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Unitary dilations of contraction operators

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Preliminaries

The present paper attempts to give a brief outline of the basic concepts of the theory of unitary dilations in Hilbert spaces. We try here to present an introduction into the foundations of this theory. Mostly we deal with unitary dilations of a single contraction. A considerable number of theorems presented in this paper may serve as a model for further extensions and modifications. The knowledge of those theorems may be treated as the first step in the study of the theory of unitary dilations.

Let H and K be Hilbert spaces and suppose that $K \supset H$. Suppose we are given a family T_a of bounded linear operators defined in H. Denote by P the orthogonal projection of K onto H. The family S_a of operators in K is called the *dilation* of T_a if

$$T_a P = P S_a P \quad \text{for every } a.$$

H is called the *initial space* and K the *dilation space*. We say also that T_a *are dilated to* S_a. In the whole paper P will always stand for the orthogonal projection of the dilation space onto the initial space.

The term "dilation" was introduced by P. Halmos in [11].

The basic problem in the dilation theory may be roughly formulated as follows:

Given a family T_a of linear bounded operators in H, find the dilation of T_a which has some required properties.

As an example we can consider the following problem:

Given a contraction T (the set of indices reduces then to a single point set), find a dilation of T which is a unitary operator.

In the last problem the required property of dilation is unitarity.

The requirements concerning dilations depens on the nature of the dilated family and it is natural to require that the dilations have a simpler structure than that of the dilated operators. In this spirit the study of dilations gives us a nice instrument in the investigation of the nature of general classes of operators. Since the dilations are required to be more "regular" than the dilated operators, we may hope to get some information about the latter by using the knowledge we have about dilations.

It is thus natural that one should have a more or less complete knowledge about special properties of operators which are dilations. Such properties are investigated in this paper for dilations which are unitary operators. We study in detail a unitary dilation of a single contraction operator and a dilation of a one-parameter semi-group of contractions. One can distinguish two groups of properties to be discussed. The first are geometric properties of a dilation space and some subspaces. The second group of properties concerns some spectral characterizations. Both kinds of properties are used in the functional calculus for contractions and a more detailed study of one-parameter semi-groups of contractions.

The earliest results concerning dilations were obtained by Neumark in paper [27]. He dealt with positive definite operator valued functions on locally compact abelian groups and studied in detail the unitary dilations of such functions. In [28] the same author proved one of the basic results in the dilation theory: every semi-spectral measure has a spectral dilation. Cooper [6] investigated semi-groups of isometries and proved in fact that they have unitary dilations which in that case are simply extensions of given isometries. Halmos [11] used a matrix construction to prove that every contraction has a simple unitary dilation. Sz.-Nagy [40] proved that the semi-group of powers of a fixed contraction and one-parameter strongly continuous semi-group of contractions do have unitary dilations. The matrix construction of the unitary dilation of a single contraction was later given by Schäffer [35]. Sz.-Nagy [41]-[43] developed his ideas and proved the generalization of Neumark's theorem about positive definite operator-valued functions to functions defined on semi-groups with involution, not necessarily abelian.

The structure of unitary dilations and the properties of their spectral functions were investigated by Sz.-Nagy [43] and continued by himself and Foiaş [45]-[52].

Several geometric and analytic properties of unitary dilations were discussed in papers [30], [36]-[39], [13]-[16] and [24]-[26].

M. Schreiber stated in 1956 that the unitary dilation of a strict contraction is a bilateral shift. His result was later improved and generalized in the papers of several authors (see [4], [13] and [47]).

Foiaş [8], [53] developed a certain kind of functional calculus for bounded linear operators in Hilbert spaces. He started from von Neumann's concept of the spectral set (see [55]) and the Riesz-Dunford functional calculus. On this general basis he proved most of the fundamental facts concerning a wide class of dilations related to spectral sets of dilated operators. The functional calculus for contractions was developed by Schreiber [37], [38] and Sz.-Nagy and Foiaş [45], [46], [48].

Unitary dilations of more than one contraction were first investigated by Sz.-Nagy [42], later by Brehmer [3] and Halperin [15], [16].

The present paper contains eleven sections. The first one is of technical character and includes the basic definitions and notational conventions.

The second section deals with a simple unitary dilation of a contraction. We present there the Halmos construction and explain the interplay between contractions and their semi-spectral measures.

Section 3 deals with the dilation theorem for operator-valued functions on groups. We also give some comments on this theorem in the case of the group in question being the additive group of integers. We then prove the Neumark's theorem on spectral dilations of semi-spectral functions on the unit circle.

Section 4 includes the Sz.-Nagy theorem on the existence of unitary dilations of a single contraction and of continuous one-parameter semi-groups of contractions. We present several different proofs of these theorems and some simple applications and examples.

In section 5 the canonical decomposition of some semi-groups of contractions is presented. Later on we prove there some decomposition formulae for the dilation space of a single contraction. Several kinds of subspaces are introduced and the investigations are performed in the spirit of the prediction theory of stationary stochastic processes. As a by-product we get the von Neumann-Halmos decomposition formula induced by an isometric operator.

Section 6 concerns some sufficient conditions which guarantee that the unitary dilation of a single contraction is a bilateral shift in all the dilation spaces or in some reducing subspaces. In this section we use the geometrical methods and the spectral multiplicity methods are not involved.

Sections 7 and 8 deal with analytic properties of semi-spectral measures and spectral-harmonic ones. We discuss in detail the eigenvalues of contractions lying on the unit circle as well as the properties of functions $\mu_f(\lambda) = (F(\lambda)f, f)$, where F stands for the spectral harmonic function of a given contraction.

In section 9 we present the fundamental theorems on the functional calculus for contractions. We use thereby unitary dilations and investigate a certain homomorphism of the algebra A (analytic functions in the open unit disc, continuous in the closed disc) into a certain algebra of operators. Later on the homomorphism is extended to some subalgebras of H^∞ (bounded analytic functions in the open unit disc). On the basis of the developed calculus one obtains a reasonable definition of an operator-valued function of a given contraction. The power of the functional calculus is illustrated by the investigation of strongly continuous one-parameter semi-groups of contractions in section 10. Here, an essential rôle is played by the concept of cogenerator of the semi-group in question. We obtain the explicit formulae for the semi-groups involved.

The last section deals with unitary dilations for families of commuting contractions. In fact, the dilations concern some special operator-valued functions defined in terms of contractions of a given family. The domain of these functions is the additive group of ultimately finite integer-valued functions defined on the set of indices of the family of contractions. The unitary dilations are unitary representations of such a group. We present the Halperin condition (necessary and sufficient) for the existence of such dilations and deduce therefore results known previously in this field.

There is a certain selection in this paper of theorems and methods which are used. Besides simple algebraic and geometrical constructions we use some classical parts of the theory of complex variable. The spectral theory enters here in the form of the Stone - von Neumann functional calculus. More advanced methods of spectral multiplicity as well related functional models are not discussed here.

1. Notation and definitions

Let T be a contraction, acting in the Hilbert space H. We say that the unitary operator U which acts in the space $K \supset H$ is the *simple unitary dilation* of T if

$$Tf = PUf \quad \text{for} \quad f \in H.$$

P stands here for the orthogonal projection of K onto H.

We say that the unitary operator U which acts in $K \supset H$ is the *unitary dilation* of the contraction T if

$$T^n f = PU^n f \quad \text{for} \quad f \in H \text{ and } n = 1, 2, 3, \ldots$$

P is the projection of K onto H.

Let T_α $(\alpha \in \Omega)$ be a family of linear bounded operators in H and let S_α be the dilation of T_α in the dilation space K. We say that S_α is a *mininal dilation* if the vectors of the form $S_\alpha f$ with $f \in H$ and $\alpha \in \Omega$ span the dilation space K. Under the same notational conventions we introduce the following definition:

We say that a certain condition (C) defines the dilation S_α *uniquely* within a unitary isomorphism if the following holds true:

If S'_α is a dilation of T_α and S'_α satisfies (C), then there is a unitary map V of K onto the whole dilation space K' of S'_α such that

(*) $$Vf = f \quad \text{for } f \in H \ (H \text{ the initial space}),$$

(**) $$VS_\alpha V^{-1} = S'_\alpha \quad \text{for all} \quad \alpha \in \Omega.$$

We then say that S_α are *essentially unique* with regard to (C).

Suppose now that T_a is an operator-valued function defined for $a \in G$ and G is a group. By a *unitary dilation* of T_a we always mean a dilation of T_a which is a unitary representation of the group G.

If G is the additive group of integers and the family T_n (we write here $a = n$ for convenience) is generated by a contraction T according to the formula

$$T_n = \begin{cases} T^n & \text{if} \quad n \geqslant 0, \\ T^{*|n|} & \text{if} \quad n \leqslant 0, \end{cases}$$

then the unitary dilation of T as defined previously is exactly the unitary dilation as defined above for such G.

A few notational conventions are now in order. The symbol $\{p\}$ stands for the set consisting of the single point p. If Z is a subset of a topological space, then $\bar{Z}$, $\mathrm{int}Z$ and ∂Z stand for the closure, interior and the boundary of Z respectively. The σ-field of Borel subsets of the topological space X is denoted by $B(X)$.

C stands for the unit circle $|z| = 1$ of the complex plane and D for the open unit disc defined as

$$D = \{z \mid |z| < 1\}.$$

The symbol dm stands for the normalized linear Lebesgue measure on the unit circle C.

The considered Hilbert spaces are denoted by H, K, L, M, N etc. possibly with indices. The vectors of such spaces are denoted by $f, g, h, \ldots$, the inner product by (f, g). We write $f \perp g$ if $(f, g) = 0$ and $Z' \perp Z''$ if $f \perp g$ for $f \in Z'$ and $g \in Z''$. The norm $|f|$ of f is $|f| = \sqrt{(f, f)}$. For scalars we use the notation $\alpha, \beta, \ldots, \lambda, \mu,$ The symbols $\to$ and lim stand, respectively, for the convergence and limit in the strong sense.

The orthogonal sum of spaces H_a $(a \in \Omega)$ is denoted by $\bigoplus_a H_a$ and in case of countable Ω by $\bigoplus_0^\infty H_n$, $\bigoplus_{-\infty}^{+\infty} H_n$ etc. *If* $H' \supset H''$, the orthogonal complement of H' within H'' is denoted by $H'' \ominus H'$. The smallest closed subspace which includes the subsets M, N of a certain Hilbert space is denoted by $M \vee N$. $\bigvee_a M_a$ stands for the smallest closed subspace which includes the union $\bigcup_a M_a$.

If Q is an operator, $D[Q]$ stands for its domain, $R[Q]$ for its range. The inverse of Q, if it exists, is denoted by Q^{-1}. The norm of a bounded linear operator is denoted by $|Q|$. $R(Q)$ stands for the resolvent set of the linear bounded operator Q, $S(Q)$ for the spectrum of Q. The point spectrum of Q is denoted by $S_p(Q)$, the continuous spectrum by $S_c(Q)$ and the residual one by $S_r(Q)$.

The approximate spectrum $S_a(Q)$ of Q is defined as

$S_a(Q) = \{z| \ |Qf_n - zf_n| \to 0$ for some sequence f_n such that $|f_n| = 1\}$.

Notice that

$$S_a(Q) \supset S_p(Q) \cup S_c(Q).$$

Let X be a topological space and suppose we are given an operator-valued function $Z(\sigma)$ defined for $\sigma \in B(X)$. Suppose that for every f belonging to the space in which $Z(\sigma)$ act the scalar-valued function $(Z(\sigma)f, f)$ is a positive measure on $B(X)$. Then $Z(\sigma)$ is called the *operator measure* or *semi-spectral measure*. In the case of $X = C$ with ordinary topology and $Z(C) = I$, we may define $Z(\lambda) = Z(\sigma_\lambda)$ where σ_λ stands for the subarc of C defined by $\sigma_\lambda = \{z|z = e^{i\mu}, 0 < \mu \leqslant \lambda\}$.

$Z(\lambda)$ is defined for $0 \leqslant \lambda \leqslant 2\pi$, $\sigma_0 =$ empty set and $\sigma_{2\pi} = C$. $Z(\lambda)$ is called the *semi-spectral function*. It is right-hand continuous in the strong topology and $Z(0) = 0, Z(2\pi) = I$. I stands for the identity operator. If $Z(\sigma)$ is a spectral measure on $B(C)$, the corresponding function $Z(\lambda)$ is called a *spectral function*. In any case, since $Z(\lambda)$ is normalized, the Fourier coefficients for $d(Z(\lambda)f, f)$ (all f) determine $Z(\lambda)$ uniquely.

The operator integral

$$A = \int_X f(z)\, dZ(z),$$

where $Z(\sigma)$ is a semi-spectral measure and f is a bounded Borel measurable function, is interpreted in the weak topology, if otherwise not stated.

There is no difficulty in defining the corresponding notions and notation for $X = (-\infty, +\infty)$ in place of $X = C$.

2. Simple unitary dilations

The problem of the existence of dilations appeared in a natural way in connection with some questions concerning the extension of certain operators. The operators in question were symmetric and closed with dense domains. Such operators can be extended to self-adjoint operators within the given initial space in the only case where the deficiency index is of the form (m, m). The Cayley transform leads us then to the following question: given an isometric linear operator V, find its unitary extension in any way.

Denote by H the space in which acts V and let

$$n = \dim(H \ominus D[V]), \qquad m = \dim(H \ominus R[V]).$$

If $m = n$, then the desired extension U is given by $U = V \oplus V'$, where V' is an arbitrary unitary mapping of $H \ominus D[V]$ onto $H \ominus R[V]$.

Let $m \neq n$. We can assume that $n \leqslant m$ and $n = 0$. In other words, we deal with the case where $D[V] = H$ and $R[V] \neq H$. Now we form the direct sum $\tilde{H} = H_1 \oplus H_2$, H_1, H_2 being the isomorphic copies of H. We define $\tilde{V} = V \oplus V^{-1}$ with $D[\tilde{V}] = H_1 \oplus R[V]$. Then $R[\tilde{V}] = R[V] \oplus H_2$. Since $\dim(\tilde{H} \ominus D[\tilde{V}]) = \dim(\tilde{H} \ominus R[\tilde{V}])$, there exists a unitary operator U on $\tilde{H}$ such that $\tilde{V} \subset U$. But $V \subset \tilde{V}$. It follows that $Vf = Uf = PUf$ for $f \in H_1$. P stands here for the orthogonal projection of $\tilde{H}$ onto H_1.

We conclude that every isometric operator V defined on all the space has a simple unitary dilation. This dilation is not a "pure" one. It is simply an extension of V. It is easy to see that the simple unitary dilation of an isometric operator must be an extension. A new feature appears: namely, we enlarge the space and the extension acts in this enlarged space.

The unitary dilation U of the isometric operator V has a spectral function, say $E(\lambda)$. Then

$$U = \int_0^{2\pi} e^{i\lambda} dE(\lambda).$$

Then, since $Vf = PUf$,

$$Vf = PUPf = \int_0^{2\pi} e^{i\lambda} dPE(\lambda)Pf = \int_0^{2\pi} e^{i\lambda} dF(\lambda)f \quad \text{for} \quad f \in H,$$

where

$$F(\lambda) = PE(\lambda)P. \tag{2.1}$$

The family $F(\lambda)$ of positive self-adjoint operators when restricted to H satisfies the following conditions:

$$F(0) = 0, \quad F(2\pi) = I, \tag{2.2}$$

$$0 \quad F(\lambda) \quad F(\mu) \quad \text{for} \quad \lambda \quad \mu, \tag{2.3}$$

$$\lim_{\lambda \to \lambda_0 +} F(\lambda) = F(\lambda_0). \tag{2.4}$$

Indeed, $F(0) \quad PE(0)P = 0$, $F(2\pi)f = PE(2\pi)f = f$ for $f \in H$. Let $\lambda \leqslant \mu$. Then $(PE(\lambda)Pf, f) \quad (E(\lambda)f, f) \leqslant (E(\mu)f, f) = (PE(\mu)f, f)$ for $f \in H$, which proves (2.3). Relation (2.4) follows from the inequalities

$$|(F(\lambda_0 + h) \quad F(\lambda_0))f|^2 \quad |P[E(\lambda_0 + h) \quad E(\lambda_0)]f|^2$$

$$(E(\lambda_0 + h) - E(\lambda_0)f, f) \quad \text{with} \quad h > 0.$$

It follows from (2.2), (2.3) and (2.4) that $F(\lambda)$ is a semi-spectral function. The operator-valued set function $F(\sigma) = PE(\sigma)P$, when restric-

ted to H, is the semi-spectral measure corresponding to the function $F(\lambda)$. We thus have

$$F(0) = 0, \quad F(C) = I, \quad F(\sigma_1 \cup \sigma_2) = F(\sigma_1) + F(\sigma_2) \quad \text{if} \quad \sigma_1 \cap \sigma_2 = 0,\ \sigma_1, \sigma_2 \in B(C) \tag{2.5}$$

$$0 \leqslant F(\sigma) \quad \text{for} \quad \sigma \in B(C), \tag{2.6}$$

$$\lim_{\sigma_n \to \sigma} F(\sigma_n) f = F(\sigma) f, f \in H, \ \{\sigma_n\} \text{ being a monotonic sequence of Borel sets.} \tag{2.7}$$

Note that every set function which satisfies (2.5), (2.6) and (2.7) is a semi-spectral measure on $B(C)$. In general, $F(\sigma)$ $(\sigma \in B(C))$ need not commute and the mapping $\sigma \to F(\sigma)$ is not multiplicative, i.e. $F(\sigma_1 \cap \sigma_2) \neq F(\sigma_1)F(\sigma_2)$ in general.

One can prove (see [5]) that if V is an isometric operator, then there is a unique semi-spectral function $F(\lambda)$ such that

$$Vf = \int_0^{2\pi} e^{i\lambda} dF(\lambda) f$$

holds true for every $f \in H$. The integral here may be interpreted in the weak sense. Going back to unitary dilations, we suppose for a moment that U is a simple unitary dilation in $K \supset H$ of the contraction T, i. e. $Tf = PUf$ for $f \in H$ and P stands as usual for the projection of K onto H. Denoting by $E(\lambda)$ the spectral function of U and using arguments similar to those used above, we get

$$(Tf, g) = \int_0^{2\pi} e^{i\lambda} d\big(F(\lambda) f, g\big) \quad \text{for} \quad f, g \in H \tag{2.8}$$

with $F(\lambda)h = PE(\lambda)h$ for $h \in H$. $F(\lambda)$ is a semi-spectral function in H. However, if for instance $|T| < 1$, then there is no non-zero element $f \in H$ such that $Tf = Uf$. Thus, U cannot be an extension of any non-trivial restriction of T. This property makes obvious the difference between dilation and ordinary extension.

We say that $F(\lambda)$ is the *semi-spectral function* of T if $F(\lambda)$ is a semi-spectral function and (2.8) holds true. The measure corresponding to the semi-spectral function of T is called the *semi-spectral measure* of T. Given a semi-spectral function $F(\lambda)$ (measure $F(\sigma)$), the integral

$$\int_0^{2\pi} e^{i\lambda} d\big(F(\lambda) f, g\big) \qquad \Big(\int_C z\, d\big(F(z) f, g\big)\Big)$$

is a bilinear form in H. Denote this form by $[f, g]$. We will show that $[f, g]$ defines a contraction T such that (2.8) holds true. Let us proceed

as follows (see [37]): the elements $f, g \in H$ being fixed, we write $\mu(\sigma) = (F(\sigma)f, g)$.

Denote by $V(\mu)$ the variation of $|\mu(\sigma)|$ over C. Let $\sigma_1, \ldots, \sigma_n$ be a partition of C into a finite number of disjoint Borel sets. Then

$$\sum |\mu(\sigma_k)| = \sum |(F(\sigma_k)f, g)| = \sum |(F(\sigma_k)^{1/2}f, F(\sigma_k)^{1/2}g)|$$

$$\sum |F(\sigma_k)^{1/2}f||F(\sigma_k)^{1/2}g| = \sum (F(\sigma_k)f, f)^{1/2}.(F(\sigma_k)g, g)^{1/2}$$

$$\leqslant \Big(\sum F(\sigma_k)f, f\Big)^{1/2}\Big(\sum F(\sigma_k)g, g\Big)^{1/2} = |f|\,|g|.$$

Hence $V(\mu) \leqslant |f|\,|g|$, which proves that the bilinear form $[f, g]$ determines a contraction T such that $[f, g] = (Tf, g)$, as was to be proved.

We see that every simple unitary dilation of contraction T defines a semi-spectral measure of T and conversely, every semi-spectral measure is a semi-spectral measure of a certain contraction. The correspondence between contractions and semi-spectral measures on $B(C)$ would be complete, although not one-to-one, if we knew that every contraction has a simple unitary dilation. The last statement holds true; namely the following theorem (see [11]) holds:

THEOREM 2.1. *Every contraction has a simple unitary dilation.*

Proof. Let T be a contraction in the space H. Then

$$0 \leqslant I - TT^* \leqslant I, \quad 0 \leqslant I - T^*T \leqslant I.$$

Define $K = H_1 \oplus H_2$ (H_1, H_2 are copies of H) and let U be the operator-valued matrix

$$U = \begin{pmatrix} T & D_{T^*} \\ -D_T & T^* \end{pmatrix},$$

where $D_T = (I - TT^*)^{1/2}$, $D_{T^*} = (I - T^*T)^{1/2}$. U defines an operator on K, just as an ordinary 2×2 matrix defines a linear operator in the 2-dimensional space. We indentify H_1 with H and easily verify that

$$Tf = PUf, \quad f \leftrightarrow (f, 0) \in H_1,$$

P being the projection of K onto H.

Using the fact that D_T and D_{T^*} are self-adjoint, as in the numerical case, we verify that

$$U^*U = \begin{pmatrix} T^* & D_{T^*} \\ D_T & T \end{pmatrix}\begin{pmatrix} T & D_T \\ D_{T^*} & T^* \end{pmatrix} = \begin{pmatrix} T^*T + D_{T^*}^2 & T^*D_T - D_{T^*}T^* \\ D_TT - TD_{T^*} & D_T^2 + TT^* \end{pmatrix},$$

$$UU^* = \begin{pmatrix} T & D_T \\ -D_{T^*} & T^* \end{pmatrix}\begin{pmatrix} T^* & -D_T \\ D_T & T \end{pmatrix} = \begin{pmatrix} TT^* + D_T^2 & -TD_{T^*} + D_TT \\ -D_{T^*}T^* + T^*D_T & D_{T^*}^2 + T^*T \end{pmatrix}.$$

The theorem will be proved if we show that

$$D_T T = T D_{T^*}. \tag{2.9}$$

This equality implies that $U^* U$ and $U U^*$ are diagonal matrices with identities on the main diagonals. Hence (2.9) implies that U is unitary.

Notice now that $D_T^2 T = T D_{T^*}^2$ and by induction $D_T^{2n} T = T D_{T^*}^{2n}$ for all $n \geqslant 0$. Consequently

$$p(D_T^2) T = T p(D_{T^*}^2) \tag{2.10}$$

for arbitrary polynomial $p(z)$. Remark that $D_T = (D_T^2)^{1/2}$ and $D_{T^*} = (D_{T^*}^2)^{1/2}$. Notice that there is a sequence of polynomials $p_n(z)$ such that for every positive bounded self-adjoint operator A (see [32])

$$\lim_{n \to \infty} p_n(A) = A^{1/2}$$

Let us write $p = p_n$ in (2.10). We get (2.9) by a limit passage, which completes the proof.

Since a multiple of a unitary operator is a normal operator, we conclude from theorem 2.1 that *every bounded operator in a Hilbert space has a normal dilation.*

Remark that a simple unitary dilation is not determined uniquely by the dilated contraction.

3. Dilation theorem

The general dilation theorem of Sz.-Nagy [42] states, roughly speaking, that a positive definite operator-valued function defined on a semi-group with involution is a dilation of a representation of the semi-group. We will present here a special case of this general theorem; namely in our case the semi-group will be simply a group (see [41]). In this context the result of Sz.-Nagy generalizes a theorem of Neumark [27], who dealed with abelian groups.

Let G be a group. We use the multiplicative notation for the inner group operation. The inverse of $\xi \in G$ is denoted by ξ^{-1}, the unit element of G by ε. Let H be a Hilbert space and let T_a be the operator-valued function defined for $a \in G$, the values T_a being bounded linear operators in H. The function T_a $(a \in G)$ is said to be *strongly positive* if

$$\sum_{\xi, \eta} \left(T_{\xi^{-1}\eta} f(\eta), f(\xi)\right) \geqslant 0 \tag{3.1}$$

for every vector-valued function $f(a)$, $a \in G$, which takes on only a finite number of values different from zero.

In the theorem below we assume that H has a real or complex or quaternionic field of scalars.

THEOREM 3.1 (Dilation theorem). *Let G be a group and let T_a be the operator-valued function defined on G. Suppose that T_a satisfies the following*

Condition (P). *The function T_a is strongly positive and*

$$T_{a^{-1}} = T_a^*, \quad T_e = I \tag{3.2}$$

Then there exists a larger space $K \supset H$ and a unitary representation U_a in K of G such that

$$T_a f = P U_a f \quad \text{for} \quad a \in G \text{ and } f \in H, \tag{3.3}$$

where P stands for the projection of K onto H.

The minimality condition $K = \bigvee_{a \in G} U_a H$ determines K and U_a in an essentially unique manner. If G is a topological group and $(T_a f, g)$ is continuous in a over G for every $f, g \in H$, then U_a is a strongly continuous representation of G.

Proof. The proof includes several steps.

(I) Let N be the set of all functions $f(a)$ on G with values in H. N becomes a linear manifold with multiplication by scalars and addition defined by $(\lambda f)(a) = \lambda f(a)$, $(f+g)(a) = f(a)+g(a)$. Denote by N_0 the submanifold of N consisting of those $f(a)$ which take only a finite number of values different from zero. Define a map φ from N_0 into N by the formula

$$(\varphi f(\ ))(a) = \sum_{\eta} T_{a^{-1}\eta} f(\eta) \tag{3.4}$$

with $f(\cdot) \in N_0$. For abbreviation we write simply $\varphi f(\cdot) = \hat{f}(\cdot)$. Let $\varphi(N_0) = N'$ and define in N' a bilinear form ,

$$[\hat{f}_1(\cdot), \hat{f}_2(\ )] = \sum_{\xi,\eta} (T_{\xi^{-1}\eta} f_1(\eta), f_2(\xi)).$$

Certainly

$$[\hat{f}_1, \hat{f}_2] = \sum_{\xi} (\hat{f}_1(\xi), f_2(\xi)).$$

On the other hand,

$$\sum_{\xi,\eta} (T_{\xi^{-1}\eta} f_1(\eta), f_2(\xi)) = \sum_{\xi,\eta} (f_1(\eta), T^*_{\xi^{-1}\eta} f_2(\xi)).$$

By (3.2), $T^*_{\xi^{-1}\eta} = T_{(\xi^{-1}\eta)^{-1}} = T_{\eta^{-1}\xi}$. Thus

$$[\hat{f}_1, \hat{f}_2] = \sum_{\xi,\eta} (f_1(\eta), T_{\xi^{-1}\eta} f_2(\xi)) = \sum_{\eta} (f_1(\eta), \hat{f}_2(\eta)).$$

It follows that $[\hat{f}_1, \hat{f}_2]$ depends only on the values of $\hat{f}_1, \hat{f}_2$ but not on their representations according to formula (3.4). It is easy to see that $[\hat{f}_1, \hat{f}_2] = \overline{[\hat{f}_2, \hat{f}_1]}$.

