } S_1, S_2 \in \mathscr{L}(H)\colon\ S_1(T_1-\lambda_1)+ +S_2(T_2^*-\bar{\lambda}_2) \neq I\},$$

$$\sigma_{\varrho\pi}(T_1, T_2) = \{(\lambda_1, \lambda_2) \in C\times C\colon \text{ for all } S_1, S_2 \in \mathscr{L}(H)\colon\ S_1(T_1^*-\bar{\lambda}_1)+ +S_2(T_2^*-\bar{\lambda}_2) \neq I\},$$

and, if $\lambda = (\lambda_1, \lambda_2) \in \sigma_\varrho(T_1, T_2)$,

$$N_\varrho^\lambda(T_1, T_2) = \{(f_n) \subset H\colon \|f_n\| = 1, \|(T_1^*-\bar{\lambda}_1)f_n\| \to 0, \|(T_2^*-\bar{\lambda}_2)f_n\| \to 0\};$$

if $\lambda \in \sigma_{\pi\varrho}(T_1, T_2)$,

$$N_{\pi\varrho}^\lambda(T_1, T_2) = \{(f_n) \subset H\colon \|f_n\| = 1, \|(T_1-\lambda_1)f_n\| \to 0, \|(T_2^*-\bar{\lambda}_2)f_n\| \to 0\};$$

if $\lambda \in \sigma_{\varrho\pi}(T_1, T_2)$,

$$N_{\varrho\pi}^\lambda(T_1, T_2) = \{(f_n) \subset H\colon \|f_n\| = 1, \|(T_1^*-\bar{\lambda}_1)f_n\| \to 0, \|(T_2-\lambda_2)f_n\| \to 0\}.$$

We say that

λ, belonging to $\sigma_\varrho(T_1, T_2)$, is in $C_\varrho(T_1, T_2) \Leftrightarrow \bar{\lambda} \in \sigma_\varrho(T_1^*, T_2^*)$ and $N_\varrho^{\bar{\lambda}}(T_1^*, T_2^*) = N_\varrho^\lambda(T_1, T_2)$,

λ, belonging to $\sigma_{\pi\varrho}(T_1, T_2)$, is in $C_{\pi\varrho}(T_1, T_2) \Leftrightarrow \bar{\lambda} \in \sigma_{\pi\varrho}(T_1^*, T_2^*)$ and $N_{\pi\varrho}^{\bar{\lambda}}(T_1^*, T_2^*) = N_{\pi\varrho}^\lambda(T_1, T_2)$,

λ, belonging to $\sigma_{\varrho\pi}(T_1, T_2)$, is in $C_{\varrho\pi}(T_1, T_2) \Leftrightarrow \bar{\lambda} \in \sigma_{\varrho\pi}(T_1^*, T_2^*)$ and $N_{\varrho\pi}^{\bar{\lambda}}(T_1^*, T_2^*) = N_{\varrho\pi}^\lambda(T_1, T_2)$.

Then the following corollary is true:

COROLLARY 1. *If $\lambda \in C_\varrho(T_1, T_2)\big(C_{\pi\varrho}(T_1, T_2), C_{\varrho\pi}(T_1, T_2)\big)$, then there exists a character $\chi\colon C^*(T_1, T_2) \to C$ such that $\chi(T_1^*) = \bar{\lambda}_1$, $\chi(T_2^*) = \bar{\lambda}_2$ (and, respectively: $\chi(T_1) = \lambda_1$, $\chi(T_2^*) = \bar{\lambda}_2$ and $\chi(T_1^*) = \bar{\lambda}_1$, $\chi(T_2) = \lambda_2$)).*

We notice that the following inclusions are true: $\{(\lambda, \lambda) \in C\times C\colon \lambda \in \sigma_\pi(T)\} \subset \sigma_\pi(T, T)$ for every $T \in \mathscr{L}(H)$ and, for all $S, T \in L(H)$, $\sigma_\pi(S, T) \subset \sigma_\pi(S)\times\sigma_\pi(T)$.

In the case when we consider the algebra $C^*(T)$ generated by one operator $T \in \mathscr{L}(H)$, we have:

COROLLARY 2. *If $\lambda \in C$ is a normal approximate proper value of T, then there exists a character $\chi\colon C^*(T) \to C$ such that $\chi(T) = \lambda$.*

Proof. Let λ be a normal approximate proper value of T. Then there exists a sequence (f_n) of unit vectors in H such that $\|(T-\lambda)f_n\| \to 0$ and $\|(T^*-\bar{\lambda})f_n\| \to 0$. Hence (λ, λ) is in $C_\pi(T, T)$ and, by Theorem 2, there exists a character χ on $C^*(T, T)$ such that $\chi(T) = \lambda$, which completes the proof.

It has been proved by Bunce [3], that if χ is a character on $C^*(A)$, then $\chi(A)$ is in $\sigma_\pi(A)$. Since the character χ on $C^*(T_1, T_2)$ is a character both on $C^*(T_1)$ and $C^*(T_2)$, it follows that $\chi(T_1) \in \sigma_\pi(T_2)$ and $\chi(T_2) \in \sigma_\pi(T_2)$.

It may be proved in a similar way that Theorem 2 is true in the case of the finitely generated algebra $C^*(T_1, \ldots, T_n)(T_i \in \mathscr{L}(H)$ for $i = 1, \ldots, n)$, when, by definition, 0 is in the joint approximate point spectrum of $T_1, \ldots, T_n$, if for all $S_1, \ldots, S_n \in \mathscr{L}(H)$ it holds $S_1T_1 + \ldots + S_nT_n \neq I$.

Example. We take the l_2^+ space of square summable sequences $(z_1, z_2, \ldots)$, $\sum_{i=1}^{\infty} |z_i|^2 < \infty$. Let T be the unilateral shift in l_2^+, i.e. $Tx = (0, x_1, x_2, \ldots)$ and U – a diagonal unitary operator of the form $Ux = (a_1 x_1, a_2 x_2, \ldots)$ $(a_i \in C,\ |a_i| = 1)$. The approximate point spectrum of T is the unit circle [5]. We choose $a_i \in C$, $|a_i| = 1$ such that $a_n \to 1$, $a_n \neq 1$. Then 1 is in the approximate point spectrum of U. The algebra $C^*(T, U)$ is not commutative. Consider the sequence: $z_1 = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, $z_2 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots), \ldots, z_i = \left(\frac{1}{i}, \frac{1}{i+1}, \ldots\right), \ldots$ of vectors from l_2^+. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, thus $\lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{i^2} = 0$. Hence

$$\|T^* z_n - z_n\|^2 \leqslant \|Tz_n - z_n\|^2 = \sum_{i=n}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right)^2 + \frac{1}{n^2} \to 0, \quad \text{when } n \to \infty$$

and

$$\|U^* z_n - z_n\|^2 \leqslant \|Uz_n - z_n\|^2 = \sum_{i=1}^{\infty} \left| \frac{a_{i-1}}{n+i-1} \right|^2$$

$$\leqslant M \sum_{i=0}^{\infty} \frac{1}{(n+i)^2} = M \sum_{i=n}^{\infty} \frac{1}{i^2} \to 0.$$

It follows that $(1, 1)$ is in $C_\pi(T, U)$. By Theorem 2 there exists a character χ on $C^*(T, U)$ such that $\chi(T) = 1 = \chi(U)$.

References

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