The positivity of T_a implies that

$$[\hat{f}, \hat{f}] = \sum_{\xi,\eta} \left(T_{\xi-1,\eta} f(\eta), f(\xi)\right) \geqslant 0.$$

It follows that Schwartz's inequality holds true, $|[\hat{f}, \hat{g}]|^2 \leqslant |[\hat{f}, \hat{f}]||[\hat{g}, \hat{g}]|$, and consequently $\hat{f}(a)$ is identically zero in the only case when $[\hat{f}, \hat{f}] = 0$. Indeed, if $[\hat{f}, \hat{f}] = 0$, then $[\hat{f}, \hat{g}] = 0$ for $\hat{g} \in N'$. Consequently, if $g \in H$, $a \in G$ and $g(\eta) = g$ for $\eta = a$ and $g(\eta) = 0$ if $\eta \neq a$, then $[\hat{f}, \hat{g}] = \left(\hat{f}(a), g\right) = 0$.

Since g is arbitrary we get $\hat{f}(a) = 0$ for $a \in G$.

We conclude that N' is a prehilbertian space with the $[\hat{f}, \hat{g}]$ as a candidate for the inner product. Let K be the completion of N' in the metric induced by the norm $|\hat{f}| = \sqrt{[\hat{f}, \hat{f}]}$. K becomes a complete space.

(II) The initial space H may be embedded into K as follows. The correspondence

$$f \in H, \quad f \leftrightarrow \tilde{f}(a) = T_{a-1} f, \quad \tilde{f}(a) \in N' \tag{3.5}$$

is one-to-one and linear. $\tilde{f}(a)$ equals $g_f(a)$ with

$$\begin{aligned} &g_f(\varepsilon) = f, \\ &g_f(a) = 0 \quad \text{for} \quad a \neq \varepsilon. \end{aligned} \tag{3.6}$$

We see that H is linearly isomorphic to a submanifold of K. The correspondence (3.5) is unitary. Indeed, the formulae of (I) imply

$$[\hat{f}_1, \hat{f}_2] = \sum_{\xi} \left(\hat{g}_{f_1}(\xi), g_{f_2}(\xi)\right) = \left(\hat{g}_{f_1}(\varepsilon), g_{f_2}(\varepsilon)\right) = (T_\varepsilon f_1, f_2).$$

By (3.2), $T_\varepsilon = I$. Hence $[\hat{f}_1, \hat{f}_2] = (f_1, f_2)$, which was to be proved.

Let $\tilde{H}$ be the image of H under the mapping established by (3.5). We now take $\hat{f} \in N'$ and $\tilde{h} \in \tilde{H}$. Since $P\hat{f}$ belongs to $\tilde{H}$ (P-projection of K onto $\tilde{H}$), there is a $\tilde{u} \in \tilde{H}$ such that $P\hat{f} = \tilde{u}$. Then $[\hat{f}, \tilde{h}] = [P\hat{f}, \tilde{h}] = [\tilde{u}, \tilde{h}] = (u, h)(u, h \in H)$. On the other hand, by (3.6)

$$[\hat{f}, \tilde{h}] = \sum_{\xi,\eta} \left(T_{\xi-1,\eta} f(\eta), g_h(\xi)\right) = \left(f(\varepsilon), h(\varepsilon)\right) = \left(f(\varepsilon), h\right).$$

Consequently $(u, h) = (f(\varepsilon), h)$ for every $h \in H$. This implies $u = f(\varepsilon)$. It follows that

$$(P\hat{f})(\alpha) = T_{\alpha^{-1}} f(\varepsilon) \quad (\alpha \in G). \tag{3.7}$$

(III) Define now

$$(U_\alpha \hat{f})(\xi) = \hat{f}(\alpha^{-1}\xi) \quad \text{for} \quad \hat{f} \in N'.$$

Since $f(\xi) = \sum_\eta T_{\xi^{-1}\eta} f(\eta)$, we get

$$\hat{f}(\alpha^{-1}\xi) = \sum_\eta T_{(\alpha^{-1}\xi)^{-1}\eta} f(\eta) = \sum_\eta T_{\xi^{-1}\alpha\eta} f(\eta) = \sum_\eta T_{\xi^{-1}\eta} f(\alpha^{-1}\eta).$$

Certainly $f(\alpha^{-1}\eta)$ belongs to N_0 for a fixed α. Hence $U_\alpha \hat{f}$ belongs to the manifold N'. Moreover,

$$(U_\varepsilon \hat{f})(\xi) = \hat{f}(\varepsilon^{-1}\xi) = \hat{f}(\xi). \tag{3.8}$$

Let $\hat{g}(\xi) = (U_\beta \hat{f})(\xi) = \hat{f}(\beta^{-1}\xi)$. Then

$$(U_\alpha \hat{g})(\xi) = \hat{g}(\alpha^{-1}\xi) = \hat{f}(\beta^{-1}\alpha^{-1}\xi) = \hat{f}\big((\alpha\beta)^{-1}\xi\big) = (U_{\alpha\beta}\hat{f})(\xi).$$

Hence

$$U_\alpha U_\beta \hat{f} = U_{\alpha\beta}\hat{f} \quad \text{for} \quad \hat{f} \in N'. \tag{3.9}$$

It follows that $U_\alpha N' = N'$ for all $\alpha \in G$. Next

$$[U_\alpha \hat{f}, \hat{g}] = \sum_{\xi,\eta} \big(T_{\xi^{-1}\alpha\eta} f(\eta), g(\xi)\big) = \sum_{\xi,\eta} \big(f(\eta), T_{\eta^{-1}\alpha^{-1}\xi} g(\xi)\big) = [\hat{f}, U_{\alpha^{-1}}\hat{g}].$$

Using (3.9) we get therefore

$$[U_\alpha \hat{f}, U_\alpha \hat{f}] = [U_{\alpha^{-1}} U_\alpha \hat{f}, \hat{f}] = [\hat{f}, \hat{f}].$$

Consequently U_α are one-to-one mappings of N' and preserve the inner product in K. Since N' is dense in K, U_α has a unique extension on all of K to the operator function taking on unitary operators as values. This new function is denoted also by U_α.

(IV) Let $\hat{f}(\xi) = T_{\xi^{-1}} f \in \tilde{H}$. Then $\hat{g}(\xi) = (U_\alpha \hat{f})(\xi) = \hat{f}(\alpha^{-1}\xi) = T_{\xi^{-1}\alpha} f$ and, by (3.7), $(P\hat{g})(\xi) = T_{\xi^{-1}}(T_{\xi^{-1}\alpha} f) = T_{\xi^{-1}} T_\alpha f = h(\xi)$ with $h = T_\alpha f$. It follows that the copy of T_α in $\tilde{H}$, say $\tilde{T}_\alpha$, satisfies

$$P U_\alpha \tilde{f} = \tilde{T}_\alpha \tilde{f}, \quad \tilde{f} \in \tilde{H}.$$

Consequently U_α is a unitary dilation of T_α.

We will show that $K = \bigvee_{\alpha \in G} U_\alpha \tilde{H}$. To begin with, we notice that, for $\hat{f} \in N'$,

$$\hat{f}(\xi) = \sum_\eta T_{\xi^{-1}\eta} f(\eta) = \sum_\eta (U_\eta \tilde{f}_\eta)(\xi) = \Big(\sum U_\eta \tilde{f}_\eta\Big)(\xi),$$

where $\tilde{f}_\eta(\zeta) = T_{\zeta^{-1}} f(\eta)$. Hence $\hat{f} = \sum U_\eta \hat{f}_\eta$. Since $\hat{f}$ span K, the space $\bigvee_{a \in G} U_a H$ equals K, q. e. d. We infer therefore that the dilation U_a constructed is minimal.

Next we prove that every two minimal unitary dilations of T_a are unitarily equivalent.

Let U'_a, U''_a be two unitary dilations of T with

$$K' = \bigvee_{a \in G} U'_a H, \qquad K'' = \bigvee_{a \in G} U''_a H \tag{3.10}$$

being the dilation spaces. Both U'_a and U''_a are representations of G. Let

$$f'^{(\prime)} = \sum_a U'^{(\prime)}_a f(a)$$

with $f(a) \in H$ and let $f(a)$ take only a finite number of values different from zero. We define the map V by $V\colon f' \to f''$. Then

$$\begin{aligned}(f'_1, f'_2)_{K'} &= \Big(\sum_\xi U'_\xi f_1(\xi), \sum_\eta U'_\eta f_2(\eta)\Big) \\ &= \sum_{\xi,\eta} \big(U'_{\eta^{-1}\xi} f_1(\xi), f_2(\eta)\big) = \sum_{\xi,\eta} \big(T_{\eta^{-1}\xi} f_1(\xi), f_2(\eta)\big).\end{aligned}$$

By similar arguments

$$(f''_1, f''_2)_{K''} = \sum_{\xi,\eta} \big(T_{\eta^{-1}\xi} f_1(\xi), f_2(\eta)\big) H$$

Formulae (3.10) imply that V can be extended to the unitary isomorphism of K' and K''. Certainly V leaves the elements of H invariant.

Since U'_a and U''_a are representations of G,

$$V\Big(U'_a \sum_\eta U'_\eta f(\eta)\Big) = V\Big(\sum_\eta U'_{a\eta} f(\eta)\Big) = V \sum_\zeta U'_\zeta f^a(\zeta)$$

with $f^a(\zeta) = f(a^{-1}\zeta)$. But

$$V \sum_\zeta U'_\zeta f^a(\zeta) = \sum_\zeta U''_\zeta f^a(\zeta) = U''_a \sum_\eta U''_\eta f(\eta).$$

We conclude therefore that U'_a and U''_a are unitarily equivalent representations of G, as was to be proved.

(V) Suppose now that G is a topological group and let $(T_a f, g)$ be continuous for every $f, g \in H$. Let U_a be the minimal unitary dilation of T_a. We understand that U_a is a representation of G. Write

$$u = \sum_\eta U_\eta f(\eta), \qquad v = \sum_\xi U_\xi g(\xi),$$

where $f(\alpha), g(\alpha) \in H$ and $f(\alpha), g(\alpha)$ take a finite number of values different from zero. We have

$$(U_\alpha u, v) = \sum_{\xi,\eta} \left(T_{\xi-1,\alpha\eta} f(\eta), g(\xi)\right).$$

The right-hand member is continuous in α. Since the elements as u and v span K and U_α have uniformly bounded norms, the standard arguments prove that $(U_\alpha u, v)$ is continuous in α for all of $u, v \in K$. Using the formula

$$|U_\alpha f - U_\beta f|^2 = 2|f|^2 - (U_\alpha f, U_\beta f) - (U_\beta f, U_\alpha f)$$

one gets that $U_\alpha f$ is strongly continuous in α for $f \in K$. This completes the proof of the last part of the assertion of the dilation theorem.

Remark. If H is a complex Hilbert space, then the positivity condition (3.1) implies that $T_{\xi-1} = T_\xi^*$.

Consider now the function

$$f(\alpha) = \begin{cases} f & \text{if} \quad \alpha = \varepsilon, \\ -T_{\xi-1} f & \text{if} \quad \alpha = \xi, \\ 0 & \text{elsewhere}, \end{cases}$$

the element ξ being fixed. If the positivity condition holds true, then

$$\Delta = \sum_{\gamma,\eta} \left(T_{\gamma^{-1},\eta} f(\eta), f(\gamma)\right) \geqslant 0$$

for the function $f(\alpha)$. An easy calculation shows that

$$\Delta = |f|^2 - |T_{\xi-1} f|^2 \geqslant 0$$

provided $T_\varepsilon = I$, $T_{\xi-1} = T_\xi^*$. It follows that under the condition (P) all T_α are contractions. This conclusion was deduced here without using the dilation theorem, which implies this conclusion immediately.

As an illustration of the general dilation theorem we will give several comments in the case where G has a simple structure. Let G be the additive group of integers. The operator-valued function on G now reduces simply to a sequence. Let $\{T_n\}$ be a sequence of linear bounded operators on H. $\{T_n\}$ is infinite in both directions, $+\infty$ and $-\infty$.

The positivity property of $\{T_n\}$ means that for every sequence

$$\ldots, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3, \ldots, \tag{3.11}$$

which has only a finite number of elements different from zero, the, inequality

$$\sum_{\nu,\mu=-\infty}^{+\infty} (T_{\nu-\mu} f_\nu, f_\mu) \geqslant 0 \tag{3.12}$$

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holds true. If $T_0 = I$ and $T_{-n} = T_n^*$, then $\{T_n\}$ satisfies condition (P), and condition (3.12) has a simple matrix interpretation.

Let us consider the infinite operator-valued matrix

$$\tau = \begin{pmatrix} I & T_{-1} & T_{-2} & T_{-3} & \cdots \\ T_1 & I & T_{-1} & T_{-2} & \\ T_2 & T_1 & I & T_{-1} & \\ T_3 & T_2 & T_1 & I & \\ \cdot & & & & \cdot \end{pmatrix}$$

with constant diagonals, the main diagonal consisting of identities. The above matrix is hermitian symmetric in the obvious sense. We call τ the *Toeplitz matrix* of $\{T_n\}$. Consider the sum $\bar{H}_n = \bigoplus_{i=0}^{n} H_i$ with H_i, the copies of H, and let τ_n be an $(n+1)\times(n+1)$ matrix which is obtained from τ by the restriction to the first $n+1$ columns and the first $n+1$ rows. Then τ_n may be interpreted as an operator on $\bar{H}_n$ which acts on $\bar{H}_n$ according to the same formulae which hold for the finite dimensional matrix calculus. Let $g = (g_0, \ldots, g_n) \in \bar{H}_n$ and consider $(\tau_n g, g)$. Since the (i, k)-entry of τ_n is T_{i-k}, we have

$$(\tau_n g, g) = \sum_{i,k=0}^{n} (T_{i-k} g_i, g_k)$$

and by (3.12)

$$(\tau_n g, g) \geqslant 0 \quad \text{for} \quad n = 0, 1, 2, \tag{3.13}$$

Suppose now that (3.13) holds for every n. The form appearing in (3.12) equals in fact to

$$\sum_{\nu,\mu=-p}^{p} (T_{\nu-\mu} f_\nu, f_\mu)$$

with a finite p. Then

$$\sum_{\nu,\mu=-p}^{p} (T_{\nu-\mu} f_\nu, f_\mu) = (\tau_{2p} g, g) \geqslant 0$$

with $g_i = f_{i-p}$. We have thus proved that (3.13) implies the positivity of $\{T_n\}$. It follows that (P) is in this case equivalent to the following statement: every matrix τ_n is hermitian symmetric and positive definite.

The dilation theorem implies the following

THEOREM 3.2. *Let $\{T_n\}$ be a strongly positive doubly infinite sequence of bounded linear operators in the space H. Assume that $T_0 = I$ and $T_{-n} = T_n^*$. Then there exists a larger space $K \supset H$ and a unitary operator U in K such that*

$$T_n f = P U^n f, \quad n = 0, \pm 1, \pm 2, \ldots, f \in H. \tag{3.14}$$

If $K = \bigvee_{-\infty}^{+\infty} U^n H$, *then* K *and* U *are determined uniquely within a unitary isomorphism.*

Proof. Let G be the additive group of integers. The dilation theorem and the assumptions imply that T_n is a dilation of the representation of G. Let U_n be this representation. Then $U = U_1$ satisfies the assertion of our theorem.

It was proved in section 2 that every semi-spectral function is a semi-spectral function of a suitable contraction. On the other hand, theorem 2.1 implies that every contraction has at least one semi-spectral function of its own. This semi-spectral function of the prescribed contraction may be obtained from the spectral function of the simple unitary dilation of the contraction. The natural problem now arises whether every semi-spectral function may be dilated to a spectral one. This problem is solved completely by the theorem of Neumark, which we will prove now. Neumark's theorem states that every semi-spectral measure has an essentially unique spectral dilation. Since we do not need further the most general version of this theorem, we will prove a particular case of this statement, namely when the σ-field is simply $B(C)$. It suffices to consider the semi-spectral function instead of a measure. The initial space is supposed to be a complex Hilbert space.

We will prove the following theorem (see [22]) by using the arguments applied by Sz.-Nagy in [11]:

THEOREM 3.3. *Let* H *be a complex Hilbert space and let* $F(\lambda)$ *be a semi-spectral function on* $\langle 0,2\pi\rangle$. *Then* $F(\lambda)$ *has an essentially unique minimal dilation* $E(\lambda)$ *which is a spectral function.*

Proof. The integrals

$$\int_0^{2\pi} e^{in\lambda} dF(\lambda)$$

interpreted in the weak sense define linear bounded operators in H. This can be verified by using the arguments we have applied in section 2 for $n = 1$. Define

$$T_n = \int_0^{2\pi} e^{in\lambda} dF(\lambda).$$

Since $F(\lambda)$ are symmetric, $T_{-n} = T_n^*$ and obviously $T_0 = I$. We will check that $\{T_n\}$ is positive definite. Let $g \quad (g_0, \ldots, g_n)$ and denote by τ_n the n-th matrix of $\{T_n\}$. Then

$$(\tau_n g, g) = \int_0^{2\pi} \sum_{s,k=0}^{n} e^{i(s-k)\lambda} d\big(F(\lambda) g_s, g_k\big).$$

Let

$$S = \sum_{p} \sum_{s,k=0}^{n} e^{i(s-k)\lambda_p} \big([F(\lambda_{p+1}) - F(\lambda_p)]g_s, g_k\big) \qquad (\lambda_p < \lambda_{p+1})$$

be the approximate sum for the last integral. Then

$$S = \sum_{p} \Big([F(\lambda_{p+1}) - F(\lambda_p)] \sum_{s|0}^{n} e^{is\lambda_p} g_s, \sum_{k|0}^{n} e^{ik\lambda_p} g_k\Big) \geqslant 0$$

for the differences $[F(\lambda_{p+1}) - F(\lambda_p)]$ are positive operators. This establishes the positivity of the sequence $\{T_n\}$.

Denote by U the unitary operator acting in $K \supset H$ such that $T_n f = PU^n f$ for all $f \in H$ and all n. The existence of such a U follows from theorem 3.2. Now

$$\int_0^{2\pi} e^{in\lambda} d\big(F(\lambda)f, g\big) = \int_0^{2\pi} e^{in\lambda} d\big(PE(\lambda)f, g\big) \tag{3.15}$$

for $f, g \in H$ and $E(\lambda)$ is the spectral function of U. The n-th Fourier coefficients of $d\big(F(\lambda)f, g\big)$, $d\big(PE(\lambda)f, g\big)$ being equal and the functions being normalized, we conclude that $F(\lambda)f = PE(\lambda)f$ for $f \in H$ and all λ. Hence $E(\lambda)$ is a spectral dilation of $F(\lambda)$. Since

$$\bigvee_{\sigma \in B(C)} E(\sigma)H = \bigvee_{-\infty}^{+\infty} U^n H,$$

we conclude by theorem 3.1 that if $E(\lambda)$ is minimal then it is essentially unique.

Let $F(\lambda)$ be a semi-spectral function. Its spectral dilation constructed in theorem 3.3 will be called the *Neumark dilation* of $F(\lambda)$ and denoted by $E_F(\lambda)$. The corresponding unitary operator

$$\int_0^{2\pi} e^{i\lambda} dE_F(\lambda),$$

which is a simple unitary dilation of the contraction

$$T = \int_0^{2\pi} e^{i\lambda} dF(\lambda),$$

will be denoted by U_F.

In general, if $E(\sigma)$ is a minimal spectral dilation of the semi-spectral measure $F(\sigma)$, the dilation space being K, then no non-trivial subspace of $K \ominus H$ is reduced by all of $E(\sigma)$. Indeed, let $M \subset K \ominus H$ be such a subspace and denote by P_M the projection onto M. Then $\big(P_M E(\sigma)f, g\big) = \big(E(\sigma)P_M f, g\big)$ for all $f \in K$ and $g \in H$. Hence $\big(P_M f, E(\sigma)g\big) = 0$ for such f and g, which by the minimality of $E(\sigma)$ proves that $P_M = 0$, q. e. d.

Conversely, if $K \ominus H$ does not include non-trivial subspaces reducing all $E(\sigma)$, then K is minimal and $E(\sigma)$ is minimal.

It follows from theorem 3.3 that the weak integrals

$$\int_0^{2\pi} e^{in\lambda} dF(\lambda)$$

are in fact integrals in the uniform operator topology. This conclusion is a consequence of the corresponding property for unitary operators.

It is a simple matter to transfer our discussion to a more general ground. In place of the additive group of integers we can consider an abelian locally compact topological group G. Denote by $\hat{G}$ the group of multiplicative characters of G and assume that the initial space H is a complex Hilbert space. Suppose that the operator-valued function T_a defined on the group G satisfies condition (P). Then, by the dilation theorem $T_a f = PU_a f$ where U_a is a unitary representation of G. If this representation is strongly continuous, then by the generalized Stone theorem there is a unique spectral measure $E(\gamma)$ on Baire sets such that

$$U_a = \int_{\hat{G}} \gamma(a)\, dE(\gamma)$$

for all $a \in G$, $\gamma(a)$ being the value of the character γ at a. Now we get the representation (see [27])

$$T_a f = \int_{\hat{G}} \gamma(a)\, dF(\gamma) f, \qquad f \in H,$$

where $F(\gamma) = PE(\gamma)$, the right-hand side being restricted to H. $F(\gamma)$ is a semi-spectral measure on Baire sets. Obviously

$$(T_a f, f) = \int_{\hat{G}} \gamma(a)\, d(F(\gamma) f, f) \qquad (a \in G, f \in H) \tag{3.16}$$

and $(F(\gamma)f, f)$ is a positive measure on Baire sets. Since every continuous function with compact support on $\hat{G}$ can be approximated uniformly by linear combinations of characters, equalities (3.16) determine $F(\gamma)$ uniquely.

Suppose that G contains a subsemi-group G_+ which includes the unit element ε. We say that G satisfies the condition (R) if the following holds true:

$$G_+ \cup G_+^{-1} = G \quad \text{and} \quad G_+ \cap G_+^{-1} = \{\varepsilon\}. \tag{R}$$

Under condition (R) the function $T(\xi)$ defined merely on G_+ may be extended naturally to all of G by the formulae

$$T_\xi = \begin{cases} T(\xi) & \text{if } \xi \in G_+, \\ T(\xi^{-1})^* & \text{if } \xi^{-1} \in G_+. \end{cases} \tag{3.17}$$

Suppose now that $T(\xi)$ is a strongly continuous representation of G_+, i. e. for $\xi, \eta \in G_+$

$$T(\xi\eta) = T(\xi)T(\eta), \quad \lim_{\xi \to e} T(\xi)f = f \quad \text{for} \quad f \in H, \tag{3.18}$$

and let T_ξ be the extension of $T(\xi)$ according to formulae (3.17). If (R), (3.17) and (3.18) hold true, then the semi-spectral measure $F(\gamma)$ which satisfies (3.16) is called the *spectral-harmonic measure* of $T(\xi)$. Such terminology is a natural generalization of the terminology introduced by Foiaș in [8] if G is the additive group of integers.

Notice that (R) holds true for subgroups of the additive group of reals with $G_+ = \{\xi | \xi \geqslant 0, \xi \in G\}$.

If (R) holds true and $U(\xi)$ is a unitary representation of G_+ and

$$T(\xi)f = PU(\xi)f \quad \text{for} \quad \xi \in G_+ \text{ and } f \in H, \tag{3.19}$$

then $U(\xi)$ extended onto G by (3.17) becomes the unitary dilation of T_ξ defined by (3.17) by $T(\xi)$. It follows that there is no confusion in calling $U(\xi)$ the unitary dilation of $T(\xi)$ for $\xi \in G_+$ provided (R) and (3.19) hold true. This applies in particular to one-parameter semi-groups of contractions.

4. Unitary dilation of a single contraction

We will investigate some necessary conditions for the existence of a unitary dilation of a single contraction. The conditions in question are of analytical character and involve the spectral theorem for unitary operators.

Suppose we are given a complex Hilbert space H. Let T be a contraction in H and assume for a moment that U is the unitary dilation of T in the dilation space $K \supset H$, i. e.

$$T^n f = PU^n f \quad \text{for} \quad n = 0, 1, 2, \quad f \in H, \tag{4.1}$$

where P stands for the projection of K onto H. It follows that $(T^{*n}f, g) = (T^{*n}f, Pg) = (f, T^n Pg) = (f, PU^n g) = (PU^{*n}f, g)$ for $f \in H$, $g \in K$, that is (4.1) implies

$$T^{*n} f = PU^{-n} f \quad \text{for} \quad n = 0, 1, 2, \quad f \in H \tag{4.2}$$

Hence (4.1) and (4.2) are equivalent.

Let $E(\lambda)$ be the spectral function of U. By (4.1), (4.2) we have

$$(T^{(n)}f, g) = \int_0^{2\pi} e^{in\lambda} d(F(\lambda)f, g) \quad \text{for all } n \text{ and } f, g \in H, \tag{4.3}$$

where

$$T^{(n)} = \begin{cases} T^n & \text{if} \quad n \geqslant 0, \\ T^{*|n|} & \text{if} \quad n \leqslant 0, \end{cases}$$

and $F(\lambda)h = PE(\lambda)h$ for $h \in H$. We know that the integrals

$$\int_0^{2\pi} e^{in\lambda} dF(\lambda)$$

can be interpreted in the uniform operator topology. Then (4.3) will imply

$$\int_0^{2\pi} e^{in\lambda} dF(\lambda) = \left[\int_0^{2\pi} e^{i\lambda} dF(\lambda)\right]^{(n)} \tag{4.4}$$

According to general terminology, the semi-spectral function F which satisfies (4.4) will be called the spectral-harmonic function.

The function F which satisfies (4.3) is called the *spectral harmonic function* of contraction T. The corresponding semi-spectral measure is called the *spectral-harmonic measure* of contraction T. It follows from the discussion of section 3 that every contraction may have at most one spectral-harmonic measure.

The question of the existence of a spectral-harmonic function of the contraction T can be stated in several equivalent forms. First of all, if $F(\lambda)$ is the spectral-harmonic function for T, then for every fixed $f \in H$ the function $(F(\lambda)f, f)$ is a normalized solution of the trigonometrical moment problem

$$c_n(f) = \int_0^{2\pi} e^{in\lambda} d\alpha(\lambda : f) \qquad (\text{all } n) \tag{4.5}$$

with $c_n(f) = (T^{(n)}f, f)$.

Suppose now that for each $f \in H$ the moment problem (4.5) has a normalized solution $\alpha(\lambda : f)$. Hence

$$0 \leqslant \alpha(\lambda : f) \leqslant \alpha(2\pi : f) = (f, f)$$

and the family of bilinear forms $\alpha(\lambda : f, g)$,

$$\alpha(\lambda : f, g) = \tfrac{1}{4}[\alpha(\lambda : f+g) + i\alpha(\lambda : f+ig) - \alpha(\lambda : f-g) - i\alpha(\lambda : f-ig)],$$

defines uniquely a semi-spectral function $F(\lambda)$ such that $\alpha(\lambda : f) = (F(\lambda)f, f)$. It follows by (4.5) that (4.3) holds true, i. e. $F(\lambda)$ is a spectral-harmonic function of T.

Let us now consider the scalar Toeplitz matrix $\tau(f)$ associated with $c_n(f)$. The (i, k)-entry of $\tau(f)$ is $c_{i-k}(f)$. It is known that a necessary and sufficient condition for the existence of the normalized solution of (4.5) is the following one (see [34]):

For every m the Toeplitz form

$$\tau_m(f:u) = \sum_{i,k=0}^{m} c_{i-k} u_i \bar{u}_k \tag{4.6}$$

is non-negative.

In our context this last property means that the hermitian symmetric matrices with (i,k)-entry, equal to $(T^{(i-k)}f, f)$, should define positive forms for every $f \in H$.

If forms (4.6) are positive, then

$$\det \begin{pmatrix} c_0 & c_n \\ c_{-n} & c_0 \end{pmatrix} \geqslant 0 \tag{4.7}$$

and consequently $|c_n| \leqslant c_0$ for every n. It follows that the function

$$\hat{f}(z) = c_0(f) + 2\sum_{n=1}^{\infty} c_n(f) z^n$$

is regular for $|z| < 1$. It is known (see [34]) that the forms $\tau_m(f)$ are positive for $m = 0, 1, 2, \ldots$ if and only if $\operatorname{Re} \hat{f}(z) \geqslant 0$ for $|z| < 1$. It follows that (4.5) is solvable if and only if, for every $f \in H$,

$$\operatorname{Re} F_f(z) \geqslant 0, \qquad |z| < 1, \tag{4.8}$$

where

$$F_f(z) = (f,f) + 2\sum_{n=1}^{\infty} z^n (T^n f, f).$$

Hence the construction of the spectral-harmonic function will be established if we show that (4.8) holds for every $f \in H$.

We proceed as follows (see [40]).

Since T is a contraction

$$I + 2\sum_{n=1}^{\infty} z^n T^n = (I+zT)(I-zT)^{-1} \quad \text{for} \quad |z| < 1$$

Write $g = (I-zT)^{-1} f$. Then

$$F_f(z) = (g,g) + z(Tg,g) - \bar{z}(g,Tg) - z\bar{z}(Tg,Tg) \tag{4.9}$$

and consequently

$$\operatorname{Re} F_f(z) = |g|^2 - |z|^2 |Tg|^2 \geqslant 0 \quad \text{for} \quad |z| < 1. \tag{4.10}$$

Since (4.10) holds for every $f \in H$, we conclude that (4.5) is solvable for each $f \in H$. Summing up we get the following theorem (see [40]):

THEOREM 4.1. *Every contraction has a spectral-harmonic function.*

We will now prove the basic theorem of Sz.-Nagy [40]:

THEOREM 4.2. *Every contraction T has an essentially unique unitary dilation.*

Proof. Denote by $F(\lambda)$ the spectral-harmonic function of T. Let $E(\lambda)$ be the Neumark dilation of $F(\lambda)$ with K as a dilation space. We have

$$(T^{(n)}f, g) = \int_0^{2\pi} e^{in\lambda} d\big(F(\lambda)f, g\big) = \int_0^{2\pi} e^{in\lambda} d\big(E(\lambda)f, g\big) \quad \text{for} \quad f, g \in H.$$

Write

$$U = \int_0^{2\pi} e^{i\lambda} dE(\lambda).$$

Then $(T^{(n)}f, g) = (U^n f, g) = (PU^n f, g)$. The element g is arbitrary. It follows that $T^{(n)}f = PU^n f$, which proves that U is a dilation of T. Since $E(\lambda)$ is minimal,

$$K = \bigvee_{\sigma} E(\sigma)H = \bigvee_{-\infty}^{+\infty} U^n H.$$

Hence U is minimal and, by the dilation theorem, essentially unique.

It is an interesting question how some properties of T are reflected in the properties of its spectral-harmonic function. For an illustration we will present some propositions concerning normal contractions.

The following theorem holds true (see [38]):

THEOREM 4.3. *Let T be a contraction in the complex Hilbert space H, and let $F(\lambda)$ be the spectral-harmonic function of T. Suppose that Q is a linear bounded operator in H. Then Q commutes with T and T^* if and only if Q commutes with all of $F(\lambda)$, $0 \leqslant \lambda \leqslant 2\pi$.*

Proof. If Q commutes with $F(\lambda)$ (all λ), then Q commutes with approximate sums for the integrals

$$\int_0^{2\pi} e^{i\lambda} dF(\lambda) = T, \qquad \int_0^{2\pi} e^{-i\lambda} dF(\lambda) = T^*.$$

Since the integrals converge in the uniform operator topology, we are done.

Suppose now that Q commutes with T and T^*. Let $f, g \in H$. Then $(T^{(n)}Qf, g) = (QT^{(n)}f, g)$. It follows that the n-th Fourier coefficients of $d\big(F(\lambda)Qf, g\big)$ and $d\big(F(\lambda)f, Q^*g\big)$ are equal. Hence $F(\lambda)Q = QF(\lambda)$, q. e. d.

COROLLARY. *T commutes with its own spectral-harmonic function for all λ if and only if T is normal* (see [38]).

This is a consequence of theorem 4.3 with $Q = T$.

The next theorem (see [38]) also deals with normal contractions:

THEOREM 4.4. *Let T be a contraction in the complex Hilbert space and let $F(\lambda)$ be the spectral-harmonic function of T. Then T is normal if and only if $F(\lambda)$ and $F(\mu)$ commute for every $\lambda, \mu \in [0, 2\pi]$.*

Proof. $F(\lambda)$ and $F(\mu)$ commute for arbitrary λ and μ, so do the approximate sums for the integrals

$$\int_0^{2\pi} e^{i\lambda} dF(\lambda) = T, \qquad \int_0^{2\pi} e^{-i\lambda} dF(\lambda) = T^*.$$

Consequently T and T^* commute, q. e. d.

Suppose now that T is normal. Let U be the minimal unitary dilation of T and let $E(\lambda)$ be the spectral function of U. For a fixed λ, $E(\lambda)$ is a strong limit of polynomials of the form

$$p_{n,\lambda}(U) = \sum_{-n}^{n} c_{k,n} U^k$$

It follows that

$$Pp_{n,\lambda}(U)f \to PE(\lambda)f = F(\lambda)f \tag{4.11}$$

for f belonging to the initial space in which T acts. Since T is normal, $Pp_{n,\lambda}(U)$ and $Pp_{m,\mu}(U)$ commute for every m, n, λ, μ. By (4.11) we get therefore that

$$PE(\lambda)PE(\mu)f = PE(\mu)PE(\lambda)f,$$

which completes the proof.

Some examples are now in order.

Example 1. Let H^2 be the space of analytic functions $f(z)$ in D such that

$$\lim_{r\to 1-} \int_0^{2\pi} |f(re^{is})|^2 ds < +\infty.$$

If $f(z) \in H^2$, then the boundary values

$$f(e^{i\lambda}) = \lim_{r\to 1-} f(re^{i\lambda})$$

exist for almost all λ. H^2 becomes a Hilbert space under the inner product defined by

$$(f, g) = \int_0^{2\pi} f(e^{i\lambda}) \overline{g(e^{i\lambda})}\, dm(\lambda).$$

Let T be the isometric operator in H^2 defined by

$$T : f(e^{i\lambda}) \to e^{i\lambda} f(e^{i\lambda}).$$

Denote by $F(\lambda)$ the spectral-harmonic function of T. Notice now that

$$(T^{(n)}f, g) = \int_0^{2\pi} e^{in\lambda} f(e^{i\lambda}) \overline{g(e^{i\lambda})}\, dm(\lambda) \qquad \text{for all } n.$$

Since

$$(T^{(n)}f, g) = \int_0^{2\pi} e^{in\lambda} d(F(\lambda)f, g),$$

we conclude that

$$\frac{d}{d\lambda}(F(\lambda)f, g) = \frac{1}{2\pi} f(e^{i\lambda})\overline{g(e^{i\lambda})} \quad \text{for almost all } \lambda.$$

Consequently, the spectral-harmonic measure of T is given by the formula

$$(F(\sigma)f, g) = \int_\sigma f(e^{i\lambda})\overline{g(e^{i\lambda})}\, dm(\lambda).$$

Example 2. Suppose that H is a complex Hilbert space and define T by $T: f \to \varrho f$ $(f \in H)$ where ϱ is a a fixed complex number such that $|\varrho| < 1$. Let $P_r(\lambda)$ be the Poisson kernel

$$P_r(\lambda) = \mathrm{Re}\left[\frac{1+re^{i\lambda}}{1-re^{i\lambda}}\right].$$

Write $\varrho = re^{i\theta}$ $(0 \leqslant r < 1,\ 0 \leqslant \theta \leqslant 2\pi)$ and consider the function

$$p(\lambda) = \frac{1}{2\pi} P_r(\theta - \lambda).$$

Since

$$\int_0^{2\pi} e^{in\lambda} p(\lambda)\, d\lambda = r^{|n|} e^{in\theta} = \varrho^{(n)}$$

for all n and

$$\varrho^{(n)} = \begin{cases} \bar{\varrho}^{|n|} & \text{for} \quad n \leqslant 0, \\ \varrho^n & \text{for} \quad n \geqslant 0, \end{cases}$$

the spectral-harmonic measure $F(\sigma)$ of T is simply the multiplication by

$$F(\sigma) = \int_\sigma P_r(\theta - \lambda)\, dm(\lambda),$$

the harmonic measure of ϱ with respect to $\sigma \subset C$.

Here we have an explanation of the use of the term spectral-harmonic measure. Other suggestions in this subject are included in other sections of this paper.

The proof we have presented of theorem 4.2 is valid in complex spaces and makes use of analytic functions, the spectral theory and the Neumark theorem. Using only the Fourier series (see [42]) one can verify that $\{T^{(n)}\}$ defined for a prescribed contraction T is strongly positive. The case of the real space can be reduced to the complex case. By [42] we argue as follows.

Write

$$T(r, \varphi) = \sum_{-\infty}^{+\infty} r^{|n|} e^{in\varphi} T^{(n)}.$$

Formula (4.10) proves that

$$(T(r, \varphi)f, f) \geqslant 0 \quad \text{for every } f \in H. \tag{4.12}$$

Let $\{f_n\}$ be a sequence of vectors with only a finite number of elements different from zero. Writing

$$f(\varphi) = \sum_{-\infty}^{+\infty} e^{-in\varphi} f_n$$

an easy calculation shows that

$$\big(T(r, \varphi)f(\varphi), f(\varphi)\big) = \sum_{l} e^{il\varphi} \sum_{m,n} r^{|l+n-m|} (T^{(l+n-m)} f_n, f_m). \tag{4.13}$$

The integrating in φ of (4.13) proves that

$$\sum_{m,n} r^{|n-m|} (T^{(n-m)} f_n, f_m) = \frac{1}{2\pi} \int_0^{2\pi} \big(T(r, \varphi)f(\varphi),\ f(\varphi)\big)\, d\varphi \geqslant 0. \tag{4.14}$$

The limit passage $r \to 1-$ in (4.14) proves that $\{T^{(n)}\}$ is strongly positive. Now we can apply directly the dilation theorem and, omitting the Neumark theorem, prove theorem 4.2.

Theorem 4.2 may be proved by using other arguments. We mention here the matrix method due to Schäffer [35]. It generalizes the method of Halmos used in the proof of theoren 2.1. The Schäffer construction gives a unitary dilation of a given contraction, but this dilation need not be minimal. Sz.-Nagy improved the Schäffer method to get minimal dilation. The direct geometrical construction of the minimal unitary dilation of a single contraction was given in [13] by Halperin. The method of [13] is closely related to the matrix construction. It was noticed by Halperin in [13] that the matrix construction as well the geometrical one is valid for spaces with real, complex and quaternionic field of scalars. The simple direct proof of the positivity of $\{T^{(n)}\}$ for a single contraction, valid in all these three cases, belongs also to Halperin. We will present this proof now. What we will prove is exactly the following: the operator-valued Toeplitz matrix

$$\tau_n = \begin{pmatrix} I & T & T^2 & \dots & T^n \\ T^* & I & T & \dots & T^{n-1} \\ T^{*2} & T^* & I & \dots & T^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ T^{*n} & T^{*n-1} & \dots & \dots & I \end{pmatrix}$$

admits a factorization of the form

$$\tau_n = W^*DW, \tag{4.15}$$

where the entries of D are $D_{00} = I$, $D_{ii} = (I - T^*T)$ for $i = 1, \ldots, n$ and $D_{ik} = 0$ for $i \neq k$ and the entries of W are given by the formulae

$$W_{ij} = \begin{cases} T^{j-i} & \text{if} \quad j \geqslant i, \\ 0 & \text{if} \quad j < i. \end{cases}$$

To begin with, notice that τ_n is hermitian symmetric. It follows that we need only to show that the (i, j) entries of τ_n and W^*DW are equal for $i \leqslant j$.

Since $(W^*)_{i,j} = (W_{ji})^* = 0$ if $j > i$, we have

$$(W^*DW)_{i,j} = \sum_{h=0}^{i} (W^*)_{i,h} D_{h,h} W_{h,j} = (T^*)^i T^j + \sum_{h=1}^{i} (T^*)^{i-h} (I - T^*T) T^{j-h}$$

$$= (T^*)^i T^j + \sum_{h=1}^{i} \left((T^*)^{i-h} T^{j-h} - (T^*)^{i-h+1} T^{j-h+1} \right) = (T^*)^0 T^{j-i} = (\tau_n)_{i,j}$$

and the proof of (4.15) is complete.

Let $H^n = H \oplus H \oplus \ldots \oplus H$ ($n+1$ times) and let $g \in H^n$. It follows from (4.15) that $(\tau_n g, g) = (DWg, Wg) \geqslant 0$ since D is positive definite; $(I - T^*T) \geqslant 0$ for T is a contraction. We know from section 3 that the positivity of all τ_n is equivalent to the strong positivity of the sequence $\{T^{(n)}\}$. It follows now from the dilation theorem that theorem 4.2 holds true for every contraction in H, where the field of scalars in H is real, complex or quaternionic.

To complete our considerations we will prove that every weakly continuous semi-group of contractions has a unitary dilation. The weak continuity of a semi-group means that $(T(t)f, g)$ is a continuous function for fixed but arbitrary f and g. Given the semi-group $T(t)$ ($T(0) = I$, $t \geqslant 0$), we embed it in the family

$$T_t = \begin{cases} T(t) & \text{if} \quad t \geqslant 0, \\ T(-t)^* & \text{if} \quad t \leqslant 0. \end{cases}$$

If $T(t)$ is weakly continuous, so is T_t. Next we prove (see [41])

LEMMA 4.1. *If $T(t)$ is a weakly continuous semi-group of contraction operators, then T_t is strongly positive on $(-\infty, +\infty)$.*

Proof. Let $f(t)$ be a vector-valued function and let it take on only a finite number of values different from zero. Denote by $t^1, t^2, \ldots, t^n$ the

sequence of all those t for which $f(t) \neq 0$. Suppose that t_ν^r $(r = 1, 2, \ldots, n)$ is a sequence of rational numbers such that

$$t^r = \lim_{\nu \to \infty} t_\nu^r \qquad (r = 1, \ldots, n).$$

Since T_t is weakly continuous, we infer that

$$\sum_{t,s} \left(T_{t-s} f(t), f(s)\right) = \lim_{\nu \to \infty} \sum_{m,k=1}^{n} \left(T_{t_\nu^m - t_\nu^k} f(t^m), f(t^k)\right).$$

The numbers t_ν^r being rational there are integers τ_ν^r, and $t_\nu^r - \tau_\nu^r d_\nu$ and $d_\nu > 0$. Consequently, by the semi-group property

$$T_{t_\nu^m - t_\nu^k} = \begin{cases} (T_{d_\nu})^{\tau_\nu^m - \tau_\nu^k} & \text{if} \quad \tau_\nu^m \geq \tau_\nu^k, \\ (T_{d_\nu}^*)^{\tau_\nu^k} & \text{if} \quad \tau_\nu^m < \tau_\nu^k. \end{cases}$$

Hence

$$\Delta_\nu = \sum_{m,k=1}^{n} \left(T_{t_\nu^m - t_\nu^k} f(t^m), f(t^k)\right) = \sum_{m,k=1}^{n} \left(T_\nu^{(\tau_\nu^m - \tau_\nu^k)} f(t^m), f(t^k)\right),$$

where $T_\nu = T_{d_\nu}$. Since T_ν is a contraction, it generates a strongly positive sequence $\{T_\nu^{(s)}\}$. It follows that $\Delta_\nu \geq 0$, as was to be proved.

Using lemma 4.1 and the dilation theorem we get the following theorem (see [40], [41]):

THEOREM 4.5. *Let $T(t)$ be a weakly continuous semi-group $(t \geq 0)$ of contractions in a Hilbert space H. Then $T(t)$ has an essentially unique minimal unitary dilation which is strongly continuous.*

Notice that this theorem is valid for real, complex and quaternionic fields of scalars.

In the case of a complex space the spectral-harmonic measure F of T_t is determined by the formula

$$T_t = \int_{-\infty}^{+\infty} e^{it\lambda} dF(\lambda)$$

and the Neumark dilation of F is the spectral measure corresponding to the unitary dilation of $T(t)$.

5. Decomposition theorems

We present here several theorems concerning some decompositions of the initial space and the dilation space. Spectral consequences of the theorems of the present section will be discussed in the next sections. We begin with the following lemma:

LEMMA 5.1. *The minimal unitary dilation of the sum* $\bigoplus_\alpha T_\alpha$ *of contractions* T_α *equals the sum* $\bigoplus_\alpha U_\alpha$, *where* U_α *is a minimal unitary dilation of* T_α. *The dilation space of* $\bigoplus_\alpha T_\alpha$ *equals* $\bigoplus_\alpha K_\alpha$, *where* K_α *is the dilation space of* T_α.

Proof. Let T_α act on the space H_α ($\alpha \in \Omega$) and denote by U_α the minimal unitary dilation of T_α with K_α as a dilation space. Write now $K = \bigoplus_\alpha K_\alpha$, $H = \bigoplus_\alpha H_\alpha$ and define $U = \bigoplus_\alpha U_\alpha$. Certainly U is a unitary dilation of $\bigoplus_\alpha T_\alpha$. We will show that U is minimal. Denote by P_α the projection of K onto K_α. Suppose that $g \in K$ and assume that $(U^n f, g) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$ and $f \in H$. If $f \in H_\alpha \subset K_\alpha$, then $U^n f = U_\alpha^n f = P_\alpha U_\alpha^n f$ and consequently $(U_\alpha^n f, P_\alpha g) = 0$ for such f. Since U_α is minimal and f is an arbitrary vector of H_α, we conclude that $P_\alpha g = 0$. It follows that $g = 0$, which completes the proof.

We now introduce the following definition: the family T_α ($\alpha \in \Omega$) of contractions acting in the space H is called *completely non-unitary* if $\inf_\alpha |T_\alpha f| < |f|$ for every $f \in H$, $f \neq 0$.

Now we will prove the theorem on the canonical decomposition (see [20], [46]):

THEOREM 5.1. *Let* G *be a group and suppose that* T_α *is an operator-valued function defined for* $\alpha \in G$. *Assume that* T_α *satisfies condition* (P) *of the dilation theorem. Then there exists a unique decomposition* $H = H_1 \oplus H_2$ *of the space* H *in which act* T_α, *such that the following conditions hold true*:

(5.1) H_1 *and* H_2 *reduce all of* T_α.

(5.2) *The part* T'_α *of* T_α *in* H_1 *is a unitary representation of the group* G.

(5.3) *The part* T''_α *of* T_α *in* H_2 *is a completely non-unitary family.*

Proof. Denote by U_α the minimal unitary dilation of T_α. We write

$$H_1 = \bigcap_{\alpha \in G} U_\alpha H .$$

If $f \in H_1$, then $U_\alpha f \in H$ for every $\alpha \in G$ and consequently $T_\alpha f = U_\alpha f$ for $\alpha \in G$. Since H_1 reduces all of U_α and T_α coincide on H_1 with U_α, the space H_1 reduces all of T_α. Since U_α is a unitary representation of G, we conclude that the part T'_α of T_α in H_1 is a unitary representation of G. Let us define $H_2 = H \ominus H_1$. H_2 reduces all of T_α. Notice now that if $|T_\alpha h| = |h|$ for all $\alpha \in G$, then $U_\alpha h = T_\alpha h$, which implies $h \in H_1$. Consequently, the part T''_α of T_α in H_2 is completely non-unitary.

Let H' and H'' reduce all of T_α and suppose that $H = H' \oplus H''$. Assume that the part of T_α in H' is a unitary representation of G and

the part of T_α in H'' is completely non-unitary. We have to show that $H' = H_1$ and $H'' = H_2$. Notice first that $H' \subset H_1$. Let $h \in H_1 \ominus H'$. Then $h \in H''$ and $h \in H_1$. Hence $|T_\alpha h| = |h|$ for all $\alpha \in G$. If $h \neq 0$ then, since $h \in H''$, $\inf |T_\alpha h| < |h|$. It follows that h must be the zero vector. This completes the proof of uniqueness. (For a generalization of Theorem 5.1 see [54].)

The decomposition $T_\alpha = T'_\alpha \oplus T''_\alpha$ of theorem 5.1 is called the *canonical decomposition* of T_α. T'_α is called the *unitary part* and T''_α the *completely non-unitary part* of T_α. Now let T be a contraction and U its minimal unitary dilation. Consider the canonical decomposition of the sequence $\{T^{(n)}\}$. Let T_1 be the value of the unitary part for $n = 1$ and T_2 the value of the completely non-unitary part for $n = 1$. It is easy to see that the canonical decomposition of $T^{(n)}$ is given by $T^{(n)} = T_1^{(n)} \oplus T_2^{(n)}$. Let $H = H_1 \oplus H_2$ be the corresponding decomposition of the space H. If a certain element $T^{(n)}$ $(n \neq 0)$ is reduced by a subspace $\bar{H} \subset H$ and the restrictions of $T^{(n)}$ and $T^{(-n)}$ to $\bar{H}$ are isometries, then $\bar{H} \subset H_1$. Indeed, let $h \in \bar{H}$ and consider $T^{(n)}h$, $T^{(-n)}h$. Both these elements are in $\bar{H}$ and $|T^{(n)}h| = |h| = |T^{(-n)}h|$. Even more is true, namely $|T^{(kn)}h| = |h| = |T^{(-kn)}h|$ for $k = 1, 2, 3, \ldots$ This is a consequence of the fact that both operators are reduced by $\bar{H}$ and behave as isometries. It follows that $|T^{(s)}h| = |h|$ for every integer s. Hence $h \in H_1$, q. e. d. We conclude that no subspace of H_2 can reduce any $T^{(n)}$ $(n \neq 0)$ in such a way that the restriction of $T^{(n)}$ to this subspace is unitary.

Now we introduce the following definition: the contraction T is called *completely non-unitary* if the only vector f which satisfies $|T^{(s)}f| = |f|$ (all s) is a zero vector.

Since, by previous notation, $T = T_1 \oplus T_2$, we see that T is completely non-unitary if and only if $T_1 = 0$. The equivalent condition is

$$\bigcap_{-\infty}^{+\infty} U^n H = \{0\}, \tag{5.4}$$

where U stands for the minimal unitary dilation of T. The decomposition $T = T_1 \oplus T_2$ is called the *canonical decomposition* of T. It follows from theorem 5.1 that $T_1^{(n)}$ is a unitary representation of the additive group of integers and T_2 is a completely non-unitary contraction. T_1 is called the *unitary part* of T, T_2 the *completely non-unitary part* of T. It is a simple matter to check that $\bigcap_{-\infty}^{+\infty} U^n H$ is the largest space in which T behaves as a unitary operator. The canonical decomposition of a contraction is unique.

Certainly, every contraction T which satisfies $|Th| < |h|$ for all $h \in H$, $h \neq 0$, is completely non-unitary. If the spectral radius of T is less than 1, then T is completely non-unitary. Consequently, if $|T| < 1$, then T is completely non-unitary. But there is no difficulty in finding a completely

non-unitary isometry. Indeed, let T be the unilateral shift in the space l_2^+ of square summable sequences $\{\xi_n\}_1^\infty$, i. e. let T map $h = \{\xi_1, \xi_2, \ldots\}$ onto $Th = \{0, \xi_1, \xi_2, \ldots\}$. Then $|Th| = |h|$ and $|T| = 1$ but, for every h, $|T^{*n}h| < |h|$ $(h \neq 0)$ for sufficiently large n. Hence T is completely non-unitary.

The final remark is the following one: if $T = T_1 \oplus T_2$ is the canonical decomposition of the contraction T, then, by lemma 5.1, the unitary dilation U of T is $U = T_1 \oplus U_2$, where U_2 is the minimal unitary dilation of T_2.

Let $T(t)$ be a strongly continuous one-parameter semigroup of contractions. Denote by $U(t)$ the minimal unitary dilation of $T(t)$. Then, by theorem 5.1,

$$T(t) = T_1(t) \oplus T_2(t),$$

where $T_1(t)$ and $T_2(t)$ are parts of $T(t)$ and $T_1(t)$ act in

$$H_1 = \bigcap_{-\infty<t<+\infty} U(t)H$$

and $T_2(t)$ act in $H_2 = H \ominus H_1$. H_i reduces all of $T_i(t)$ $(t \geqslant 0)$ and $T_1(t)$ are unitary operators. $T_2(t)$ is embedded in the family

$$T_t = \begin{cases} T_2(t) & \text{if} \quad t \geqslant 0, \\ T_2(-t)^* & \text{if} \quad t \leqslant 0, \end{cases}$$

and T_t is a completely non-unitary family when considered on $(-\infty, +\infty)$, $T_1(t)$ is called the *unitary part* of $T(t)$ and $T_2(t)$ the *completely non-unitary part* of $T(t)$. The decomposition $T(t) = T_1(t) \oplus T_2(t)$ is unique and is called the *canonical decomposition* of $T(t)$.

Let V be a linear bounded operator in the Hilbert space H. We say that the closed subspace $\hat{H} \subset H$ is *wandering* (see [12]) for V if $V^n\hat{H} \perp \hat{H}$ for $n = 1, 2, 3, \ldots$ If V is an isometry, then $\hat{H}$ is wandering for V if and only if $V^n\hat{H} \perp V^m\hat{H}$ for $m \neq n$, $m, n \geqslant 0$. If V is unitary, then $\hat{H}$ is wandering for V if and only if $V^n\hat{H} \perp \hat{H}$ for all $n \neq 0$ or $V^n\hat{H} \perp V^m\hat{H}$ for all $m \neq n$.

Suppose that T is a contraction in H and U is its minimal unitary dilation. Let S be a subspace of H. We introduce the following notation:

$$L_+^0(S) = (U - T)S, \qquad L_-^0(S) = (U^* - T^*)S, \qquad L_\pm(S) = \overline{L_\pm^0(S)}.$$

In case $S = H$ we write simply $L_\pm(S) = L_\pm$.

We have the following theorem (see [13], [47]):

THEOREM 5.2. *Let T be a contraction in H and let U be its unitary dilation. Then the spaces L_+ and L_- are wandering for U, i. e.*

$$U^n L_+ \perp L_+, \qquad U^n L_- \perp L_- \qquad (n \neq 0). \tag{5.5}$$

Moreover,

$$U^n L_+ \perp U^{*m} L \quad \text{for} \quad m, n \geqslant 0 \tag{5.6}$$

and

$$U^n L_+ \perp H, \quad U^n L_- \perp H \quad \text{for} \quad n \geqslant 0. \tag{5.7}$$

Proof. Since $(U-T)H$ is dense in L_+, it suffices to prove that $U^n(U-T)H \perp (U-T)H$ for $n \geqslant 1$. Let $f, g \in H$. We have

$$\big(U^n(U-T)f, (U-T)g\big) = (U^n f, g) - (U^{n+1}f, Tg) - (U^{n-1}Tf, g) + (U^n Tf, Tg) = (T^n f, g) - (T^{n+1}f, Tg) - (T^n f, g) + (T^{n+1}f, Tg) = 0.$$

The second relation in (5.5) is proved by replacing U and T by adjoints in the first relation.

The equalities

$$\big(U^n(U-T)f, U^{*m}(U-T)g\big) = (U^{n+m+2}f, g) - (U^{n+m+1}f, Tg) - (U^{n+m+1}Tf, g) + (U^{n+m}Tf, Tg),$$

$$\big(U^n(U-T)f, g\big) = (U^{n+1}f, g) - (U^n Tf, g)$$

prove (5.6) and (5.7) respectively.

The wandering spaces L_+ and L_- are isomorphic copies of some subspaces of the initial space H. Let us write $D_T = (I-T^*T)^{1/2}$, $D_{T^*} = (I-TT^*)^{1/2}$ and $L_T = \overline{(I-T^*T)H}$, $L_{T^*} = \overline{(I-TT^*)H}$. We will prove the following lemma (see [47]):

Lemma 5.2. *Let T be a contraction and U its simple unitary dilation. Then*

$$\delta_+ = \dim L_T = \dim L_+ = \dim \overline{(I-T^*T)H} = \dim \overline{R[D_T]},$$

$$\delta_- = \dim L_{T^*} = \dim L_- = \dim \overline{(I-TT^*)H} = \dim \overline{R[D_{T^*}]}.$$

Proof. Let $f \in H$. Then

$$|(U-T)f|^2 = (f,f) - (Tf, Tf) = |D_T f|^2. \tag{5.8}$$

Since D_T is positive, the closure of $R[D_T^2]$ equals to $\overline{R[D_T]}$. Hence $\overline{D_T H} = \overline{D_T^2 H}$, which together with the unitary correspondence established by (5.8) proves the first equality of the assertion. The second equality may be proved by using similar arguments.

Remark. Suppose that $h \in H \ominus \overline{R[D_T]}$. Then $D_T^2 h = 0$, which implies $|Th| = |h|$. Conversely, if $|Th| = |h|$, then $\big((I-T^*T)h, h\big) = 0$ and consequently $D_T^2 h = 0$. It follows that the space of zeros of D_T^2 is exactly the space $H \cap U^{-1}H$, U being an arbitrary simple unitary dilation of T. By similar arguments $D_{T^*}^2 h = 0$ if and only if $h \in H \cap UH$. Since

$H \cap UH = \{0\}$ if and only if $U^{-1}(H \cap UH) = U^{-1}H \cap H = \{0\}$, we conclude that $|Th| < |h|$ for all $h \neq 0$ if and only if $|T^*h| < |h|$ for all $h \neq 0$.

The basic decomposition theorem for the dilation space in case of a single contraction is the following one (see [13], [47]):

THEOREM 5.3. *Let U be the unitary minimal dilation of the contraction T and let $K \supset H$ be the dilation space. Then*

$$(5.9)\quad K = \ldots \oplus U^n L_+ \oplus \quad \oplus UL_+ \oplus L_+ \oplus H \oplus L_- \oplus U^{-1}L_- \oplus \ldots \oplus U^{-n}L_- \ldots$$

Proof. Denote by K_0 the right-hand side of (5.9). Then

$$UK_0 = \quad \oplus U^2 L_+ \oplus UL_+ \oplus UH \oplus UL_- \oplus L_- \oplus U^{-1}L_- \oplus \ldots$$

We will show that

$$(5.10)\qquad UH \oplus UL_- = L_+ \oplus H.$$

Let $f = Uh_1 + U(U^* - T^*)h_2$ $(h_1, h_2 \in H)$. Then for $h' = h_1 - T^*h_2$, $h'' = Th_1 + (I - TT^*)h_2$ we get $f = (U-T)h' + h''$. Conversely, given $h', h'' \in H$ we take

$$h_1 = (I - TT^*)h' + T^*h'', \qquad h_2 = h'' - Th'$$

and just obtain $f = Uh_1 + U(U^* - T^*)h_2$. Thus, both sides of (5.10) have a common dense subset. This completes the proof of (5.10). We conclude therefore that $UK_0 = K_0$. But $H \subset K_0$ and U is minimal. Hence $K_0 = K$, q. e. d.

Remark. Formula (5.9) may be obtained as a by-product of the direct construction of the unitary dilation of T (see [13]).

If T is a projection operator, then $L_+ = L_- = R[T]^\perp$. One can prove that if two projections have unitarily equivalent unitary dilations, then the dilated projections are unitarily equivalent (see [36]).

Let U be the minimal unitary dilation of the contraction T. We write

$$R = \bigcap_{-\infty}^{+\infty} U^n H, \qquad S_+ = \bigoplus_{-\infty}^{+\infty} U^n L_+, \qquad S_- = \bigoplus_{-\infty}^{+\infty} U^n L_-.$$

The following theorem (see [13]) characterizes the canonical decomposition of a single contraction:

THEOREM 5.4. *Let U be the minimal unitary dilation of the contraction T with the dilation space $K \supset H$. Then*

$$R = K \ominus [S_+ \vee S_-].$$

Proof. Let $f \in R$. Then $U^k f = T^{(k)} f$ for all k. It follows from (5.7) of theorem 5.2 that

$$U^n L_+ \perp H \ni U^k f \quad \text{for} \quad n \geqslant 0 \text{ and all } k, \tag{5.11}$$

$$U^{-n} L_- \perp H \ni U^k f \quad \text{for} \quad n \geqslant 0 \text{ and all } k. \tag{5.12}$$

It follows that $f \perp S_+$ and $f \perp S_-$.

Suppose that $f \in K \ominus [S_+ \vee S_-]$. Then, by formula (5.9) of theorem 5.3, $f \in H$. Notice now that $S_+ \vee S_-$ reduces U. Hence $K \ominus [S_+ \vee S_-]$ reduces U. It follows that $U^n f \in K \ominus [S_+ \vee S_-]$ for all n. Applying our previous arguments to $U^n f$ in place of f we obtain $U^n f \in H$ for all n, which completes the proof.

The canonical decomposition of a single contraction and theorem 5.4 imply that the contraction is completely non-unitary if and only if the dilation space is of the form $K = S_+ \vee S_-$.

Let S be a subspace of the initial space H and write

$$M_+(S) = \bigvee_0^\infty U^n S, \quad M_-(S) = \bigvee_0^\infty U^{-n} S, \quad R_+(S) = \bigcap_0^\infty U^n M_+(S),$$

$$R_-(S) = \bigcap_0^\infty U^{-n} M_-(S).$$

If S is a one-dimensional space spanned by the element $f \neq 0$, we write $M_+(f)$, $M_-(f)$, $R_+(f)$ and $R_-(f)$ in place of $M_+(S)$, $M_-(S)$, $R_+(S)$ and $R_-(S)$ respectively. If $S = H$, the symbol S is omitted in the above-mentioned notation.

Using theorem 5.3 we will prove (see [25], [51])

THEOREM 5.5. *Let U be the minimal unitary dilation of the contraction T. Let K be the dilation space, and H the initial one. Then*

$$M_+ = \bigoplus_{+\infty}^{0} U^n L_+ \oplus H, \quad M_- = H \oplus \left(\bigoplus_0^\infty U^n L_-\right), \tag{5.13}$$

$$K = R_+ \oplus \left(\bigoplus_{-\infty}^{+\infty} U^{-n} L_-\right), \quad K = R_- \oplus \left(\bigoplus_{-\infty}^{+\infty} U^n L_+\right), \tag{5.14}$$

$$UL_- = M_+ \ominus U M_+, \quad U^{-1} L_+ = M_- \ominus U^{-1} M_- \tag{5.15}$$

Proof. Certainly $L_+ \subset M_+$. Write $Q = \bigoplus_{+\infty}^{0} U^n L_+ \oplus H$. Hence $Q \subset M_+$. Using (5.7) of theorem 5.2 we conclude that $U^{-k} L_+ \perp M_+$ for $k \geqslant 0$. Consequently, by formula (5.9) of theorem 5.3 we get $M_+ \subset Q$, q. e. d.

Using again formula (5.7) we get $U^{-n} L_- \perp H$ for $n \geqslant 0$ and consequently $S_- \perp R_+$. If $h \perp S_-$, then $U^k h \perp S_-$ for arbitrary k and by (5.13) and the decomposition formula (5.9) we obtain $U^k h \in M_+$ (all k), which proves that $h \in R_+$.

Suppose that $h \in UL_-$. Then $h \in M_+$. On the other hand, $UL_- \perp UH$ and, by (5.9) of theorem 5.3, $UL_- \perp U^k L_+$ for $k \geqslant 1$. Using (5.13) we infer that $h \perp UM_+ = UH \oplus (\bigoplus_1^\infty U^n L_+)$. Consequently $h \in M_+ \ominus UM_+$. Suppose now that $h \in M_+ \ominus UM_+$. Then $h \perp UH$ and $h \perp U^n L_+$ for $n \geqslant 1$. Since $h \in M_+$ we get by (5.13) that $h \in L_+ \oplus H$. Using relation (5.10) we conclude that $h \in UL_-$, q. e. d.

Symmetric relations are proved in a similar way.

COROLLARY. *The equalities of theorem* 5.5 *imply that*

$$H \cap R_+ = \bigcap_0^\infty U^n H, \quad H \cap R_- = \bigcap_0^\infty U^{-n} H, \tag{5.16}$$

$$H = M_+ \cap M_-. \tag{5.17}$$

One can prove that

$$(I-P)R_+ \perp M_-, \quad (I-P)R_- \perp M_+ \tag{5.18}$$

(P — the projection of K onto H),

$$R_+ \perp (\overline{U^{*s} - T^{*s}})H, \quad R_- \perp (U^s - T^s)H \tag{5.19}$$

for $s \geqslant 1$.

It follows from (5.16) that

$$R_+ \cap H \cap R_- = \bigcap_{-\infty}^{+\infty} U^n H.$$

We will show that something more is true, namely that every $h \in R_- \cap R_-$ belongs necessarily to H. Actually, the following theorem (see [25]) holds true:

THEOREM 5.6. *Let U be the minimal unitary dilation of the contraction T which acts in the space H. Then* $R_+ \cap R_- = \bigcap_{-\infty}^{+\infty} U^n H$.

Proof. Let $f \in R_+ \cap R$ Then by formula (5.14) we get $f \perp S_+$ and $f \perp S_-$ It follows by theorem 5.4 that $f \in U^n H$ for all n. Suppose now that $f \in \bigcap_{-\infty}^{+\infty} U^n H$. Then, by theorem 5.4 and formula (5.14), $f \in R_+$ and $f \in R_-$. This completes the proof.

The decomposition formulae take on a simple form in case the contraction is an isometry. For, if T is an isometric operator, then $T^*T = I$, which implies that $L_T = \{0\}$ and, by lemma 5.2, $L_+ = \{0\}$. Since $f = T^n f = U^n f$ $(n > 0)$, if T is isometric, the space M_+ reduces simply to H, i. e. $M_+ = H$. By formula (5.9) we get $K = H \oplus (\bigoplus_0^\infty U^{-n} L_-)$ and by formula (5.14)

$$K = R_+ \oplus (\bigoplus_{-\infty}^{+\infty} U^{-n} L_-).$$

It follows that

$$H = R_+ \oplus (\bigoplus_1^\infty U^n L \ ).$$

Since $T \subset U$, we get

$$H = R_+ \oplus (\bigoplus_0^\infty T^n N),$$

where $N = H \ominus TH$. Summing up we get the following theorem (see [12]):

THEOREM 5.7. *Let T be an isometric operator in the Hilbert space H. Then $N = H \ominus TH$ is a wandering subspace for T and*

$$H = \bigcap_0^\infty T^n H \oplus (\bigoplus_0^\infty T^n N). \tag{5.20}$$

Remark. The part of T in $\bigcap_0^\infty T^n H$ is unitary and the part of T in the space $\bigoplus_0^\infty T^n N$ is simply a unilateral shift. It is a simple matter to check that $f \in \bigcap_0^\infty T^n N$ if and only if $T^{*n} f \to 0$. It follows that decomposition (5.20) is the canonical decomposition of H with respect to the isometry T. The restriction of T to $\bigcap_0^\infty T^n H$ is unitary and the restriction of T to $\bigoplus_0^\infty T^n N$ is completely non-unitary.

We can complete the collection of decomposition formulae by (see [51])

$$M_+ = R_+ \oplus (\bigoplus_1^\infty U^n L_-); \tag{5.21}$$

U is the unitary minimal dilation of a certain contraction.

The proof of (5.21) reduces to two observations. The first observation is that the restriction of U to M_+ is an isometric operator. The second observation is that the corresponding space N of this isometric operator is simply $M_+ \ominus UM_+ = UL$ Now, applying (5.20), we get the desired result.

6. Bilateral shifts

Let S be a wandering subspace of the unitary operator U which acts in the Hilbert space K. We say that S is a *complete wandering subspace* for U if

$$K = \bigoplus_{-\infty}^{+\infty} U^n S,$$

If U has a complete wandering subspace, then it is called a *bilateral shift*. The corresponding complete wandering subspace is called a *shift space*. Since for every wandering subspace S of a unitary operator U

$$\bigvee_{-\infty}^{+\infty} U^n S = \bigoplus_{-\infty}^{+\infty} U^n S,$$

complete wandering subspaces are those wandering subspaces which satisfy $K = \bigvee_{-\infty}^{+\infty} U^n S$.

We will prove the following lemma (see [47]):

LEMMA 6.1. *Let S_1 and S_2 be wandering subspaces of the unitary operator U. Suppose that S_2 is a complete wandering subspace for U. Then*

$$\dim S_1 \leqslant \dim S_2. \tag{6.1}$$

Proof. It follows from the assumptions that

$$\bigoplus_{-\infty}^{+\infty} U^n S_1 \subset \bigoplus_{-\infty}^{+\infty} U^n S_2 = K,$$

hence

$$\aleph_0 \dim S_1 \leqslant \aleph_0 \dim S_2. \tag{6.2}$$

Relation (6.1) easily follows from (6.2) if $\dim S_2 \geqslant \aleph_0$.

Suppose that $\dim S_2 < \aleph_0$. After Halperin (see [47], p. 108) we proceed in this case as follows. Denote by $f_\nu^{(i)}$ the orthonormal basis of S_i $(i = 1, 2)$. Since S_2 is a complete wandering subspace for U, the system $U^\mu f_k^{(2)}$ $(\mu = 0, \pm 1, \pm, 2, \quad \text{and } k = 1, \ldots, a_2)$, where

$$a_i = \dim S_i = \sum_k |f_k^{(i)}|^2,$$

is an orthonormal basis in K. We get therefore

$$|f_\nu^{(1)}|^2 = \sum_{\mu,k} |(f_\nu^{(1)}, U^\mu f_k^{(2)})|^2$$

and consequently

$$a_1 = \sum_{\nu,\mu,k} |(f_\nu^{(1)}, U^\mu f_k^{(2)})|^2 = \sum_{\nu,\mu,k} |(U^{*\mu} f_\nu^{(1)}, f_k^{(2)})|^2.$$

Now, by the Bessel inequality,

$$\sum_{\mu,\nu,k} |(U^{*\mu} f_\nu^{(1)}, f_k^{(2)})|^2 \leqslant \sum_k |f_k^{(2)}|^2 = a_2,$$

which completes the proof.

It follows from lemma 6.1 that two bilateral shifts U_1 and U_2 are unitarily equivalent if and only if their wandering complete spaces have

the same dimension. Indeed, if S_1 and S_2 are complete wandering spaces for U_1 and U_2 respectively, and $\dim S_1 = \dim S_2$, then U_1 and U_2 are equivalent. On the other hand, if U_1 and U_2 are equivalent we may simply assume that $U_1 = U_2$. Then, by the above lemma $\dim S_{1(2)} = \dim S_{2(1)}$ which proves that $\dim S_1 = \dim S_2$, as was to be proved.

The following definition is now in order: the *multiplicity of the bilateral shift* is the dimension of its (arbitrary!) complete wandering subspace.

It is thus seen that the only property which can distinguish two bilateral shifts are their multiplicities.

This section deals with unitary dilations which are bilateral shifts. We are interested here which properties of contractions ensure that their unitary dilations are bilateral shifts. The first result in this field was obtained by Schreiber in [36]. It was proved in [36] that if $|T| < 1$, then the unitary dilation (minimal) of T is a bilateral shift of multiplicity equal to the dimension of the initial space.

Theorems of the present section use geometrical methods and the construction of the shift space for a dilation if it is a bilateral shift, is pretty effective.

Let H be an initial space and T a contraction acting in H. Let U stand for the minimal unitary dilation of T and denote by K the dilation space. To begin with we will prove the following lemma (see [13]):

LEMMA 6.2. *Let S be a closed subspace of H and suppose that*

$$TS \perp T(H \ominus S), \tag{6.3}$$

$$T(H \ominus S) \subset H \ominus S. \tag{6.4}$$

Then

$$U^n L_+(S) \perp H \ominus S \quad \text{for all } n.$$

Proof. Since $H \ominus S \subset H$ and $L_+(S)$ is a subspace of the wandering subspace $L_+ \perp H$, we have to prove only that $L_+(S) \perp U^n(H \ominus S)$ for $n \geqslant 1$. Using again the property that L_+ is wandering for U, we infer that

$$L_+^0(S) \subset L_+ \perp U^n L_+ \supset U^n L_+^0(H \ominus S) \quad \text{for} \quad n \quad 1$$

Hence

$$L_+^0(S) \perp U^n L_+^0(H \ominus S) \quad \text{for} \quad n \quad 1.$$

On the other hand, if $f \in L_+^0(S)$ and $g \in L_+^0(H \ominus S)$, then $f = (U - T)f_1$, $f_1 \in S$, $g = (U - T)g_1$, $g_1 \in H \ominus S$, and, by (6.3), $(f, g) \quad ((U - T)f_1, (U - T)g_1) = 0$. It follows that $L_+^0(S) \perp L_+^0(H \ominus S)$ and consequently

$$L_+^0(S) \perp U^n L_+^0(H \ominus S) \quad \text{for} \quad n \geqslant 0. \tag{6.5}$$

Suppose now that $n \geqslant 1$ and $h \in H \ominus S$. Then

$$U^n h = T^n h + \sum_{k=0}^{n-1} U^{n-k-1}(U-T)T^k h.$$

Because of (6.4) we have $T^k h \in H \ominus S$ for $k \geqslant 0$. Hence, by (6.5),

$$\sum_{k=0}^{n-1} U^{n-k-1}(U-T)T^k h \in \bigoplus_{k=0}^{n-1} U^{n-k-1} L_+^0(H \ominus S) \perp L_+^0(S).$$

Obviously $T^n h \mid L_+^0$ (see theorem 5.3). We conclude that $U^n h \perp L_+^0(S)$, which proves the assertion of the lemma.

Next we need the following (see [13])

LEMMA 6.3. *Let T be a contraction in H and U its minimal unitary dilation. Suppose that*

$$H = S_1 \oplus M \oplus S_2 \quad \text{and} \quad TS_1 \subset S_1 \oplus M,\ T(M \oplus S_2) \subset S_2, \tag{6.6}$$

where S_1, M and S_2 are closed subspaces of H.

Under these assumptions the spaces

$$U^\nu L_+(S_1), \quad U^\mu M, \quad U^\varrho L_-(S_2) \quad (\nu, \mu, \varrho = 0, \pm 1, \pm 2, \ldots)$$

are mutually orthogonal.

Proof. Since the subspaces $L_+(S_1)$, $L_-(S_2)$ are wandering for U, we need only to show that

$$U^\nu L_+(S_1) \perp M \oplus S_2 \quad \text{for all } \nu, \tag{6.7}$$

$$U^\nu L_-(S_2) \perp S_1 \oplus M \quad \text{for all } \nu, \tag{6.8}$$

$$U^\nu L_+(S_1) \perp L_-(S_2) \quad \text{for all } \nu, \tag{6.9}$$

$$U^\nu M \perp M \quad \text{for} \quad \nu \neq 0. \tag{6.10}$$

We write $S = S_1$ in lemma 6.2. Then

$$T(H \ominus S) = T(M \oplus S_2) \subset S_2 \subset H \ominus S$$

and

$$TS \subset S_1 \oplus M \perp S_2 \supset T(M \oplus S_2) = T(H \ominus S).$$

The assumptions of the lemma being satisfied, we conclude that

$$U^\nu L_+(S_1) \perp M \oplus S_2 \quad (\text{all } \nu), \tag{6.11}$$

which proves (6.7).

We now write $S \quad S_2$ in lemma 6.2 and in place of T we take T^*. Then, by (6.6),

$$T^* S = T^* S_2 \subset M \oplus S_2. \tag{6.12}$$

Also $T^*(S_1 \oplus M) \subset S_1$. Consequently

$$T^*S \subset M \oplus S_2 \perp S_1 \supset T^*(S_1 \oplus M) = T^*(H \ominus S),$$

which implies $T^*(H \ominus S) \subset S_1 \subset S_1 \oplus M = H \ominus S$.

We are now in a position to use lemma 6.2 and infer therefore $U^n L_-(S_2) \perp S_1 \oplus M$. This completes the proof of (6.8).

Next

$$UL_-^0(S_2) = U(U^* - T^*)S_2 \subset (I \quad UT^*)S_2.$$

Since $T^*S_2 \subset M \oplus S_2$, every element $h \in UL_-^0(S_2)$ is of the form $h = h_1 + h_2$, where $h_1 \in S_2$ and $h_2 \in U(M \oplus S_2)$. Hence $U^n h = U^n h_1 + U^n h_2$. But, by (6.11), $U^n h_1 \perp L_+(S_1)$ and $U^n h_2 \perp L_+(S_1)$ for all n, for both h_1 and $U^{-1}h_2$ belong to $M \oplus S_2$. It follows that $U^\nu L_-(S_2) \perp L_+(S_1)$ for all ν and the proof of (6.9) is complete.

Now remark that (6.6) implies that $TS_2 \subset S_2$, $TM \subset S_2$. Consequently $T^n M \subset S_2 \perp M$ for $n \geqslant 1$. Hence $(T^n f, g) = (U^n f, g) = 0$ for $f, g \in M$ and $n \geqslant 1$. This proves (6.10).

We are now able to prove the following theorem (see [13]):

THEOREM 6.1. *Let U be the minimal unitary dilation of the contraction T. Suppose that the initial space H may be decomposed into the form $H = S_1 \oplus M \oplus S_2$ and the following conditions hold true:*

(6.13) $TS_1 \subset S_1 \oplus M$, $T(M \oplus S_2) \subset S_2$.

(6.14) *P_1 is the projection of H onto S_1 and*

$$(TP_1)^n f \to 0 \quad \text{for every} \quad f \in H.$$

(6.15) *P_2 is the projection of H onto S_2 and*

$$(T^*P_2)^n f \to 0 \quad \text{for every} \quad f \in H.$$

Under these assumptions U is a bilateral shift with $R = L_+(S_1) \oplus M \oplus L_-(S_2)$ as a shift space.

Proof. It follows from lemma 6.3 that R is a wandering subspace for U. Denote by K the dilation space and write

$$Q = \bigoplus_{-\infty}^{+\infty} U^n R.$$

Since $K = \bigvee_{-\infty}^{+\infty} U^n H$ and $Q \subset K$, it is sufficient to prove that $H \subset Q$. Notice that $M \subset Q$. We infer that we need only to show that $S_1 \subset Q$, $S_2 \subset Q$.

Suppose that $f \in S_1$. Then $Uf = Tf + g$, where $g \quad (U - T)f \in L_+(S_1)$. Notice that $Tf \in S_1 \oplus M$. Hence $P_2 Tf = 0$. Denote by P_M the projection of H onto M. We then get

$$Uf = P_1 Tf + P_M Tf + g = P_1 Tf + h$$

where $h \in M \oplus L_+(S_1)$. We just proved that if $f \in S_1$, then $(U - P_1 T)f \in L_+(S_1) \oplus M$. Since $(P_1 T)^k f \in S_1$ for $k \geqslant 0$ provided $f \in S_1$ we get

$$g_m = \sum_{k=0}^{n-1} U^{k-1}(U - P_1 T)(P_1 T)^k f \in \bigoplus_{1}^{\infty} U^{-n}\big(L_+(S_1) \oplus M\big).$$

But $f - g_m = U^{-m}(P_1 T)^m f$. Using (6.14) we infer that

$$|f - g_m| \leqslant |(P_1 T)^m f| \leqslant |(TP_1)^m f| \to 0.$$

It follows that

$$S_1 \subset \bigoplus_{1}^{\infty} U^{n}\big(L_+(S_1) \oplus M\big).$$

Taking T^*, U^* and S_2 in place of T, U and S_1 respectively, one proves that

$$S_2 \subset \bigoplus_{1}^{\infty} U^{n}\big(M \oplus L_+(S_2)\big),$$

which, by the previous remarks, completes the proof.

A few comments are now in order. Theorem 6.1 generalizes some results obtained by several methods.

If we take $S_1 = H$ ($S_2 = H$) and $M \oplus S_2 = \{0\}$ ($S_1 \oplus M = \{0\}$) and assume that $T^n \to 0$ ($T^{*n} \to 0$) strongly, then theorem 6.1 applies. As a consequence we get the following proposition, proved by de Bruijn [4]:

If $T^n \to 0$, then the minimal unitary dilation of T is a bilateral shift of multiplicity equal to $\dim L_+$.

Notice that L_+ is the shift space in this case. The theorem of de Bruijn generalizes the following one, proved by Schreiber [36] for $\dim H \leqslant \aleph_0$ and later generalized by Sz.-Nagy [43] to arbitrary dimensions:

If $|T| < 1$, then the unitary minimal dilation of T is a bilateral shift of multiplicity equal to $\dim H$.

In this case L_+ and L_- are complete wandering subspaces. If $|T| < 1$, the operator $(I - T^*T)$ establishes a one-to-one continuous map of H onto itself. Lemma 5.2 then implies $\dim L_+ = \dim H$. This explaines why in this case the minimal unitary dilation of T has multiplicity equal to $\dim H$. It follows also that two strict contractions which act in spaces of equal dimensions have unitarily equivalent minimal unitary dilations.

Suppose now that

$$T^n \to 0, \quad T^{*n} \to 0 \text{ strongly.} \tag{6.16}$$

Then L_+ and L_- are both complete wandering subspaces for the minimal unitary dilation U of T. It follows by lemma 6.1 that (6.16) implies $\dim L_+ = \dim L_-$. Consequently, by lemma 5.2, $\dim \overline{D_T[H]} = \dim \overline{D_{T^*}[H]}$ in this case.

If H is finite-dimensional and T is completely non-unitary, then (6.16) holds true. This is a consequence of the fact that in this case the spectra of T and T^* are included entirely in the open unit disc. The de Bruijn theorem implies that U is then a bilateral shift of multiplicity equal to $\dim H$, which is a finite number.

A more advanced characterization of the structure of unitary dilations of completely non-unitary contractions was given in [47]. In particular, it was proved in [47] that if

$$\delta = \max\left(\dim \overline{(I-TT^*)H}, \dim (I-T^*T)H\right)$$

is infinite, then U, the minimal unitary dilation of T, is equivalent to a bilateral shift of multiplicity equal to δ. The proof of this statement uses the methods of spectral multiplicity.

Also the case δ finite and $\dim H$ infinite can be characterized (see [47]). Notice that the minimal unitary dilation of a completely non-unitary contraction need not be a bilateral shift (see [47], where a suitable example is given).

Theorem 6.1 and also its consequences are of a rather global character. They deal with unitary dilation when considered on the whole space. It is, however, of some interest to consider the restrictions of unitary dilation to some reducing subspaces and ask under what conditions these restrictions are bilateral shifts. We will deal with this question and, in this way, complete theorem 6.1 and related propositions.

To begin with we start with the following lemma:

LEMMA 6.4. *Let T be a contraction in the space H and let U be its minimal unitary dilation in the dilation space K. Let P be the projection of K onto H.*

Suppose that S is a closed subspace of H. Then

$$T^n P U^{-n} h = Ph \quad \text{for} \quad h \in U^n M_+(S), \ n \geq 1, \tag{6.17}$$

$$T^{*n} P U^n h = Ph \quad \text{for} \quad h \in U^{-n} M_-(S), \ n \geq 1. \tag{6.18}$$

Proof. Suppose that $h \in U^n M_+(S)$. Then there is a sequence $h_\nu \to h$ such that

$$h_\nu = \sum_{p=n}^{r_\nu} U^p f_p^\nu \quad (f_p^\nu \in S).$$

Hence $Ph_\nu \to Ph$. On the other hand,

$$hP = \sum_{p=n}^{r_\nu} T^p f_p^\nu = T^n \sum_{p=n}^{r_\nu} T^{p-n} f_p^\nu = T^n P U^{-n} h_\nu. \tag{6.19}$$

Let $\nu \to \infty$ in (6.19). Then

$$Ph_\nu \to Ph = \lim_{\nu\to\infty} T^n P U^{-n} h_k = T^n P U^{-n} h,$$

which completes the proof.

By similar arguments we prove (6.18).

Let S be a subspace of H and consider the space $M(S) = \bigvee_{-\infty}^{+\infty} U^n S$. We denote by P_n the projection (within $M(S)$) onto $U^n M_+(S)$. Since $M_+(S)$ is invariant under U, the projection-valued sequence P_n decreases and $Q = \lim_{n\to\infty} P_n$ is the projection onto $R_+(S) = \bigcap_0^\infty U^n M_+(S)$. Suppose that $f \in S$. The element $P_n f$ belongs to $U^n M_+(S)$. By lemma 6.4 we conclude that

$$(P U^{-n} P_n f, T^{*n} f) = (P P_n f, f) = |P_n f|^2$$

or, which is the same,

$$(P_n f, U^n T^{*n} f) = |P_n f|^2. \tag{6.20}$$

Assume that S is invariant under T^*. Let $f \in S$ and write $T^{*n} f = P U^{-n} P_n f + P U^{-n}(I - P_n) f$. It follows that

$$|T^{*n} f|^2 = (P U^{-n} P_n f, T^{*n} f) + (P U^{-n}(I - P_n) f, T^{*n} f).$$

Since $T^{*n} f \in S$, we conclude by using (6.20) that

$$|T^{*n} f|^2 = |P_n f|^2. \tag{6.21}$$

(6.21) holds provided $T^* S \subset S$.

THEOREM 6.2. *Let S be a closed subspace of H. If $T^{*n} f \to 0$ for $f \in S$, then $R_+(S) = \{0\}$. If S is invariant under T^*, then $R_+(S) = \{0\}$ if and only if $T^{*n} f \to 0$ for $f \in S$.*

Proof. If $T^{*n} f \to 0$, then, by (6.20), $Qf = 0$. Since $R_+(S)$ reduces U, it follows that if $Qf = 0$ for all $f \in S$, then $R_+(S) = \{0\}$.

Suppose that $T^* S \subset S$ and $R_+(S) = \{0\}$. Then $P_n \to 0$ strongly. By (6.21) we infer that $T^{*n} f \to 0$ for $f \in S$, q. e. d.

Remark. Theorem 6.2 is a slight extension of de Bruijn result. If $S = H$, then obviously $T^* S \subset S$ and we conclude that if $R_+ = \{0\}$, then $T^{*n} f \to 0$ strongly for every $f \in H$. Using the formula

$$K = R_+ \oplus \Big(\bigoplus_{-\infty}^{+\infty} U^n L_-\Big)$$

we infer that if U is a bilateral shift, L_- being the complete wandering subspace, then $T^{*n} \to 0$ strongly. This last statement was proved in [47].

7. Some spectral relations

We will now discuss the relationship between the semi-spectral measure of a contraction and the spectral properties of this contraction. If the measure is spectral-harmonic, some of the results can be suitably sharpened.

Throughout the present section T stands for a fixed but arbitrary contraction T acting in the complex Hilbert space H. $F(\sigma)$ stands for an arbitrary semi-spectral measure of T. The Neumark dilation of F is denoted by E_F and the corresponding simple unitary dilation of T is given by the formula

$$U_F = \int_0^{2\pi} e^{i\lambda}\, dE_F(\lambda).$$

The first proposition states, roughly speaking, that F and E_F are mutually absolutely continuous (see [28]):

LEMMA 7.1. *Let $\sigma \in B(C)$. Then $F(\sigma) = 0$ if and only if $E_F(\sigma) = 0$.*

Proof. Obviously, if $E_F(\sigma) = 0$, then $F(\sigma) = 0$. Suppose that $F(\sigma) = 0$. Denote by P the projection of the dilation space onto the initial one. Then $PE_F(\sigma)P = 0$. But $PE_F(\sigma)P = \big(E_F(\sigma)P\big)^*\big(E_F(\sigma)P\big)$. It follows that $E_F(\sigma)P = 0$. Let $f \in H$ and let $\gamma \in B(C)$. Then

$$E_F(\sigma)E_F(\gamma)f = E_F(\gamma)E_F(\sigma)f = E_F(\gamma)E_F(\sigma)Pf = 0.$$

Since the dilation space is spanned by elements of the form $E_F(\gamma)f$ with $f \in H$ and $\gamma \in B(C)$, we conclude that $E_F(\sigma) = 0$, q. e. d.

Since the Neumark dilation of the spectral-harmonic measure of the contraction T is the spectral measure of the minimal unitary dilation of T, we conclude from lemma 7.1 that the following statement holds true:

If F is the spectral-harmonic measure of T and E is the spectral measure of the minimal unitary dilation of T, then $F(\sigma) = 0$ if and only if $E(\sigma) = 0$ for every Borel subset of the unit circle.

It follows also from lemma 7.1 that if the semi-spectral function $F(\lambda)$ is constant on the open subarc $\sigma \subset C$, then the Neumark dilation $E_F(\lambda)$ is constant on σ. Thus σ belongs to the resolvent set of U_F.

Let Q be a linear operator in H. Write

$$B_z(Q) = \{f \mid Qf = zf\};$$

$B_z(Q)$ is the eigenspace in H of the operator Q, corresponding to the eigenvalue z.

We have the following theorem (see [8], [32], [49]):

THEOREM 7.1. *Let T be a contraction in H. Then $B_z(T) = B_{\bar{z}}(T^*)$ for $z \in C$ ($|z| = 1$). If $z \notin S_p(T)$ and $z \in C$, then the domain of $(zI-T)^{-1}$ is dense in H.*

Proof. Let U be a unitary dilation of T, which may be a simple one. If $Tf = zf$, then $|Tf| = |f| = |Uf| = |PUf|$. Hence $Tf = Uf$ and consequently $Uf = zf$. It follows that $f = zU^{-1}f$. Multiplying the last equality by $\bar{z}$ and using the equality $z\bar{z} = 1$, we get $U^{-1}f = \bar{z}f$. Hence $T^*f = PU^{-1}f$ $\bar{z}f$. We have just proved that $B_z(T) \subset B_{\bar{z}}(T^*)$. Taking T^* in place of T we get $B_{\bar{z}}(T^*) \subset B_z(T)$, which establishes the proof.

If $D[(zI$ $T)^{-1}]$ $(z \notin S_p(T))$ were not dense for some $z \in C$, then $\bar{z} \in S_p(T^*)$ and, by the first part of the assertion, $z \in S_p(T)$, which is a contradiction.

It follows from theorem 7.1 that there is no residual spectrum of T on the unit circle C.

As a by-product of the proof of theorem 7.1 we get the following conclusion:

If U is a simple unitary dilation of the contraction T, then

$$S_p(T) \cap C \subset S_p(U), \tag{7.1}$$

$$B_z(T) \subset B_z(U) \quad \text{for} \quad z \in C. \tag{7.2}$$

If U is a dilation with respect to the semi-spectral measure F of T, i. e. if $U = U_F$, then by (7.1) for every $e^{i\lambda} \in S_p(T)$ we have $E_F(\{\lambda\}) \neq 0$. By lemma 7.1 we have therefore the following

LEMMA 7.2. *If $e^{i\lambda} \in S_p(T)$ $(\lambda \in [0, 2\pi])$, then $F(\{\lambda\}) \neq 0$ for every semi-spectral measure F of T.*

The next theorem shows that there is also a close relationship between $S_a(T) \cap C$ and the approximate spectrum of any simple unitary dilation U of T. There is a natural restriction on approximating vectors for U in this case, namely we restrict ourselves to those points of $S_a(U)$ which belong to

$$S_{a,H}(U) \quad \{e^{i\lambda} | \lambda \in [0, 2\pi] \text{ and there is a sequence}$$
$$|f_n| \quad 1,\ f_n \in H \text{ such that } |Uf_n - e^{i\lambda}f_n| \to 0\}.$$

We now formulate (see [39])

THEOREM 7.2. *Let U be a simple unitary dilation of the contraction T. Then $S_a(T) \cap C$ $S_{a,H}(U)$.*

Proof. It is sufficient to prove that $1 \in S_a(T)$ if and only if $1 \in S_{a,H}(U)$. If $1 \in S_{a,H}(U)$ and if f_n is the corresponding sequence of approximate eigenvectors belonging to H, then

$$|Tf_n - f_n| \quad |P[Uf_n - f_n]| \leqslant |Uf_n - f_n| \to 0,$$

and this establishes the inclusion $1 \in S_a(T)$.

The converse implication may be proved, as observed by Sz.-Nagy, by using the following arguments. For every $f \in H$

$$|Uf-f|^2 = |Uf|^2+|f|^2-2\mathrm{Re}(Uf,f) = 2|f|^2-2\mathrm{Re}(Tf,f)$$
$$= 2\mathrm{Re}(f,f-Tf) \leqslant 2|f|\,|Tf-f|.$$

Hence, if $|Tf_n-f_n| \to 0$ ($|f_n| = 1$), then $|Uf_n-f_n| \to 0$, q. e. d.

THEOREM 7.3 (see [39]). *Let F be the semi-spectral measure of the contraction T and let $F(\sigma) = 0$ for the open arc $\sigma \subset C$. Then σ lies in the open set which is the complement (to the whole plane) of $S_a(T)$.*

Proof. Suppose the assertion is false. Then, since $S_a(T)$ is closed, $S_a(T) \cap \sigma \subset S_a(T) \cap C$ is non-empty. By theorem 7.2 there is a $\lambda \in [0, 2\pi]$ such that $e^{i\lambda} \in S_{a,H}(U_F) \cap \sigma \subset S(U_F) \cap \sigma$. On the other hand, it follows from lemma 7.1 that the arc σ lies in the resolvent set of U_F. Since U_F is normal, we have $S(U_F) = S_a(U_F)$. Thus $\sigma \subset C-S_a(U_F)$, which contradicts the inclusion $e^{i\lambda} \in S(U_F)$. This completes the proof.

In the case where U is the minimal unitary dilation of T the inclusions (7.1) and (7.2) hold true, but even more is true: they may be replaced by equalities. To complete our discussion we will prove (see [45] and [49])

THEOREM 7.4. *Let U be the minimal unitary dilation of the contraction T. Then*

$$S_p(T) \cap C = S_p(U) \quad \text{and} \quad B_z(T) = B_z(U) \quad \text{for } z \in C.$$

Proof. There is no loss of generality if we restrict ourselves to the case $z = 1$. Since the inclusions (7.1) and (7.2) hold for U, we need only to show that if $1 \in S_p(U)$ and $Uf = f$, then $f \in H$. In the proof we shall use theorem 5.3 adopting the notation of section 5.

Let us define

$$L_\nu = \begin{cases} U^{-(\nu+1)}L_+ & \text{if} \quad \nu < 0, \\ U^{-(\nu-1)}L_- & \text{if} \quad \nu > 0, \end{cases} \qquad L_0 = H.$$

Denote by K the dilation space and let $(g)_\nu$ stand for the projection of $g \in K$ onto L_ν. By theorem 5.3

$$g = \sum_{-\infty}^{+\infty} (g)_\nu, \qquad (g)_\nu \perp (g)_\mu \quad \text{for} \quad \nu \neq \mu.$$

Also $UL_\nu = L_{\nu-1}$ for $\nu < 0$ and $\nu > 1$. If $Uf = f$, then $U(f)_\nu \quad (f)_{\nu-1} = (f)_\nu$ for $\nu < 0$ or $\nu > 1$. Consequently, $(f)_{\nu-1} = (f)_\nu = 0$ for $\nu < 0$ or $\nu > 1$, i. e. $(f)_\mu = 0$ for $\mu \neq 0$. It follows that $f \in H$, which implies in an obvious way that $Tf = f$, as was to be proved.

In terms of a spectral-harmonic measure theorem 7.4 is completed by the following one (see [8]):

THEOREM 7.5. *Let $F(\lambda)$ be the spectral-harmonic function of the contraction T. Then $z_0 = e^{i\lambda_0} \in S_p(T)$ $(\lambda_0 \in [0, 2\pi])$ if and only if λ_0 is the discontinuity point of $F(\lambda)$, i. e. $F(\lambda_0 - 0) - F(\lambda_0) \neq 0$, and in this case the difference $F(\lambda_0 - 0) - F(\lambda_0)$ equals to the projection P_{z_0} onto $B_{z_0}(T)$. Moreover, if $z_0 = e^{i\lambda_0} \neq e^{i\lambda_1} = z_1$, then $P_{z_0} P_{z_1} = 0$.*

Proof. If $z_0 = e^{i\lambda_0} \in S_p(T)$, then, by lemma 7.4, $F(\lambda_0 - 0) - F(\lambda_0) \neq 0$. Conversely, if $F(\lambda_0 - 0) - F(\lambda_0) \neq 0$, then the spectral function $E(\lambda)$ of the minimal unitary dilation of T has a discontinuity at λ_0. Then $e^{i\lambda_0} \in S_p(U)$ and by theorem 7.4 we conclude that $e^{i\lambda_0} \in S_p(T)$, which completes the proof of the first part of the assertion.

We have to show that $\Delta_{\lambda_0} = F(\lambda_0 - 0) - F(\lambda_0) = P_{z_0}$, where $z_0 = e^{i\lambda_0}$. By theorem 7.4, $E(\{\lambda_0\})K \subset B_{z_0}(T)$. Thus $P_{z_0} E(\{\lambda_0\}) f = E(\{\lambda_0\}) f$ and $E(\{\lambda_0\}) P_{z_0} f = P_{z_0} f$ for $f \in H$. It follows that for all f and g in H

$$\left(P_{z_0} E(\{\lambda_0\}) f, g\right) = \left(f, E(\{\lambda_0\}) P_{z_0} g\right) = \left(f, E(\{\lambda_0\}) g\right)$$

and consequently $E(\{\lambda_0\}) P_{z_0} f = E(\{\lambda_0\}) f$ for $f \in H$. We conclude that

$$P_{z_0} f = E(\{\lambda_0\}) f \quad \text{for} \quad f \in H, \tag{7.3}$$

which implies

$$(P_{z_0} f, g) = \left(E(\{\lambda_0\}) f, g\right) = \left(E(\lambda_0 - 0) - E(\lambda_0) f, g\right) = \left((F(\lambda_0 - 0) - F(\lambda_0) f, g\right),$$

i. e. $P_{z_0} = F(\lambda_0 - 0) - F(\lambda_0)$.

If $z_0 \neq z_1$, then $B_{z_0}(U) \perp B_{z_1}(U)$, which proves that $P_{z_0} P_{z_1} = 0$.

We see from theorem 7.5 that the spectral-harmonic measure characterizes the part of the point spectrum of T on the unit circle exactly in the same way as the spectral measure does in the case of unitary operators.

If the contraction T is not itself unitary, then there cannot be any gaps in the spectrum of its minimal unitary dilation. To show this we need the following lemma:

LEMMA 7.3. *Let V be a unitary operator in a complex Hilbert space and let V have a non-zero wandering subspace. Then the spectrum of V covers the whole unit circle C.*

Proof. Let f be a non-zero vector of the wandering subspace. The restriction V' of V to the reducing subspace spanned by $V^n f$ $(n = 0, \pm 1, \pm, 2, \ldots)$ is a bilateral shift of multiplicity 1. It follows that every point of C belongs to the approximate spectrum of V'. Since the approximate eigenvectors belong to the space in which V' acts, they are also approximate vectors for V, which completes the proof.

Now we are able to prove the following theorem (see [38], [45]):

THEOREM 7.6. *Let T be a contraction in a complex Hilbert space and let U be its minimal unitary dilation. Suppose that T is not a unitary operator. Then $S(U) = C$.*

Proof. If T is not unitary, then at least one of the subspaces L_T, L_{T^*} is non-trivial. It follows by lemma 7.2 that L_+ or L_- is non-trivial. Consequently, by theorem 5.2, U has a wandering subspace. Using lemma 7.3 we get the assertion of the theorem.

Theorem 7.6 can be expressed in terms of spectral-harmonic measure as follows (see [8]):

THEOREM 7.7. *Let $F(\sigma)$ be the spectral-harmonic measure of the contraction T. If $F(\sigma) = 0$ for some open subarc of the unit circle C, then T is unitary.*

Proof. If T were not unitary, then $E_F(\sigma) = 0$ by lemma 7.1, which contradicts theorem 7.6.

8. Properties of spectral-harmonic measures

Throughout the present section we consider only complex Hilbert spaces.

Let U be a unitary operator in the space K and let $E(\sigma)$ be its spectral measure. Let us take an element f and denote by $L_2(f)$ the space spanned by $U^n f$ $(n = 0, \pm 1, \pm 2, \ldots)$. Consider the positive measure $\mu_f(\sigma) = (E(\sigma)f, f)$ on the unit circle C. The function $\mu_f(\lambda) = (E(\lambda)f, f)$ is the distribution function of the measure $\mu_f(\sigma)$. It is known that the correspondence $e^{in\lambda} \leftrightarrow U^n f$ can be extended to the unitary isomorphism of $L_2(f)$ and $L_2(\mu_f)$, the space of complex-valued functions on C, square summable with respect to $\mu_f(\sigma)$. The unitary operator U is then equivalent to the operator V of multiplication by $e^{i\lambda}$ in $L_2(\mu_f)$. If $d\mu_f(\sigma) = \varphi_M d\sigma$, where φ_M is a characteristic function of a measurable set M, we write simply $L_2(M) = L_2(\mu_f)$. In this case U is equivalent simply to the restriction of V in $L_2(m)$ (m—the Lebesgue measure) to the reducing subspace with support M.

Under the equivalence established above, the copy of the space $M_{(\pm)}(f)$ spanned by $U^n f$ with $n \geqslant 0$ $(n \leqslant 0)$ is the space $M_{(\pm)}(\mu_f)$ spanned in $L_2(\mu_f)$ by $1, e^{i\lambda}, e^{2i\lambda}, \ldots$ $(1, e^{-i\lambda}, e^{-2i\lambda}, \ldots)$.

Suppose that the vector g belongs to $L_2(f)$. The corresponding function in $L_2(\mu_f)$ will be denoted by $\hat{g}(\lambda)$. Notice that every reducing subspace for V in $L_2(\mu_f)$ consists exactly of those functions which vanish outside a fixed measurable subset of C which corresponds to this subspace uniquely (modulo μ_f). It follows that if $\hat{g}(\lambda)$ belongs to such a reducing subspace, so does its complex conjugate function $\hat{g}(\lambda)$.

Since the space

$$R_{(\pm)}(f) = \bigcap_0^\infty U^{(-n)} M_{(\pm)}(f)$$

reduces U, its isomorphic copy $\hat{R}_{(\pm)}(f)$ reduces V. Suppose now that $\varepsilon > 0$

and n is a positive integer. If $g \in R_+(f)$, then $\overline{g(\lambda)} \in \hat{R}_+(f)$, and consequently there is a polynomial

$$p_{n,\varepsilon}(\lambda) = \sum_{\nu=n}^{q} a_\nu e^{i\nu\lambda} \qquad (q \geqslant n)$$

such that

$$r_{n,\varepsilon} = \int_0^{2\pi} |\overline{g(\lambda)} - p_{n,\varepsilon}(\lambda)|^2 d\mu_f(\lambda) < \varepsilon.$$

Hence

$$\left| g - \sum_{\nu=n}^{q} \overline{a}_\nu U^{-n} f \right|^2 = \int_0^{2\pi} |g(\lambda) - \overline{p_{n,\varepsilon}(\lambda)}|^2 d\mu_f(\lambda) = r_{n,\varepsilon} < \varepsilon.$$

The parameters n, ε being arbitrary, we infer that

$$g \in \bigcap_0^\infty U^{-n} M_-(f) = R_-(f).$$

We have just proved that

$$R_+(f) = R_-(f) \quad \text{for every} \quad f \in H. \tag{8.1}$$

We have (see [25])

THEOREM 8.1. *Let U be the minimal unitary dilation of the contraction T acting in the complex Hilbert space H. Then for every $f \in H$,*

$$R_+(f) = R_-(f) \subset \bigcap_{-\infty}^{+\infty} U^n H.$$

Proof. It is easy to see that

$$R_{(\pm)}(f) \subset R_{(\pm)} = \bigcap_0^\infty U^{(-n)n} M_{(\pm)}, \quad \text{where} \quad M_{(\pm)} = \bigvee_0^\infty U^{(-n)n} H.$$

It follows now from (8.1) that $R_+(f) = R_-(f) \subset R_+ \cap R_-$. By theorem 5.6, $R_+ \cap R_- = \bigcap_{-\infty}^{+\infty} U^n H$ and the assertion follows.

It follows from theorem 5.7 that

$$M_+(f) = R_+(f) \oplus \Big(\bigoplus_0^\infty U^n \big(M_+(f) \ominus U M_+(f) \big) \Big),$$

$$M_-(f) = R_-(f) \oplus \Big(\bigoplus_0^\infty U^{-n} \big(M_-(f) \ominus U^{-1} M_-(f) \big) \Big),$$

and consequently

$$L_2(f) = R_+(f) \oplus \Big(\bigoplus_{-\infty}^{+\infty} U^n \big(M_+(f) \ominus U M_+(f) \big) \Big),$$

$$L_2(f) = R_-(f) \oplus \Big(\bigoplus_{-\infty}^{+\infty} U^{-n} \big(M_-(f) \ominus U^{-1} M_-(f) \big) \Big).$$

It is easy to check that $\dim\big(M_+(f)\ominus UM_+(f)\big)\leqslant 1$. Suppose now that $f\neq 0$ and $R_+(f)=\{0\}$. Then U restricted to $L_2(f)$ is a bilateral shift of multiplicity 1. Let e be the unit basis vector in $M_+(f)\ominus UM_+(f)$. Since $f\in M_+(f)$, we have

$$f=\sum_{\nu=0}^{\infty}c_\nu U^\nu e,\qquad |f|^2=\sum_{\nu=0}^{\infty}|c_\nu|^2 \tag{8.2}$$

The sequence $U^\nu e$ $(\nu\geqslant 0)$ is an orthonormal basis in $M_+(f)$ and the correspondence

$$g\in M_+(f),\qquad g=\sum_{\nu=0}^{\infty}a_\nu U^\nu e\leftrightarrow\{a_\nu\}\in l_2^+$$

establishes the unitary equivalence between $M_+(f)$ and l_2^+. The restriction of U to $M_+(f)$ is interpreted as the unilateral shift

$$\{a_0,a_1,a_2,\ldots\}\to\{0,a_0,a_1,a_2,\ldots\}. \tag{8.3}$$

Since the elements U^nf, $n\geqslant 0$, span $M_+(f)$, we conclude that the right-hand translates of $\{c_\nu\}$, that is sequences of the form

$$\{0,0,0,\ldots c_0,c_1,c_2,\ldots\}, \tag{8.4}$$

span l_2^+. The sequence $\{c_\nu\}$ is determined by (8.2).

One can give a function-theoretical interpretation of the condition $R_+(f)=\{0\}$. For a detailed study of the function-theoretical properties which we need we refer to [18] and [21].

With every sequence $\{a_\nu\}\in l_2^+$ we can associate a function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \tag{8.5}$$

which is analytic for $|z|<1$ and satisfies

$$\int_0^{2\pi}|f(re^{i\lambda})|^2d\lambda\qquad M_f<+\infty,\qquad 0\qquad r<1, \tag{8.6}$$

where M_f is a constant independent of r but depending only on $\{a_\nu\}$. Conversely, if $f(z)$ is analytic for $|z|<1$ and satisfies (8.6), then the Taylor coefficients a_n of $f(z)$ forms a sequence belonging to l_2^+. The class of all such functions $f(z)$ forms a linear space, which becomes a Hilbert space under the inner product

$$(f,g)=\int_0^{2\pi}f(e^{i\lambda})\,g(e^{i\lambda})\,dm(\lambda).$$

The boundary values

$$f(e^{i\lambda}) = \lim_{r\to 1} f(re^{i\lambda})$$

for such f exist almost everywhere on $[0, 2\pi]$ and

$$f(e^{i\lambda}) \neq 0 \quad \text{for almost all } \lambda \in [0, 2\pi] \tag{8.7}$$

unless f is identically zero.

The unitary isomorphism

$$l_2^+ \ni \{a_n\} \leftrightarrow \sum_{n|0}^{\infty} a_n z^n$$

gives the interpretation of the uniteral shift (8.3) as the multiplication by z; namely this shift is equivalent to the map $\leftrightarrow$ which maps $f(z)$ onto $zf(z)$. The space of analytic functions with the inner product defined above will be denoted by H^2. Now, the fact that the elements (8.4) span l_2^+ means that, for $f(z) = \sum_{n|0}^{\infty} c_n z^n$,

(8.8) *The functions* $z^n f(z)$ $(n \geqslant 0)$ *span* H^2.

Following Beurling [2] we call $f(z)$ an *outer function in* H^2 if it satisfies (8.8).

Going back to the space $L_2(f)$ and assuming that $R_+(f) = R_-(f) = \{0\}$ we infer that

$$f(z) = \sum_{n|0}^{\infty} c_n z^n$$

with c_n determined by (8.2) is an outer function. Next (see [21])

$$\begin{aligned} \int_0^{2\pi} e^{ik\lambda} d\mu_f(\lambda) &= (e^{ik\lambda}, 1)_{L_2(\mu_f)} = (U^k f, f)_{L_2(f)} \\ &= \Big(\sum_{\nu=0}^{\infty} c_\nu U^{k+\nu} f, \sum_{\mu=0}^{\infty} c_\mu U^\mu f\Big) = \sum_{j=k}^{\infty} c_{j-k}\bar{c}_j \\ &= \int_0^{2\pi} e^{ik\lambda} f(e^{i\lambda})\,\overline{f(e^{i\lambda})}\, dm(\lambda) = (z^k f, f)_{H^2} \qquad \text{for} \qquad k \geqslant 0. \end{aligned} \tag{8.9}$$

Since the measures $d\mu_f(\lambda)$ and $|f(e^{i\lambda})|^2 dm$ are real, we infer that (8.9) hold for $k < 0$. Hence $d\mu_f(\lambda) = |f(e^{i\lambda})|^2 dm(\lambda)$. Summing up we get the following useful lemma (see [21]):

LEMMA 8.1. *Suppose that* $R_+(f) = \{0\}$. *Then* $d\mu_f(\lambda) = |f(e^{i\lambda})|^2 dm(\lambda)$, *where* $f(z)$ *is an outer function in* H^2.

The converse of the lemma also holds true.

As indicated in (8.7) the boundary values of non-trivial function in H^2 cannot vanish on a subset of positive measure of the unit circle. Thus $\mu_f(\sigma)$ and the linear Lebesgue measure on C are equivalent provided $R_+(f) = \{0\}$.

We will now prove the following lemma:

LEMMA 8.2. *Let T be a contraction and U its minimal unitary dilation. Let*

$$H = \bigcap_{-\infty}^{+\infty} U^n H \oplus \hat{H}$$

be the canonical decomposition of the space H corresponding to the contraction T. Then, if $f \in \hat{H}$ then $R_+(f) = \{0\}$.

Proof. If $f \in \hat{H}$ then, by theorem 8.1, $f \perp R_+ \cap R_- \supset R_+(f)$. It follows that

$$f \in \bigoplus_0^\infty U^n \big(M_+(f) \ominus U M_+(f)\big) = Y.$$

Consequently $M_+(f) \subset Y$. But $R_+(f) \subset M_+(f)$ and $R_+(f) \perp Y$. We infer that $R_+(f) = \{0\}$, as was to be proved.

THEOREM 8.2. *The contraction T in the complex Hilbert space H is completely non-unitary if and only if*

$$R_+(f) = \{0\} \quad \text{for every} \quad f \in H. \tag{8.10}$$

Proof. The necessity of (8.10) follows from lemma 8.2. Suppose now that (8.10) holds true and T is not completely non-unitary. Then there is an $h \in H$, $h \neq 0$, such that $U^n h \in H$ for all n. U stands here for the minimal unitary dilation of T. It follows that $L_2(h) \subset H$. Let $\varphi(\lambda)$ be the characteristic function of $[0, \pi]$ ($= 1$ for $\lambda \in [0, \pi]$ and $= 0$ elsewhere). The measure

$$a(\sigma) = \int_\sigma \varphi(s)\, d\big(E(s)h, h\big),$$

where E stands for the spectral measure of U, is absolutely continuous with regard to $\mu_h(\sigma)$. This being established, we conclude that there is a vector $g \neq 0$, $g \in L_2(h)$ such that $a(\sigma) = \big(E(\sigma)g, g\big)$. Since $L_2(h) \subset H$, the vector $g \in H$. By lemma 8.1 the functions $\big(E(\lambda)g, g\big)$, $\big(E(\lambda)h, h\big)$ are both absolutely continuous in the ordinary sense. It follows that $d\mu_g(\lambda)/d\lambda = 0$ almost everywhere with respect to the Lebesgue measure on the subinterval $(\pi, 2\pi]$. This, however, is in contradiction to $R_+(g) = \{0\}$ ($g \in H$!). Indeed, by lemma 8.1 and (8.7) the linear Lebesgue measure on C is absolutely continuous with regard to the measure $\mu_g(\sigma)$. The desired contradiction completes the proof.

We have explained the geometrical and analytical sense of the condition $R_+(f) = \{0\}$. Some properties of measures $\mu_f(\sigma)$ have been involved. Since $\mu_f(\sigma) = (F(\sigma)f, f)$, where F is the spectral-harmonic measure of the considered contraction, we have proved in fact some propositions concerning the spectral-harmonic measures. Going further in this direction, we will need a more advanced study of space $L_2(f)$. To begin with we present here the classical result of Szegö [18], [34], improved later by Kolmogoroff and Krein.

Let μ be a positive measure on C, or, which is equivalent, a corresponding measure on $[0, 2\pi]$. The Szegö theorem states that

$$\inf_p \int_0^{2\pi} |1-p(\lambda)|^2 d\lambda = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log k(\lambda)\, d\lambda\right), \tag{8.11}$$

where $k(\lambda)$ is the derivative of $d\mu$ with respect to the normalized Lebesgue measure, i. e $k(\lambda) = 2\pi\, d\mu/d\lambda$ and p ranges over the set of polynomials of the form

$$p(\lambda) = \sum_{\nu=1}^{q} a_\nu e^{i\nu\lambda} \qquad (q \geqslant 1).$$

If $\log k(\lambda)$ is not summable, i. e.

$$\int_0^{2\pi} \log k(\lambda)\, d\lambda = -\infty,$$

the right side of (8.11) is to be interpreted as zero and inf on the left is in fact zero. Taking $\mu = \mu_f$ in the Szegö theorem, we conclude that $f \in UM_+(f)$ if and only if $\log d\mu_f(\lambda)/d\lambda$ is not summable over $[0, 2\pi]$. If $R_+(f) = \{0\}$ and $f \neq 0$, then, by (8.11), $\log d\mu_f(\lambda)/d\lambda$ is summable over $[0, 2\pi]$. This last property may be understood also by using lemma 8.1. For, if $f \in H^2$ and $f \neq 0$, then the modulus $|f(e^{i\lambda})|$ of the boundary values on the unit circle of $f(z)$ has a logarithm summable over $[0, 2\pi]$ (see [18]). Now, if $R_+(f) = \{0\}$, then, by lemma 8.1, $d\mu_f(\lambda)/d\lambda = |f(e^{i\lambda})|^2$, where $f(z)$ is an outer function in H^2. Thus $\log d\mu_f(\lambda)/d\lambda$ is summable in this case. We can summarize our discussion in the following lemma:

LEMMA 8.3. *Suppose that* $R_+(f) = \{0\}$. *Then the function* $(E(\lambda)f, f) = \mu_f(\lambda)$ *is absolutely continuous in the ordinary sense over* $[0, 2\pi]$ *and* $\log d\mu_f(\lambda)/d\lambda$ *is summable over* $[0, 2\pi]$.

We will now apply the previous properties to unitary dilations. Henceforth U will stand for the minimal unitary dilation of a contraction T. The function $E(\lambda)$ is the spectral function of U and $\mu_f(\lambda)$ is defined as before.

THEOREM 8.3 ([24], [25], [46]). *Suppose that*

$$H = \bigcap_{-\infty}^{+\infty} U^n H \oplus \hat{H}$$

is the canonical decomposition of the initial space corresponding to the contraction T. If $f \in \hat{H}$, then the function $\mu_f(\lambda)$ is absolutely continuous over $[0, 2\pi]$ and $\log d\mu_f(\lambda)/d\lambda$ is summable over $[0, 2\pi]$. If $f \in H$ and $f \in UM_+(f)$, then

$$f \in \bigcap_{-\infty}^{+\infty} U^n H,$$

and this holds if and only if $\log d\mu_f(\lambda)/d\lambda$ is not summable over $[0, 2\pi]$.

Proof. The first part of the assertion follows immediately from lemma 8.2 and lemma 8.3. To prove the rest we notice that if $f \in UM_+(f)$, then $f \in R_+(f)$ and consequently, by theorem 8.1,

$$f \in \bigcap_{-\infty}^{+\infty} U^n H.$$

The last part of the assertion is a trivial consequence of the Szegö formula (8.1) and our previous considerations.

Remark. Theorem 8.3 was generalized in [45] by using the so-called Szegö measures.

As a by-product of theorem 8.3 we get the following conclusion (see [46], [47], and [24]):

If $f \in \hat{H}$, then $\mu_f(\sigma)$ and the Lebesuge measure are mutually absolutely continuous.

It follows from this conclusion that T is completely non-unitary if and only if for every $f \in H$ the Lebesgue measure and $\mu_f(\sigma)$ are mutually absolutely continuous. Denote by $m(\sigma)$ the Lebesgue measure on C. We conclude that if $m(\sigma) = 0$ for some $\sigma \in B(C)$, then $F(\sigma) = 0$, where F stands for the spectral-harmonic measure of the completely non-unitary contraction T. Now, if $E(\sigma)$ is the spectral measure of the minimal unitary dilation U of T, then, by lemma 7.1, we conclude that $(E(\sigma)f, f)$ is absolutely continuous with respect to m for every f in the dilation space in which U acts. In other words, *the minimal unitary dilation of a completely non-unitary contraction has an absolutely continuous spectrum* (see [46]). More precisely, the following theorem holds true (see [46]):

THEOREM 8.4. *Let $E(\sigma)$ be the spectral measure of the minimal unitary dilation of a completely non-unitary contraction. Let K stand for the dilation space. Then for every $f \in K$ the measure $(E(\sigma)f, f)$ is absolutely continuous with regard to the linear Lebesgue measure on the unit circle.*

The significance of theorem 8.4 may be explained as follows (see [46]): The absolute continuity of the spectrum of

$$U = \int_0^{2\pi} e^{i\lambda} dE(\lambda)$$

implies that there is an indexed set of vectors f_α ($\alpha \in \Omega$) such that the whole space K may be written as

$$K = \bigoplus_{\alpha \in \Omega} L_2(f_\alpha),$$

where every $L_2(f_\alpha) = L_2(M_\alpha)$ with $M_\alpha = \{\lambda | d\mu_{f_\alpha}(\lambda)/d\lambda \neq 0\}$. The restriction U_α of U to $L_2(M_\alpha)$ is the operator of multiplication by $e^{i\lambda}$.

The method of investigation measures $\mu_f(\sigma)$ which we have presented above admits of some modifications. The assumptions of the previous theorems dealt rather with the global behaviour of the contraction T on the whole space H. However, it is possible to consider certain individual properties of a single measure $\mu_f(\sigma)$ provided some assumptions are made merely about the sequence $\{T^{(n)}f\}$.

Let us assume that for some $f \in H$

$$\inf_n (|T^n f|, |T^{*n} f|) < |f| \quad (f \neq 0). \tag{8.12}$$

Only three cases are possible:

I. There are positive integers p_1, p_2 and non-negative constants $K_1 < 1, K_2 < 1$ such that

$$|T^{*s} f| = |f| = |T^q f|$$

for $s = 1, 2, \ldots, p_1 - 1$, $q = 1, 2, \ldots, p_2 - 1$ and

$$|T^{*p_1} f| \quad K_1 |T^{*p_1 - 1} f| = K_1 |f|, \quad |T^{p_2} f| \leqslant K_2 |T^{p_2 - 1} f| = K_2 |f|.$$

II. There are a positive integer p_1 and a non-negative constant $K_1 < 1$ such that $|T^s f| \quad f = |T^{*q} f|$ for $s \geqslant 1$ and $q = 1, 2, \ldots, p_1 - 1$ and $|T^{*p_1} f| \leqslant K_1 |T^{*p_1 - 1} f| \quad K_1 |f|$.

III. There are a positive integer p_2 and a non-negative constant $K_2 < 1$ such that $|T^{*s} f| = |f| = |T^q f|$ for $s \geqslant 1$ and $q = 1, 2, \ldots, p_2 - 1$ and $|T^{p_2} f| \quad K_2 |T^{p_2 - 1} f| \quad K_2 |f|$.

Now we define

$$K(f) \begin{cases} \min(K_1, K_2) & \text{in case I,} \\ K_1 & \text{in case II,} \\ K_2 & \text{in case III.} \end{cases}$$

To simplify the notation we denote by $\hat{h}(\lambda)$ the derivative of $\mu_h(\lambda)$ with respect to the normalized Lebesgue measure (for every $h \in H$).

THEOREM 8.5. *Suppose that* $f \in H$ *and* (8.12) *holds true. Then* $\log \hat{f}(\lambda)$ *is summable over* $[0, 2\pi]$ *and*

$$\frac{1}{2\pi}\int_0^{2\pi} \log \hat{f}(\lambda)\, d\lambda \geqslant \log\left|\left(1-K^2(f)\right)|f|^2\right|. \tag{8.13}$$

If additionally $T^n f \to 0$ *or* $T^{*n} f \to 0$, *then* $\mu_f(\lambda)$ *is absolutely continuous in the ordinary sense over* $[0, 2\pi]$.

Proof. Without loss of generality we can assume that $K(f) = K_1$. Otherwise we take U in place of U^* and T in place of T^* and the arguments apply as well.

Write $g = T^{*p_1-1} f$ and consider the space $M_+(g)$. Let g_1 be the projection of g onto $UM_+(g)$. It follows from lemma 6.4 that $TPU^{-1}g_1 = Pg_1$. Then $(TPU^{-1}g_1, g) = (g_1, g)$ and consequently

$$(g_1, UT^*g) = (g_1, g) = |g_1|^2. \tag{8.14}$$

But $UT^*g = UT^{*p_1} f$. Thus $|UT^*g| \leqslant K(f)|g|$ and by (8.14) and by Schwartz's inequality $|g_1|^2 \leqslant K(f)|g|\,|g_1|$. Hence

$$|g_1| \leqslant K(f)|g| = K(f)|f|. \tag{8.15}$$

By Szegö's theorem (8.11) we get

$$|g - g_1|^2 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log \hat{g}(\lambda)\, d\lambda\right). \tag{8.16}$$

On the other hand, by (8.15)

$$\left(1-K^2(f)\right)|f|^2 \leqslant |g|^2 - |g_1|^2 = |g-g_1|^2. \tag{8.17}$$

Since $K^2(f) < 1$, we conclude that $\log \hat{g}(\lambda)$ is summable over $[0, 2\pi]$. Notice now that $g = U^{*p_1-1} f = T^{*p_1-1} f$. It follows that

$$\left(E(\lambda)g, g\right) = \left(E(\lambda)U^{*p_1-1}f,\ U^{*p_1-1}f\right) = \left(E(\lambda)f, f\right).$$

Thus $\hat{g}(\lambda) = \hat{f}(\lambda)$ and (8.13) follows from (8.16) and from (8.17). This establishes the proof of the first part of the assertion.

Let $T^{*n} f \to 0$. Then, by theorem 6.2, $R_1(f) \quad \{0\}$. Using lemma 8.1 or lemma 8.3 we infer that $\mu_f(\lambda)$ is absolutely continuous in the ordinary sense over $[0, 2\pi]$, q. e. d.

Remark. A considerable part of the results proved here for the spectral-harmonic measure of a single contraction can be proved, with suitable changes, for the spectral-harmonic measure associated with a one-parameter semi-group of contractions. One has to use suitable

prediction-theoretical arguments based on theorems concerning mean continuous stationary stochastic processes. For example, one can prove that if F is the spectral-harmonic measure associated with a completely non-unitary semi-group $T(t)$ $(t \geqslant 0)$ acting in the initial space H, then for every $f \in H$, $f \neq 0$, the integral

$$\int_{-\infty}^{+\infty} \left(\log d\big(F(\lambda)f, f\big)/d\lambda\right) \cdot (1/1+\lambda^2)\, d\lambda$$

converges.

A natural problem now arises: under what conditions the spectral-harmonic measure of completely non-unitary contraction is differentiable, strongly or uniformly, to an operator function. We will discuss this problem briefly.

Let the spectral radius

$$r(T) = \lim_{n\to\infty} |T^n|^{1/n} < 1.$$

Then $r(T^*) < 1$ and consequently the series

$$K(e^{i\lambda}) = I + 2\mathrm{Re}\Big(\sum_{k|1}^{\infty} e^{ik\lambda} T^{*k}\Big) = \sum_{-\infty}^{+\infty} e^{-ik\lambda} T^{(k)}$$

converges in the uniform operator topology uniformly in λ. It follows that $K(e^{i\lambda})$ is continuous in λ in the uniform operator topology. Let $m(\sigma)$ be the normalized linear Lebesgue measure on C. Then, for $f, g \in H$,

$$(8.18) \qquad \int_0^{2\pi} e^{in\lambda} \big(K(e^{i\lambda})f,\, g\big)\, dm(\lambda) = (T^{(n)}f,\, g) = \int_0^{2\pi} e^{in\lambda}\, d\big(F(\lambda)f,\, g\big),$$

where $F(\lambda)$ stands as usual for the spectral-harmonic function of T. We are now able to prove the following theorem (see [37]):

THEOREM 8.6. *If $r(T) < 1$, then $F(\lambda)$ is continuously differentiable in λ in the uniform operator topology.*

Proof. It follows from (8.18) that

$$F(\lambda) = \int_0^{\lambda} K(e^{is})\, dm(s).$$

Since the integral on the right is in the uniform operator topology, we are done.

Notice that under the assumption $r(T) < 1$ we have

$$F(\sigma) = \int_{\sigma} K(e^{i\lambda})\, dm(\lambda),$$

which is analogous to the classical formula for harmonic measure with K in place of the Poisson kernel.

It is possible to prove some estimates of $K(e^{i\lambda})$ and $F(\lambda)$ provided $r(T)<1$ holds true (see [36]). This will be illustrated here for the case where T is a strict contraction, that is when $|T|<1$. This being assumed, we easily check that

$$K(e^{i\lambda}) = \mathrm{Re}[(I+e^{-i\lambda}T)(I-e^{-i\lambda}T)^{-1}]. \tag{8.19}$$

Setting $f=(I-e^{-i\lambda}T)^{-1}g$ we get

$$\big(K(e^{i\lambda})g, g\big) = \mathrm{Re}\big((I+e^{-i\lambda}T)f, (I-e^{-i\lambda}T)f\big).$$

Simple calculations show that $\big(K(e^{i\lambda})g, g\big) = (1-|T|^2)|f|^2$. Since

$$|f|^2 = |(I-e^{-i\lambda}T)^{-1}g|^2 \geqslant \frac{|g|^2}{(1+|T|)^2},$$

we get by integrating

$$\big(F(\sigma)g, g\big) = \int_\sigma \big(K(e^{i\lambda})g, g\big)\,dm(\lambda) \geqslant \frac{1-|T|^2}{[1+|T|]^2}\,|g|^2 m(\sigma)$$

for every $\sigma\in B(C)$. On the other hand, using (8.19) and the inequality $|T|<1$ one verifies that

$$|K(e^{i\lambda})| \leqslant \frac{1+|T|}{1-|T|} \quad (0\leqslant\lambda \quad 2\pi).$$

Summing up we get the following inequality (see [36]):

$$\frac{1-|T|^2}{[1+|T|]^2}\,|g|^2 m(\sigma) \leqslant \big(F(\sigma)g, g\big) \leqslant \frac{1+|T|}{1-|T|}\,|g|^2 m(\sigma)$$

valid for every $g\in H$, provided $|T|<1$.

The existence almost everywhere of a strong derivative of the spectral-harmonic function may take place in the case of $r(T)=1$. An example (see [37]) on this subject will be presented below.

Let ϱ_n be a sequence of complex numbers such that $|\varrho_n|$ 1. Define a map T in l_2^+ by

$$T \quad \{\xi_n\} \quad \succ \{\varrho_n \xi_n; \}\in l_2^+$$

Assume now that $\limsup_n |\varrho_n| = 1$. It easily follows that $r(T)=1$. Let $\hat{P}(\lambda)$ be the operator defined in l_2^+ by

$$\hat{P}(\lambda) \quad \{\xi_n\} \to \{P_{r_n}(\theta_n - \lambda)\xi_n\},$$

where $\varrho_n = r_n e^{i\theta_n}$ and $P_r(s)$ stands for the Poisson kernel. If $f=\{\xi_n\}$ and $g=\{\eta_n\}$ belong to l_2^+, then

$$|(\hat{P}(\lambda)f, g)| = \Big|\sum_1^\infty P_{r_n}(\theta_n \quad \lambda)\,\xi_n\bar{\eta}_n\Big| \quad \sum_{n\rfloor 1}^\infty P_{r_n}(\theta_n - \lambda)\,|\xi_n\bar{\eta}_n|, \tag{8.20}$$

since the Poisson kernel is non-negative. Notice that

$$\int_0^{2\pi} P_{r_n}(\theta_n - \lambda)\, dm(\lambda) = 1 \quad \text{for} \quad n \geqslant 1. \tag{8.21}$$

Consequently, the function $(\hat{P}(\lambda)f, g)$ is summable over $[0, 2\pi]$. We will show that

$$(T^{(n)}f, g) = \int_0^{2\pi} e^{in\lambda}(\hat{P}(\lambda)f, g)\, dm(\lambda) \quad \text{for all } n. \tag{8.22}$$

Consider the partial sums

$$w_k(\lambda) = \sum_{j=1}^{k} e^{in\lambda} P_{r_j}(\theta_j - \lambda)\, \xi_j \bar{\eta}_j$$

and assume that $n \geqslant 0$. Since $w_k(\lambda) \to (\hat{P}(\lambda)f, g)e^{in\lambda}$ pointwise, (8.20) and (8.21) imply that

$$\lim_{k\to\infty} \int_0^{2\pi} w_k(\lambda)\, dm(\lambda) = \int_0^{2\pi} e^{in\lambda}(\hat{P}(\lambda)f, g)\, dm(\lambda) \tag{8.23}$$

On the other hand,

$$\int_0^{2\pi} w_k(\lambda)\, dm(\lambda) = \sum_{i=1}^{k} \varrho_i^n\, \xi_i \bar{\eta}_i \to (T^n f, g) \quad \text{as} \quad k \to \infty,$$

which implies by (8.23) that (8.22) holds true for $n \geqslant 0$. Since for every λ the operator $\hat{P}(\lambda)$ is self-adjoint, we infer that (8.22) holds true for $n \leqslant 0$. It follows that $F(\lambda)$ is weakly differentiable a. e. to $\hat{P}(\lambda)$. By Pettis's theorem

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{2\pi}\hat{P}(\lambda) \quad \text{a. e. on } [0, 2\pi]$$

in the strong sense.

9. Functional calculus

This section deals with the basis of the functional calculus for contractions. For references in this matter see [7], [8], [9], [10], [37], [38], [45], [46], [48], [50], [53], and [55].

The Hilbert spaces considered are supposed to be complex. T stands for a contraction acting in H and U for the minimal unitary dilation of T acting in the dilation space K. To begin with we start with a description of some functional classes of functions analytic in the open unit disc of the complex plane. We refer in this matter to [18].

Denote by A the class of functions analytic in the unit open disc D and continuous in the closure $\bar{D}$ of D. Every function $u(z) = \sum_{n=0}^{\infty} c_n z^n$ such that

$$\sum_{n=0}^{\infty} |c_n| < +\infty \tag{9.1}$$

belongs to A. A is a commutative Banach algebra with the sup norm and the ordinary definitions of addition and multiplication. The functions belonging to A are those which are analytic in D and have continuous boundary values on the unit circle. Every function in A is uniquely dedetermined by its boundary values. The maximum principle shows that A may be identified with the algebra of these continuous functions, which are defined on C and are boundary values of analytic functions in D.

The Lebesgue spaces on C are defined as usual and denoted by L^p $(1 \leqslant p \leqslant +\infty)$. The norm in L^∞ is defined as

$$\|u\|_\infty = \operatorname*{ess\,sup}_C |u(z)|.$$

The Hardy space H^p is by definition the closure of A in the metric of L^p. H^∞ is the set of essentially bounded functions in H^1. H^∞ becomes a commutative Banach algebra with the norm as the essential supremum and multiplication defined in the ordinary way. The Hardy space H^p consists exactly of functions which are boundary values almost everywhere on C of functions which are analytic in D and

$$\|u\|_p = \sup_{0<r<1} \int_0^{2\pi} |u(re^{i\lambda})|^p d\lambda \quad \text{if} \quad 1 \leqslant p < +\infty,$$

$$\|u\|_\infty = \sup_D |u(z)| \quad \text{if} \quad p = +\infty.$$

H^p may be identified with the Banach space of functions which are analytic in D and have p-summable boundary values, provided $1 \leqslant p < +\infty$. If $u \in H^p$ $(1 \leqslant p < +\infty)$, then

$$\|u\|_p = \int_0^{2\pi} |u(e^{i\lambda})|^p d\lambda, \quad \text{where} \quad u(e^{i\lambda}) = \lim_{r\to 1-} u(re^{i\lambda})$$

for almost all λ, namely except the points of a Borel set of Lebesgue measure zero. If $u \in H^p$, then $u_r(z) = u(rz) \in A \subset H^p$ for $0 \leqslant r < 1$ and $\|u - u_r\|_p \to 0$ as $r \to 1-$ $(1 < p < +\infty)$. If $u \in A$, then $u_r(z) \to u(z)$ uniformly on D. The class H^2 is a Hilbert space under the inner product defined by

$$(f, g) = \int_0^{2\pi} f(e^{i\lambda})\overline{g(e^{i\lambda})}\, dm(\lambda), \quad f, g \in H^2.$$

Notice that $H^p \subset H^1$ for $1 \leqslant p \leqslant +\infty$. Every function $u \neq 0$ in H^1 admits a factorization (unique up to a constant of modulus 1) of the form $u(z) = u^i(z)u^0(z)$, where $|u^i(e^{i\lambda})| = 1$ for almost all λ and

$$u^0(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\lambda}+z}{e^{i\lambda}-z} \log|u(e^{i\lambda})|\, d\lambda\right). \tag{9.2}$$

Since the logarithm of the modulus of function in H^1 not identically zero is summable over $[0, 2\pi]$, the integral on the right in (9.2) makes sense. The function $u \in H^p$ if and only if the corresponding factor $u^0 \in H^p$.

The function in H^1 which has boundary values of modulus 1 a. e. is called an *inner function*. The function $h \in H^1$ which is expressible in the form

$$h(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\lambda}+z}{e^{i\lambda}-z} \log k(e^{i\lambda})\, d\lambda\right)$$

with k and $\log$ both in L^1 is called an *outer function*. This formula implies that $h(e^{i\lambda}) = |k(e^{i\lambda})|$ a. e.

It follows from the factorization of $u \in H^1$ that every function in H^1 can be written in an essentialy unique manner as the product of an inner function and an outer one. u^i is called the *inner factor* of u, u^0 the *outer factor* of u. The geometrical definition of the outer function in H^2 given in section 8 coincides with the one given above, namely $f(z) \in H^2$ is outer in the above sense if and only if $z^n f(z)$ $(n \geqslant 1)$ span H^2.

The final property we mention here is the following one: the non-negative summable function $k(e^{i\lambda})$ on C is of the form $k(e^{i\lambda}) = |f(e^{i\lambda})|^2$, f being an outer function in H^2 if and only if $\log k(e^{i\lambda})$ is summable in λ over $[0, 2\pi]$.

Let $u(z) \in A$. Then u can be approximated uniformly in $\bar{D}$ by polynomials in z. Suppose that

$$q(z) = \sum_{k|0}^{n} a_k z^k.$$

We define the operator

$$q(T) = \sum_{k|0}^{n} a_k T^k.$$

If U is the unitary dilation of T, then

$$q(T) = Pq(U).$$

The mapping $\varphi : q \to q(T)$ is a linear mapping and

$$\varphi(q_1 q_2) = \varphi(q_1)\varphi(q_2) \tag{9.3}$$

for polynomials. In other words, φ is multiplicative on polynomials. The operator U being unitary, we conclude by the spectral theory for unitary operators that

$$|q(U)| \leqslant \sup_{\bar{D}} |q(z)| = \sup_{\sigma} |q(z)| = \varrho(q).$$

Thus $|q(T)| \leqslant \varrho(q)$. Given a function $u \in A$, we choose the sequence q_n of polynomials such that

$$q_n \to u \text{ uniformly on } \bar{D}. \tag{9.4}$$

It follows that the sequence $q_n(T)$ converges in the uniform operator topology and the limit does not depend on a special choice of q_n satisfying (9.4). We define

$$u(T) = \lim_{n\to\infty} q_n(T). \tag{9.5}$$

We are now able to extend the map φ by the formula

$$\varphi(u) = u(T) \qquad (u \in A)$$

and just conclude by (9.3) that the extended map satisfies

$$\varphi(uv) = \varphi(u)\varphi(v) \quad \text{for} \quad u, v \in A. \tag{9.6}$$

Also

$$|u(T)| \leqslant \sup_{\bar{D}} |u(z)|. \tag{9.7}$$

The algebra A has a natural involution, namely $u^*(z) = \overline{u(\bar{z})}$. If u is a polynomial, then

$$u(T)^* = u^*(T^*). \tag{9.8}$$

Extending (9.8) by a limit passage, we deduce that (9.8) holds for all of $u \in A$.

Using (9.7) we get

(9.9) *If $u_n \to u$ uniformly on $\bar{D}$ ($u_n \in A$), then $\varphi(u_n) = u_n(T) \to u(T) = \varphi(u)$ in the uniform operator topology.*

If ψ is a homomorphism of A into a certain operator algebra which includes I and T, and ψ satisfies

$$\psi(q) = \begin{cases} T & \text{if} \quad q(z) = z, \\ I & \text{if} \quad q(z) \equiv 1, \end{cases} \tag{9.10}$$

$$|\psi(u)| \leqslant \sup_{\bar{D}} |u(z)| \quad \text{for} \quad u \in A, \tag{9.11}$$

then necessarily $\psi = \varphi$. Indeed, φ and ψ coincide on polynomials and by (9.9) and (9.11) they must be equal on all $u \in A$. Summing up, we

conclude that there is a unique homomorphism of A onto a certain operator algebra satisfying (9.10) and (9.11).

We would like to extend φ onto some larger algebras than A. Here is one of the possible methods (see [48]).

Given the contraction T, we take its canonical decomposition $T = T_1 \oplus T_2$, where T_1 is the unitary part of T and T_2 is the completely non-unitary part of T. Denote by $E(\sigma)$ the spectral measure of the minimal unitary dilation of T.

Let $u \in H^\infty$. Then

$$u(re^{it}) \to u(e^{it}) \quad \text{for} \quad r \to 1- \tag{9.12}$$

except a Borel set C_u of Lebesgue measure zero. Since C_u is a Borel set, it is measurable with respect to the spectral measure $E(\sigma)$. Denote by H_T^∞ the class of those $u \in H^\infty$ for which

$$E(C_u) = 0. \tag{9.13}$$

H_T^∞ constitutes a subalgebra of H^∞ and $H_T^\infty = H_{T_1}^\infty$ because the spectrum of dilation of T_2 is absolutely continuous. If $u \in H_T^\infty$ then $u_r(z) \in A$ and $u_r(T)$ is well defined and

$$|u(re^{it})| \leqslant \|u\|_\infty \quad \text{for} \quad 0 < r \leqslant 1 \text{ and } e^{it} \notin C_u. \tag{9.14}$$

We shall prove that $\text{s-}\lim_{r \to 1-} u_r(T)$ exists and

$$\lim_{r \to 1-} u_r(T) f = Pu(U) f \quad \text{for} \quad f \in H. \tag{9.15}$$

This being done, we write by definition

$$u(T) f = Pu(U) f \quad \text{for} \quad f \in H \tag{9.16}$$

with $u \in H_T^\infty$. By the spectral theory, $u(U)$ is well defined by

$$u(U) = \int_0^{2\pi} u(e^{it}) dE(\lambda).$$

Next, for $f \in K$ (dilation space)

$$|(u(U) - u_r(U))f|^2 = \int_0^{2\pi} |u(e^{it}) - u(re^{it})|^2 d(E(\lambda)f, f). \tag{9.17}$$

By (9.13), $(E(C_u)f, f) = 0$ and by (9.12) and (9.14)

$$|u(e^{it}) - u(re^{it})|^2 \to 0,$$

$$|u(e^{it}) - u(re^{it})|^2 \leqslant \varrho < +\infty,$$

almost everywhere with respect to the scalar measure $(E(\sigma)f, f)$. It follows by the Lebesgue theorem that the limit passage in (9.17) is possible, and consequently

$$|Pu(U)f - u_r(T)f|^2 \to 0 \quad \text{for} \quad f \in H,$$

which establishes the proof of (9.15).

We say that $u_n \to u$ ($u_n, u \in H_T^\infty$) *boundedly* if u_n is bounded in the H^∞ norm and $u_n(e^{i\lambda}) \to u(e^{i\lambda})$ almost everywhere with respect to $E(\sigma)$. A slight modification of the reasoning involved in the proof of (9.5) shows that if $u_n \to u$ boundedly, then

$$\lim_{n\to\infty} u_n(T) = u(T), \qquad u_n, u \in H_T^\infty. \tag{9.18}$$

Since the map $u \to u(T)$ defined by (9.16) restricted to A is a homomorphism established by (9.5), we conclude by (9.15) that $u(T)v(T) = (uv)(T)$ for $u, v \in H_T^\infty$. Next, by (9.16) and by the spectral theory,

$$|u(T)| \leqslant \|u\|_\infty. \tag{9.19}$$

Also (9.16) implies that if $\operatorname{Re} u(z) \geqslant 0$ for $|z| < 1$, then $\operatorname{Re} u(T) \geqslant 0$ for $u \in H_T^\infty$ (Heinz theorem). Indeed, if $f \in H$, then

$$\big(\operatorname{Re} u(T)f, f\big) = \operatorname{Re}\big(u(U)f, f\big) = \int_0^{2\pi} \operatorname{Re} u(e^{i\lambda})\, d\big(E(\lambda)f, f\big) \geqslant 0.$$

It is a simple matter to verify that if $u \in H_T^\infty$, then $u^*(z)$ belongs to $H_{T^*}^\infty$ and $u(T)^* = u^*(T^*)$ for such u.

We can summarize our discussion in the following theorem:

THEOREM 9.1. *There is a unique homomorphism φ of H_T^∞ onto a certain operator algebra such that*

$$\varphi(q) = \begin{cases} T & \text{if} \quad q(z) \equiv z, \\ I & \text{if} \quad q(z) \equiv 1, \end{cases} \tag{9.20}$$

$$\lim_{n\to\infty} \varphi(u_n) = \varphi(u) \text{ if } u_n \to u \text{ boundedly in } H_T^\infty. \tag{9.21}$$

The homomorphism is given by the formula

$$\varphi(u)f = Pu(U)f = u(T)f \qquad (f \in H)$$

and the following conditions hold true:

$$|u(T)| \leqslant \|u\|_\infty, \tag{9.22}$$

$$u(T)^* = u^*(T^*), \tag{9.23}$$

$$\text{if } \operatorname{Re} u(z) \geqslant 0 \ (u \in H_T^\infty), \text{ then } \operatorname{Re} u(T) \quad 0. \tag{9.24}$$

Proof. Only the uniqueness requires a proof. Thus, let ψ be another homomorphism satisfying (9.20) and (9.21). Then ψ coincides with φ on all polynomials and by (9.21) on their uniform limits, which implies that $\psi(u) = \varphi(u)$ for $u \in A$. Given $u \in H_T^\infty$, then $u_r \to u$ boundedly and $u_r \in A$. Then

$$\varphi(u) = \lim_{r \to 1-} \varphi(u_r) = \lim_{r \to 1-} \psi(u_r) = \psi(u)$$

by (9.21), q. e. d.

As a by-product of our considerations we get

$$u(T)f = \int_0^{2\pi} u(e^{i\lambda})\, dF(\lambda) f \qquad (u \in H_T^\infty, f \in H), \tag{9.25}$$

where F stands for the spectral-harmonic measure of T. Formula (9.25) can be used as a point of departure for functional calculus. Such an approach was developed in [38]. If C_u is at most countable and no eigenvalues of T on C are in C_u, then obviously $E(C_u) = 0$. This results from theorem 7.4. Under these circumstances the functional calculus was developed in a preliminary form in [45].

One can show that if T is normal, the above-given definition of $u(T)$ coincides with the usual definition of functional calculus for normal operators. Notice, however, that if T is an isometric operator then $T^n \subset U^n$ for $n \geqslant 0$, which implies that $u(T) \subset u(U)$ for $u \in H_T^\infty$. If T has the spectral-harmonic measure absolutely continuous with regard to the Lebesgue measure, then $H_T^\infty = H^\infty$. This certainly happens when T is completely non-unitary.

The next theorem concerns a delicate spectral relation.

Theorem 9.2 ([37]). *Suppose T is a contraction and $u \in H_T^\infty$. Then*

$$u\big(S(T) \cap D\big) \subset S\big(u(T)\big).$$

Proof. Let $u \in H_T^\infty$ and consider $z_0 \in D$. We define

$$v(z) = \frac{u(z) - u(z_0)}{z - z_0} \qquad (z \neq z_0),$$

$$v(z_0) = u'(z_0).$$

For sufficiently small $\varrho > 0$

$$S_\varrho = \{z : |z - z_0| < \varrho\} \subset D$$

and

$$|v(z)| \leqslant \frac{2\|u\|_\infty}{\varrho}$$

outside of S_ϱ. Since $v(z)$ is continuous in S_ϱ, $v(z)$ is bounded and analytic in D. Obviously $v \in H_T^\infty$.

It follows from the equality $(z-z_0)v(z) = u(z)-u(z_0)$ that

$$v(T)(T-z_0I) = u(T)-u(z_0)I. \tag{9.26}$$

If $z_0 \in S_a(T)$, then for some sequence f_n, $|f_n| = 1$, $|Tf_n - z_0 f_n| \to 0$. Using (9.26) we infer that

$$[u(T)-u(z_0)I]f_n \to 0,$$

which proves that $u(z_0) \in S\big(u(T)\big)$.

If $z_0 \in S_r(T)$ then, since

$$R[u(T)-u(z_0)I] = R[(T-z_0I)v(T)] \subset R[T-z_0I],$$

we infer that

$$u(z_0) \in S_r\big(u(T)\big).$$

Consequently

$$u\big(S(T) \cap D\big) = u\big(\big(S_a(T) \cup S_r(T)\big) \cap D\big) \subset S_a\big(u(T)\big) \cup S_r\big(u(T)\big) \subset S\big(u(T)\big),$$

q.e.d.

COROLLARY. *If $u \in A$, then exactly* (see [8])

$$S\big(u(T)\big) = u\big(S(T)\big) \quad \text{and} \quad S_p\big(u(T)\big) = u\big(S_p(T)\big).$$

An immediate consequence of theorem 9.2 is the following one [37]:

THEOREM 9.3. *If $S(T)$ has a limit point in D, then the homomorphism φ of theorem* 9.1 *is an isomorphism.*

Proof. Suppose that $\varphi(u) = 0$ for some $u \in H_T^\infty$. By theorem 9.2, $u\big(S(T) \cap D\big) \subset S\big(\varphi(u)\big) = \{0\}$. It follows that $u(z) = 0$ on $S(T) \cap D$. Since the last set has a limit point in D and u is analytic, we get $u(z) \equiv 0$ as was to be proved.

Suppose that $u \in H^\infty$. Denote by C_u^0 the set of those points $e^{i\lambda}$ where the limit $\lim_{r \to 1-} u_r(e^{i\lambda})$ does not exist or equals zero. C_u^0 is a Borel set of measure zero, unless $u = 0$. Let E_T be the class of outer functions belonging to H_T^∞ such that $E(C_u^0) = 0$. It is easy to check that if $u \in E_T$, then $u^* \in E_T$. Using theorem 8.3 we verify that if T_1 is the unitary part of T, then $E_T = E_{T_1}$. If T is completely non-unitary then, by the same arguments, E_T = the class of all outer functions in H^∞. This last class is denoted by E.

Next we prove (see [48])

LEMMA 9.1. *Let T be a completely non-unitary contraction in H and suppose that u is an outer function in H^∞, i. e. $u \in E$. Then $u(T)^{-1}$ exists.*

Proof. Let $f \in H$ and $f \neq 0$ and assume that $u(T)f = 0$. Then $\big(T^n u(T)f, f\big) = 0$ for $n = 0, 1, 2, \ldots$ and consequently, by (9.16).,

$$\int_0^{2\pi} e^{in\lambda} u(e^{i\lambda})\, d\big(E(\lambda)f, f\big) \quad 0. \tag{9.27}$$

It follows from theorem 8.2 and lemma 8.1 that

$$d\big(E(\lambda)f, f\big) = f(e^{i\lambda})\overline{f(e^{i\lambda})}\,d\lambda,$$

where $f(e^{i\lambda})$ are boundary values of an outer function $f \in H^2$. Since u is outer in H^∞, the product

$$u(z)f(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\lambda}+z}{e^{i\lambda}-z}\log|u(e^{i\lambda})f(e^{i\lambda})|\,d\lambda\right)$$

is an outer function in H^2. Equalities (9.27) imply that $z^n u(z)f(z)$ and $f(z)$ are orthogonal in H^2 for $n \geqslant 0$. Since uf is outer, $z^n u(z)f(z)$ span H^2 and consequently $f(z) = 0$, which is impossible for $f \neq 0$. This contradiction proves the assertion of the lemma.

COROLLARY. *The converse of the lemma holds true in the following sense: if $u \in H^\infty$ and $u(T)^{-1}$ exists for every completely non-unitary T, then u is an outer function* (see [48]).

Lemma 9.1 will be used in the proof of the following theorem (see [48]):

THEOREM 9.4. *Let T be a contraction in H and suppose that $u \in E_T$. Then $u(T)^{-1}$ exists, $R[u(T)]$ is dense in H and*

$$[u(T^{-1})]^* = [u^*(T^*)]^{-1}. \tag{9.28}$$

Proof. Let T_1 be the unitary part of T and T_2 the completely non-unitary part of T. Then

$$u(T) = u(T_1) \oplus u(T_2).$$

$u(T_2)^{-1}$ exists by lemma 9.1. Since $E(C_u^0) = 0$ and $u \in E_T$ the limit

$$\lim_{r\to 1-} \frac{1}{u_r(e^{i\lambda})}$$

is finite for $e^{i\lambda} \in C - C_u^0$. It follows by spectral theory for unitary operators that $u(T_1)^{-1}$ exists, for

$$u(T_1)^{-1} = \int_0^{2\pi} \frac{1}{u(e^{i\lambda})}\,dE'(\lambda),$$

where E' is the spectral function of T_1. Hence, the inverse $u(T)^{-1} = u(T_1)^{-1} \oplus u(T_2)^{-1}$ exists. By similar arguments one proves that $[u^*(T^*)]^{-1}$ exists.

Suppose now that $h \perp u(T)H$. Then $u(T)^* h = 0$. Since $u(T)^* = u^*(T^*)$ and $[u^*(T^*)]^{-1}$ exists, we infer that $h = 0$. Consequently $R[u(T)]$ is dense in H. Summing up we conclude that $u(T)^{-1}$, $u(T)^*$ and $[u(T)^*]^{-1}$ exist. By the general theorems on operators it follows that (9.28) holds true.

There is no difficulty in considering unbounded operators which are functions of contraction. We will consider here functions which are of the form

$$w(z) = u(z)/v(z), \tag{9.29}$$

where $u \in H_T^\infty$ and $v \in E_T$. The class of functions of that type is denoted by H_T^∞/E_T. Since $v(T)^{-1}$ exists, we define

$$w(T)f = [v(T)]^{-1}u(T)f$$

for those f for which $u(T)f \in R[v(T)]$. Now, the first point is to check that the definition of $w(T)$ does not depend on u and v in (9.29) but merely on w. This can be done as follows.

Let $w = u/v = u'/v'$, where $u, u' \in H_T^\infty$, $v, v' \in E_T$. Then $u(T)v'(T) = v(T)u'(T)$. Suppose that $u(T)f = v(T)g$ for some necessarily unique g. It follows that

$$u(T)v'(T)f = v'(T)u(T)f = v'(T)v(T)g$$

and since $u(T)v'(T) = v(T)u'(T)$ and $v'(T)v(T) = v(T)v'(T)$, we get

$$v(T)u'(T)f = v(T)v'(T)g.$$

But $v(T)^{-1}$ exists. Hence $u'(T)f = v'(T)g$, i. e.

$$u'(T)f \in R[v'(T)] \quad \text{and} \quad v'(T)^{-1}u'(T)f = g.$$

It follows that the domains of $v(T)^{-1}u(T)$, $v'(T)^{-1}u(T)$ are equal and so do the operators themselves. This proves that the definition is well posed. We will now prove (see [48])

THEOREM 9.5. *Let $w \in H_T^\infty/E_T$. Then the operator $w(T)$ is a linear closed operator defined on a dense linear manifold in H. It commutes with every bounded linear operator S which commutes with T, that is:*

(9.30) *If $TS = ST$, then $w(T)S \supseteq Sw(T)$.*

In particular,

$$v(T)^{-1}u(T) \supseteq u(T)v(T)^{-1}, \quad u \in H_T^\infty, v \in E_T. \tag{9.31}$$

Proof. Write $u(T) = V$, $v(T) = Q$. Then $w(T) = Q^{-1}V$ is obviously linear. Let $f_n \in D[w(T)]$, $f_n \to f$, $w(T)f_n \to g$. Then $Vf_n \to Vf$. Also $Vf_n = Q(Q^{-1}Vf_n) \to Qg$. Hence $Vf = Qg$, which completes the proof that $w(T)$ is closed.

Using the above-mentioned notation we get.

$$Q^{-1}V \supseteq Q^{-1}VQQ^{-1} = Q^{-1}QVQ^{-1} = VQ^{-1},$$

which proves (9.31).

Since Q has by theorem 9.4 a dense range, we infer by (9.31) that $Q^{-1}V$ is densely defined.

Since (9.30) obviously holds for $u, v \in A$, the simple limit passage proves that it holds true for $w = u/v \in H_T^\infty/E_T$.

Theorem 9.5 gives an illustration of what can be done by the method of functional calculus for contractions. For a more detailed study of this matter, the application of this calculus to fractional powers of accretive operators, dissipative systems, etc. see [48].

An interesting but elementary application of the calculus presented here is the investigation of one-parameter continuous semi-groups of contractions in Hilbert spaces. We deal with this subject in the next section. For other selected topics in functional calculus we refer to [7], [37], [45], [50].

10. One-paramater semi-groups of contractions

In this section we adopt the notation of section 9. The spaces considered are complex. By a semi-group $T(t)$ we mean a one-parameter semi-group of contractions, strongly continuous at zero. More precisely $T(t)$ satisfies:

$$\lim_{h \to 0+} T(h)f = f \quad \text{for every } f \in H, \tag{10.1}$$

$$T(t+s) = T(t)T(s) \quad \text{for} \quad t, s \geqslant 0, \tag{10.2}$$

$$|T(t)| \leqslant 1 \quad \text{for} \quad t \geqslant 0. \tag{10.3}$$

For references in the general theory of semi-groups of linear bounded operators see [17].

We will give a brief outline of some basic facts concerning semi-groups which satisfy (10.1), (10.2) and (10.3). First of all, the infinitesimal generator A of $T(t)$ is defined by

$$Af = \lim_{h \to 0+} \frac{T(h) - I}{h} f$$

for those f for which the limit on the right exists. The domain $D[A]$ is dense in H, A is a closed operator and the bounded inverse $(A-I)^{-1}$ exists and $|(A-I)^{-1}| \leqslant 1$. Notice that if $f \in D[A]$, then $(Af, f) + (f, Af) \leqslant 0$. Indeed, since $|T(h)f|^2 \leqslant |f|^2$, we get

$$\frac{(T(h)f - f, T(h)f) + (f, T(h)f - f)}{h} \leqslant 0 \quad \text{for} \quad h > 0,$$

which proves that $\mathrm{Re}(f, Af) \leqslant 0$, q. e. d. Let f be an arbitrary vector and define $g = (A-I)^{-1}f$. Then $g \in D[A]$ and $f = Ag \quad g$. The operator T being defined by

$$T = (A+I)(A-I)^{-1}, \tag{10.4}$$

we get

$$|Tf|^2 = |(A+I)g|^2 = |Ag|^2 + |g|^2 + 2\mathrm{Re}(Ag, g).$$

Since $\mathrm{Re}(Ag, g) \leqslant 0$,

$$|Tf|^2 \leqslant |Ag|^2 + |g|^2 - (Ag, g) - (g, Ag) = |(A-I)g|^2 = |f|^2.$$

Consequently, T is a contraction.

The operator T defined by (10.4) is called the *cogenerator* of $T(t)$ (see [45]). Notice that 1 is not an eigenvalue of cogenerator T, for $Th = h$ implies by (10.4) that $h = h + 2(A-I)^{-1}h$ and this shows that $(A-I)^{-1}h = 0$ and thus $h = 0$. If the cogenerator is a strict contraction, that is if $|T| < 1$, then A is bounded. Indeed, in this case

$$|(A+I)f| \leqslant |T|\,|(A-I)f|,$$

which implies

$$|Af| \leqslant \frac{1+|T|}{1-|T|}|f|, \quad f \in D[A].$$

A being closed and densely defined we infer that A is defined everywhere and bounded.

Next, if T is a contraction and $1 \notin S_p(T)$, then, by theorem 7.1, $(T-I)^{-1}$ is densely defined and the operator

$$A = (T+I)(T-I)^{-1}$$

is closed and densely defined. This A is expected to be the infinitesimal generator of a semi-group which has T as its cogenerator. This and something more is true, and now we will investigate in detail the interplay between the semi-group, its generator and cogenerator.

Let us define

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right)$$

for $|z| < 1$ and $t \geqslant 0$,

The number t being fixed, the function $e_t(z)$ is bounded for $|z| \leqslant 1$ and $z \neq 1$, namely $|e_t(z)| \leqslant 1$. Since the only singularity of $e_t(z)$ occurs for $z = 1$, $e_t(z) \in H_T^\infty$, where T is an arbitrary contraction such that $1 \notin S_p(T)$. We argue as follows: if $1 \notin S_p(T)$, then, by theorem 7.4, 1 is not in the point spectrum of the minimal unitary dilation U of T. If follows that the spectral measure of U vanishes on $\{1\}$ and we are done. It follows

from the results of section 9 that if $1 \notin S_p(T)$, then $e_t(T)$ makes sense. From what we know about cogenerators we infer that $e_t(T)$ is well defined if T is a cogenerator of a semi-group.

We have the following theorem (see [45]):

THEOREM 10.1. *Let T be a cogenerator of the semi-group $T(t)$. Then*

$$T(t) = e_t(T) \quad (t \geqslant 0), \tag{10.5}$$

$$\lim_{s \to 0+} [T(s)-(1-s)I][T(s)-(1+s)I]^{-1} = T. \tag{10.6}$$

Proof. In the proof we apply theorem 9.1. Because of

$$e_t(z)\, e_s(z) = e_{t+s}(z), \qquad |e_t(z)| \leqslant 1$$

and

$$\lim_{t \to 0+} e_t(z) = 1 \quad \text{for} \quad z \in \overline{D} \text{ and } z \neq 1,$$

we infer that $e_t(T)$ form a semi-group of contractions. Denote by A' the generator and by T' the cogenerator of $e_t(T)$.

Let us consider the function

$$u_s(z) = \frac{e_s(z)-1+s}{e_s(z)-1-s}, \quad s > 0. \tag{10.7}$$

It is a simple matter to check that $u_s \in H^\infty_T$ and

$$|u_s(z)| \leqslant 1, \quad \lim_{s \to 0+} u_s(z) = z$$

for $z \in D$, $z \neq 1$. Using theorem 9.1 again, we infer

$$|u_s(T)| \leqslant 1 \quad (s > 0), \tag{10.8}$$

$$\lim_{s \to 0+} u_s(T) = T. \tag{10.9}$$

On the other hand, (10.7) implies that

$$u_s(T)[e_s(T)-(1+s)I] = [e_s(T)-(1-s)I].$$

Since $e_s(T)$ is a contraction,

$$u_s(T) = [e_s(T)-(1-s)I][e_s(T)-(1+s)I]^{-1} \quad \text{for} \quad s > 0. \tag{10.10}$$

Suppose now that $f \in D[A']$. Then, by (10.9), the limit passage $s \to 0+$ in the formula

$$u_s(T)\left[\frac{e_s(T)-(1+s)}{s}\right]f = \left[\frac{e_s(T)-(1-s)I}{s}\right]f$$

proves that $T(A'-I)f = (A'+I)f$. Hence $T = (A'+I)(A'-I)^{-1}$, which implies that $T = T'$ It follows that $A = A'$. But the generator determines

the semi-group uniquely. This establishes the proof of (10.5). The relation (10.6) is a consequence of (10.9), (10.10) and (10.5).

The arguments used in the above proof show that if 1 is not in the point spectrum of the contraction T, then T is a cogenerator of the unique semi-group, namely $e_t(T)$. The generator of this semi-group is determined by the relation $A = (T+I)(T-I)^{-1}$.

We will now investigate the case of unitary T. First we show that

$$\left|\frac{2-\varepsilon}{2+\varepsilon}\right| \leqslant \left|\frac{1-(1-\varepsilon)z}{(1+\varepsilon)z-1}\right| \tag{10.11}$$

for $|z| \leqslant 1$ and $0 < \varepsilon < 2$. We write

$$\varrho_\varepsilon(z) = \frac{(1+\varepsilon)z-1}{1-(1-\varepsilon)z} \tag{10.12}$$

If $0 < \varepsilon < 2$, then $\varrho_\varepsilon(z)$ is regular in the neighbourhood of $\bar{D}$. Further

$$\varrho(x) = |\varrho_\varepsilon(z)|^2 = \frac{(1+\varepsilon)^2-2(1+\varepsilon)x+1}{(1-\varepsilon)^2-2(1-\varepsilon)x+1}, \quad |z| = 1,\ z = x+iy,$$

and $\varrho(x)$ attains its maximum on $[-1, 1]$ at $x = -1$. It follows that

$$|\varrho_\varepsilon(z)| \leqslant \frac{2+\varepsilon}{2-\varepsilon} \quad \text{for} \quad |z| \quad 1 \ (0 < \varepsilon < 2),$$

which proves (10.11).

The next theorem is (see [45])

THEOREM 10.2. *Let Q be a contraction in the space K and suppose that $1 \notin S_p(Q)$. Then Q is unitary if and only if the semi-group $c_t(Q)$, $t \geqslant 0$, consists of unitary operators. Under these circumstances, if $Q(t)$, $-\infty < t < +\infty$, is a unitary group (necessarily unique) such that $Q(t) = c_t(Q)$ for $t \geqslant 0$, then*

$$\bigvee_{-\infty<t<+\infty} Q(t)H = \bigvee_{0}^{+\infty} Q^n H$$

for every subspace of K.

Proof. Suppose that Q is unitary. Then, if 1 is not in the point spectrum of Q,

$$c_s(Q) = \int_0^{2\pi} \exp\left(s\frac{e^{i\lambda}+1}{e^{i\lambda}-1}\right) dZ(\lambda),$$

where $Z(\lambda)$ is the spectral function of Q. The mapping a

$$a(z) = -i\frac{z+1}{z-1} \quad (|z| = 1,\ z \neq 1)$$

of $C-\{1\}$ onto the real line gives the representation

$$e_s(Q) = \int_{-\infty}^{+\infty} e^{is\lambda}\, dG(\lambda),$$

where $G(\lambda)$ is a spectral function on $(-\infty, +\infty)$. It follows then by Stone's theorem that $e_s(Q)$ is unitary.

Suppose now that $e_s(Q)$ are unitary for $s \geqslant 0$. Since $e_s(Q)$ form a semi-group the operators $e_s(Q)$ and $e_t(Q)$ commute and are normal. We conclude therefore that the operators $u_s(Q)$, $u_s(Q)-u_t(Q)$ with $u_s(z)$ defined by (10.7) are normal. Hence

$$|u_s(Q)h| = |u_s(Q)^* h| \tag{10.13}$$

and

$$|[u_s(Q)-u_t(Q)]h| = |[u_s(Q)^*-u_t(Q)^*]h| \tag{10.14}$$

for every h and $s, t \geqslant 0$. On the other hand, by (10.6)

$$u_s(Q) \underset{s\to 0+}{\to} Q \text{ strongly} \tag{10.15}$$

and consequently

$$u_s(Q)^* \underset{s\to 0+}{\to} Q^* \text{ weakly}. \tag{10.16}$$

On the other hand, by (10.14) and (10.15),

$$\lim_{s,t\to 0+} |[u_s(Q)^*-u_t(Q)^*]h| = 0$$

for every h. Using (10.16) we deduce therefore that $u_s(Q)^* \underset{s\to 0+}{\to} Q^*$ strongly. The limit passage in (10.13) proves that $|Qh| = |Q^*h|$, which implies that Q is normal.

Since $u_s(Q)$ is a contraction

$$|h|^2 \geqslant |u_s(Q)h|^2 = \big(u_s(Q)^* u_s(Q)h, h\big) = \int_0^{2\pi} |\varrho_s^{-1}(e^{i\lambda})|^2\, d\big(Z'(\lambda)h, h\big),$$

where Z' is the spectral measure of $e_s(Q)$ and ϱ_s is defined by (10.12). Using (10.11) we get

$$|h| \geqslant |u_s(Q)h| \geqslant \frac{2-s}{2+s}\,|h| \qquad (0 < s < 1).$$

Since $u_s(Q)h \to Qh$, we infer from this inequality that $|h| = |Qh|$, which by the normality of Q proves that Q is unitary.

Let H be a subspace of K. The unitary group $Q(t)$ which coincides with $e_t(Q)$ for $t \geqslant 0$ satisfies $Q(t) = e_{-t}(Q)^*$ for $t \leqslant 0$. Write

$$K' = \bigvee_{-\infty<t<+\infty} Q(t)H.$$

It follows that K' reduces all of $e_t(Q)$ for $t \geqslant 0$. Let P' be the projection on K'. P' commutes with $e_t(Q)$ and, by formula (10.6) of theorem 10.1, P' commutes with Q and Q^*. It follows that K' reduces Q. But $H \subset K'$. Hence $Q_n H \subset K'$ for all n, which proves that

$$K'' = \bigvee_{-\infty}^{+\infty} Q^n H \subset K'$$

On the other hand, the formula

$$e_s(Q) = \lim_{r \to 1-} e_s(rQ)$$

shows that $e_t(Q)H \subset K''$ for $t \geqslant 0$. This implies that $K' \subset K''$.

Suppose now that T is the cogenerator of the semi-group $T(t)$. Let U be the minimal unitary dilation of T. The dilation space is denoted by K and let P stand for the projection of the initial space H onto K. By theorem 10.1, $T(t)f = Pe_t(U)f$ for $f \in H$. Now denote by $U(t)$ the group of unitary operators such that $U(t) = e_t(U)$ for $t \geqslant 0$. Since, by theorem 10.2, $e_t(U)$ consists of unitary operators, $U(t)$ is well defined. We may write now

$$T(t)f = PU(t)f \quad \text{for} \quad f \in H \text{ and } t \geqslant 0. \tag{10.17}$$

It follows from theorem 10.2 that the elements $U(t)f, f \in H$, $-\infty < t < +\infty$, span K if and only if

$$U^n f \text{ span } K \quad (f \in H,\ n = 0, \pm 1, \pm 2, \ldots). \tag{10.18}$$

Since U is minimal, (10.18) holds true. Hence, by (10.17), $U(t)$ is the minimal unitary dilation of $T(t)$, and therefore determined in a unique manner. Conversely, if $U(t)$ is the minimal unitary dilation of the semi-group $T(t)$ and U is the cogenerator of $U(t)$ and T the cogenerator of $T(t)$, then U is the minimal unitary dilation of T. This follows from the uniqueness of $U(t)$ and the previous arguments. We have just proved the following theorem (see [45]):

THEOREM 10.3. *The unitary operator U is the minimal unitary dilation of the cogenerator T of the semi-group $T(t)$ if and only if U is the cogenerator of the minimal unitary dilation of $T(t)$.*

Remark. One can prove that the semigroup $T(t)$ consists of normal operators if and only if the cogenerator of $T(t)$ is normal. A similar property holds true for semi-groups of selfadjoint operators. For the proof of these statements see [45].

It is obvious that if $T(t)$ consists of isometries, then the minimal unitary dilation $U(t)$ of $T(t)$ coincides with $T(t)$ on the initial space for $t \geqslant 0$. In other words, $U(t)$ is simply an extension of $T(t)$ for $t \geqslant 0$. Cooper in [6] first proved that the semi-group of isometries has an extension which is a unitary group, proving in fact the dilation theorem for one-

parameter semi-groups of isometries. The nature of Cooper's result is completed by the following theorem (see [45]):

THEOREM 10.4. *The semi-group $T(t)$ consists of isometries if and only if its cogenerator is an isometric operator.*

Proof. We noticed in section 9 that if T is an isometry and U its minimal unitary dilation, then $u(T) \subseteq u(U)$ for every function $u \in H_T^\infty$. Now, if T is the cogenerator of $T(t)$ and T is an isometry, then $T(t) = e_t(T) \subseteq e_t(U)$. Since $e_t(U)$ are unitary and therefore isometric, it follows that $T(t)$ are isometries.

If $T(t)$ are isometries, then $T(t) \subseteq U(t)$ where $U(t)$ is the minimal unitary dilation of $T(t)$. Consequently

$$[T(t)-(1-t)I][T(t)-(1+t)I]^{-1} \subseteq [U(t)-(1-t)I][U(t)-(1+t)I]^{-1}.$$

Using formula (10.6) of theorem 10.1 we infer that $T \subseteq U$. U is the minimal unitary dilation of T. Thus T must be isometric.

Remark. In general if $\mathscr{A}$ is the infinitesimal generator of the unitary dilation of the semigroup $T(t)$ and A is the infinitesimal generator of $T(t)$ and P is the projection of the initial space onto the dilation space, then it is not true that

$$Af = P\mathscr{A}f \quad \text{for} \quad f \in D[A]. \tag{10.19}$$

A simple counterexample is $T(t) = e^{-t}$ in the one-dimensional initial space H. In this case $D[\mathscr{A}] \cap H = \{0\}$. Hovewer, if for instance iA is symmetric, then $\mathscr{A}$ is simply an extension of A and (10.19) holds true in this case. This property results easily from theorem 10.4 and the well-known properties of the Cayley transform. For details see [29] and [30].

Suppose we are given a semigroup $T(t)$. Let $T(t) = T_1(t) \oplus T_2(t)$ $(t \geqslant 0)$ be the canonical decomposition of $T(t)$, $T_1(t)$ being the unitary part of $T(t)$ and $T_2(t)$ the completely non-unitary part of $T(t)$. Let $H = H_1 \oplus H_2$ be the corresponding decomposition of the space H. Let T be the cogenerator of $T(t)$, and $T = T_1 \oplus T_2$ the canonical decomposition of T, $H = H_1' \oplus H_2'$ being the corresponding decomposition of H. T_1 is the unitary part of T and T_2 the completely non-unitary one. The relationship between these canonical decompositions is given by the following theorem (see [46]):

THEOREM 10.5. *We have $H = H_1'$, $H_2 = H_2'$, and T_i is the cogenerator of $T_i(t)$ for $i = 1, 2$.*

Proof. If $U(t)$ is the minimal unitary dilation of $T(t)$, then, by theorem 5.1,

$$H_1 = \bigcap_{-\infty<t<+\infty} U(t)H \quad \text{and} \quad H_2 = H \ominus H_1.$$

Denote by T_i' the cogenerator of $T_i(t)$. Certainly $T = T_1' \oplus T_2'$. Since $T_1(t)$ consists of unitary operators, theorem 10.2 implies that T_1' is unitary in H_1. We will show that T_2' is completely non-unitary in H_2.

Let H'' be a subspace of H_2 and suppose that H'' reduces T_2' and the restriction of T_2' to H'' is unitary. Then H'' reduces $T_2(t) = e_t(T_2')$ for $t \geqslant 0$ and the restrictions of $T_2(t)$ to H'' are unitary. We infer that $H'' \subset H_1$. It follows that $H'' = \{0\}$, which completes the proof.

11. Sz.-Nagy-Brehmer dilations

This section deals with some constructions of unitary dilations of families of contractions. The first results in this field were established in [42]. For other references see [3], [14] [15], [16], [44]. We will give a brief outline of these results. In the method we follow here [15]. The Hilbert spaces under consideration have real, complex or quaternionic fields of scalars.

Suppose that J is a linearly ordered set of indices α. Let us denote by $\tilde{J}$ the set of integer-valued functions $n(\alpha)$ defined on J such that $n(\alpha) \neq 0$ for only a finite number of α. We write $n \geqslant 0$ if $n(\alpha) \geqslant 0$ for $\alpha \in J$. We write

$$n^+(\alpha) = \max\big(n(\alpha), 0\big), \qquad n^-(\alpha) = \max\big(-n(\alpha), 0\big).$$

We say that $m, n \in \tilde{J}$ *are positive disjoint if* $(m-n)^+ = m$ *and* $(n-m)^+ = n$.

If T is an operator on H, then $T(i)$ stands for T^i if $i \geqslant 0$ and for T^{*-i} if $i \leqslant 0$, i being an integer. Suppose we are given an operator-valued function T_α defined on J. We define a new operator-valued function for $n \geqslant 0$, $n \in \tilde{J}$, by formula

$$T[n] = T_{\alpha_1}^{n(\alpha_1)} \dots T_{\alpha_r}^{n(\alpha_r)} \tag{11.1}$$

where $\alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_r$ is the sequence of all those α for which $n(\alpha) \neq 0$. $T(0) = I$ by convention.

$\tilde{J}$ forms an abelian group under addition

$$(m+n)(\alpha) = m(\alpha) + n(\alpha), \qquad \alpha \in J \tag{11.2}$$

The subset $\tilde{J}_+$ of $\tilde{J}$ consisting of $n \geqslant 0$ forms a semi-group in $\tilde{J}$. Formula (11.1) defines $T[n]$ only for $n \in \tilde{J}_+$. Suppose for a moment that T_α commute. Then

$$T[m+n] = T_{\alpha_1}^{m(\alpha_1)+n(\alpha_1)} \dots T_{\alpha_r}^{m(\alpha_r)+n(\alpha_r)}, \tag{11.3}$$

where α_i are those α for which $m(\alpha)+n(\alpha) \neq 0$. Since T_α commute, the right-hand side of (11.3) equals $T[n]T[m]$. Thus $T[n]$ is a representation of the semi-group $\tilde{J}_+$.

Since the group $\tilde{J}$ does not satisfy condition (R) (see section 3) the definition of $T[n]$ cannot be automatically extented on all of $\tilde{J}$ when starting from $\tilde{J}_+$. However, in order to be able to apply the dilation theorem of section 3 with $G = J$ we should have to deal with such an extension $T(n)$ of $T[n]$ that

$$T(-m) = T(m)^* \quad \text{for all } m \in \tilde{J} \tag{11.4}$$

For any such extension we have $(T(m))^* = T(-m)$ for $m \leqslant 0$. It is thus seen that this property can serve as the definition of the desired extension at least for $m \leqslant 0$. The question now arises what happens in the case where m takes on positive as well negative values. We may proceed as follows.

Since $m = m^+ - m$ for $m \in \tilde{J}$ we extend the definition of $T[m]$ on all of J by the formula

$$T(m) = (T[m^-])^* T[m^+], \tag{11.5}$$

where $T[n]$ is defined by (11.1). Then $T(m)$ is well defined. Notice that the $T(n)$ just defined satisfies (11.4). Remark that the extension given by (11.5) is dependent on the definition of $T[n]$ for $n \geqslant 0$. If we started with a different definition of $T[n]$ for $n \geqslant 0$, then we could produce other functions on $\tilde{J}$ generated by the function T_a. Let $m \in \tilde{J}$ and consider $T(-m)$. We have $T(-m) = T[(-m)^-]^* T[(-m)^+]$. Because of $(-m)^+ = m^-$ and $(-m)^- = m^+$ we get $T(-m) = T[m^+]^* T[m^-] = T(m)^*$.

Suppose that the $T(n)$ defined above, when considered on the group J, has the unitary dilation $U(n)$, $n \in \tilde{J}$. Then.

$$(T[m^+]f, T[m^-]g) = (U(m^+)f, U(m^-)g) \tag{11.6}$$

for all $f, g \in H$. Notice that m^+ and m^- are positive disjoint. On the other hand, if u and v both in $\tilde{J}$ are positive disjoint then for $m = u - v$ formula (11.6) implies

$$T(v)^* T(u) f = P U(v)^* U(u) f, \tag{11.7}$$

$f \in H$, u, v positive disjoint. P stands as usual for the projection of the dilation space onto the initial one. The dilation $U(n)$ which satisfies (11.7) is called the *Sz.-Nagy-Brehmer dilation* of the family T_a (see [15]).

Notice that in general J does not satisfy (R) with $\tilde{J}_+ = G_+$. Consequently the equalities $T(n) = PU(n)P$ valid for $n \in \tilde{J}_+$ do not determine the dilation on all of $\tilde{J}$. Also they do not necessarily imply (11.7).

The positivity condition for $T(n)$ with regard to the group $\tilde{J}$ is equivalent to the statement

$$\sum_{m,n} (T(m-n)g(m), g(n)) \geqslant 0, \quad m, n \in \tilde{J}, \tag{11.8}$$

for every vector-valued function $g(n)$ which takes on only a finite numbers of values different from zero.

The equivalent form of (11.8) is

$$\sum_{i,j=1}^{k} \left(T[(n_i-n_j)^+]g(n_i), T[(n_i-n_j)^-]g(n_j)\right) \geqslant 0,$$

where $n_1, \ldots, n_k$ are all those n for which $g(n) \neq 0$. Let $-q$ be the lower bound of the values of n_i, $i = 1, \ldots, k$. The positivity condition takes on the form

$$\sum_{i,j=1}^{k} \left(T\left[\left((n_i+q)-(n_j+q)\right)^+\right]g'(n_i+q), T\left[\left((n_i+q)-(n_i+q)\right)^-\right]g'(n_j+q)\right) \geqslant 0$$

with $g'(n_i+q) = g(n_i)$. We write $n_i' = n_i+q$ and obtain

$$\sum_{i,j=1}^{k} \left(T[(n_i'-n_j')^+]g'(n_i'), T[(n_i'-n_j')^-]g_j'(n_j')\right) \geqslant 0.$$

Since all n_i' are non-negative and $(n_i'-n_j')^- = (n_j'-n_i')^+$, we can assume without loss of generality that the positivity condition in our case is equivalent to the following one:

$$\sum_{i,j=1}^{k} \left(T[(n_i-n_j)^+]f(n_i), T[(n_i-n_j)^+]f(n_j)\right) \quad 0 \tag{11.9}$$

for all $n_1, \ldots, n_k \geqslant 0$ and $f(n_1), \ldots, f(n_k) \in H$.

It is convenient for technical reasons to work with a certain notationally different form of (11.9). Namely, let $a_1 \quad \ldots \leqslant a_r$ be the union of all sets $\{a|n_i(a) \neq 0\}$, $i = 1, \ldots, k$, and denote by N the integer-valued upper bound of all $n(a)$. The value of n_i at a_s is denoted by i_s and the value of n_j at a_s is denoted by j_s. Now $T[(n_i-n_j)^+]$ may be written as

$$T[(n_i-n_j)^+] = T_{a_1}^{(i_1-j_1)^+} \qquad T_{a_r}^{(i_r-j_r)^+}$$

If we define

$$p_s = p_s(i,j) = \max(0, i_s-j_s) = (i_s-j_s)^+,$$

$$q_s = q_s(i,j) = p_s(j,i),$$

then

$$T[(n_i-n_j)^+] = T_{a_1}^{p_1} T_{a_2}^{p_2} \qquad T_{a_r}^{p_r},$$

$$T[(n_j-n_i)^+] = T_{a_1}^{q_1} T_{a_2}^{q_2} \ldots T_{a_r}^{q_r}.$$

Since n_i are determined by the values $i_1, \ldots, i_r$, $f(n_i)$ depends in fact on the multi-index $i = (i_1, \ldots, i_r)$. Since p_s and q_s are functions of (i,j) and n_i and $f(n_i)$ are arbitrary, the positivity condition takes on the following form.

If r and N are arbitrary positive integers and S stands for the set of all r-tuples $i = (i_1, \ldots, i_r)$ such that $0 \leqslant i_s \leqslant N$ for $s = 1, \ldots, r$, then

$$\sum_{i,j \in S} (T_{\alpha_1}^{p_1(i,j)} \ldots T_{\alpha_r}^{p_r(i,j)} f_i, T_{\alpha_1}^{q_1(i,j)} \ldots T_{\alpha_r}^{q_r(i,j)} f_j) \geqslant 0 \tag{11.10}$$

for arbitrary vectors f_k and fixed but arbitrary $\alpha_1 \leqslant \ldots \leqslant \alpha_r$.

Given N and r there are $(N+1)^r$ functions n_i on $\alpha_1, \ldots, \alpha_r$ such that $0 \leqslant n_i(\alpha_s) = i_s \leqslant N$. In matrix terms condition (11.10) can be expressed as the positivity of an operator-valued matrix $\overline{T}$ with entries

$$T_{ij} = (T_r^*)^{p_r} \ldots (T_1^*)^{p_1} T_1^{q_1} \ldots T_r^{q_r} \tag{11.11}$$

acting in the orthogonal sum $\overline{H}_r$ of $(N+1)^r$ copies of H, the vectors in $\overline{H}_r$ being sequences with components f_i, $i = (i_1, \ldots, i_r)$. We write here for convenience $T_{\alpha_s} = T_s$.

Now let $u = (u_1, \ldots, u_r)$ and

$$0 \leqslant u_s \leqslant 1, \quad 0 \leqslant s \leqslant r. \tag{11.12}$$

Identifying u with an $n \in \tilde{J}$ such that $n(\alpha_s) = u_s$ we write $T(u)$ in place of $T(n)$.

We write

$$P(T_1, \ldots, T_r) = \sum_u (-1)^{a(u)} (T(u))^* T(u), \tag{11.13}$$

where $a(u) = \sum_{s=1}^{r} u_s$ and u_s satisfy (11.12).

We did not require that T_α should commute. If T_α do commute but J is not linearly ordered, then, since in all the formulae we deal with a finite number of operators, any linear order in J will be admissible. In other words, if T_α commute and J is arbitrary, then we may always assume that J is linearly ordered in any way.

We will now prove the following lemma (see [15]):

LEMMA 11.1. *Suppose that T_α $(\alpha \in J)$ are commuting contractions. Then the matrix $\overline{T}$ with entries defined by* (11.11) *is of the form*

$$T = W^* D W, \tag{11.14}$$

where D is a certain diagonal operator-valued matrix.

Proof. In what follows $i, j, k,$ stand for r-tuples $(i_1, \ldots, i_r)$. We define the entries of W by

$$W_{ij} = \begin{cases} T_1^{j_1 - i_1} T_2^{j_2 - i_2} \ldots T_r^{j_r - i_r} & \text{if} \quad i_s \leqslant j_s \text{ for } s = 1, \ldots, r, \\ 0 & \text{otherwise,} \end{cases}$$

and the entries of D_{ij} are $D = 0$ if $i \neq j$, $D_{ij} = P(T_{s_1}, \ldots, T_{s_k})$ if $i = j$ and $s_1, \ldots, s_k$ are all such s that $i_s > 0$. $P(T_1, \ldots, T_h)$ is defined by (11.13). An easy induction proves that, for $r > 1$,

$$P(T_1, \ldots T_r) = P(T_1 \quad T_{r-1}) - T_r^* P(T_1 \ldots T_{r-1}) T_r. \tag{11.15}$$

The theorem will be proved by induction with respect to r.

Formula (4.15) of section 4 proves that the factorization holds true for $r = 1$ with a suitable diagonal matrix D. Suppose that the factorization holds for $r-1$. Notice that $\bar{H}_r$ may be considered as a direct sum of $N+1$ copies of $\bar{H}_{r-1}$. To every i_r ($i_r = 0, 1, \ldots, N$) there corresponds a copy of $\bar{H}_{r-1}$ composed of those $f(i_1, \ldots, i_r)$ for which i_r is fixed and equal to one of the integers $0, 1, \ldots, N$. Then the matrices $\bar{T}$, W^*, D, W may be considered as $(N+1) \times (N+1)$ matrices with entries which are the operators on $\bar{H}_{r-1}$. Let $\bar{T}_r$, W_r and D_r refer to these matrices and $\bar{T}_{r-1}$, W_{r-1} and D_{r-1} refer to a similar situation for $r-1$ in place of r. Then

$$(\bar{T}_r)_{s,t} = \begin{cases} (T_r^*)^{s-t} \bar{T}_{r-1} & \text{if} \quad s \geqslant t, \\ \bar{T}_{r-1} T_r^{t-s} & \text{if} \quad s < t; \end{cases}$$

$$(W_r)_{s,t} = \begin{cases} W_{r-1} T_r^{t-s} & \text{if} \quad s \leqslant t, \\ 0 & \text{if} \quad s > t; \end{cases}$$

$$(D_r)_{s,t} = \begin{cases} D_{r-1} & \text{if} \quad s = t = 0, \\ D_{r-1} - T_r^* D_{r-1} T_r & \text{if} \quad s = t > 0, \\ 0 & \text{if} \quad s \neq t. \end{cases}$$

Since D is a diagonal matrix with hermitian symmetric diagonal elements, the product $W_r^* D_r W_r$ is hermitian symmetric. $\bar{T}_r$ is hermitian symmetric. It follows that it is sufficient to verify that the (s, t) entries of these matrices are equal for $s \leqslant t$.

Note now that $(W_r^*)_{s,h} = (W_r)_{h,s}^* = 0$ if $h > s$. Now, for $s \leqslant t$

$$(W_r^* D_r W_r)_{s,t} = \sum_{h=0}^{s} (W_r^*)_{s,h} (D_r)_{h,h} (W_r)_{h,t} \tag{11.16}$$

$$= (W_r^*)_{s,0} D_{r-1} (W_r)_{0,t} + \sum_{h=1}^{s} (W_r^*)_{s,h} (D_{r-1} - T_r^* D_{r-1} T_r) (W_r)_{h,t}$$

$$= (T_r^*) W_{r-1}^* D_{r-1} W_{r-1} T_r^t +$$

$$+ \sum_{h=1}^{s} \left((T_r^*)^{s-h} W_{r-1}^* D_{r-1} W_{r-1} T_r^{t-h} - (T_r^*)^{s-h} W_{r-1}^* T_r^* D_{r-1} T_r W_{r-1} T^{t-h} \right).$$

Since the entries of W_{r-1} are products of $T_1, \ldots, T_{r-1}$ and T_r commutes with the factors of these products, it follows that T_r commutes with the entries of W_{r-1}. By the inductive assumption $W_{r-1}^* D_{r-1} W_{r-1} = \overline{T}_{r-1}$. Hence, by (11.16),

$$(11.17) \qquad (W_r^* D_r W_r)_{s,t} = (T_r^*)^s \overline{T}_{r-1} T_r^t + \sum_{h=1}^{s} \left((T_r^*)^{s-h} \overline{T}_{r-1} T_r^{t-h} - (T_r^*)^{s-h+1} \overline{T}_{r-1} T_r^{t-h+1} \right) = (T_r^*)^0 \overline{T}_{r-1} T_r^{t-s} = (\overline{T}_r)_{s,t},$$

which completes the proof.

We will now prove (see [15])

THEOREM 11.1. *Suppose that T_α $(\alpha \in J)$ are commuting contractions. Then the positivity condition* (11.8) *holds true if and only if $P(T_1, \ldots, T_r)$ is positive for every $r \geqslant 1$.*

Proof. Using lemma 11.1. we infer that (11.8) holds if and only if W^*DW is positive. Since D is a diagonal matrix with entries $P(T_1, \ldots, T_r)$, the positivity of these diagonal elements implies the positivity of the product W^*DW, which establishes the proof of sufficiency.

To prove the necessity we totally order the set of r-tuples i, j as follows: $i \leqslant j$ means that $i_1 = j_1, i_2 = j_2, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $s = 1, \ldots, r$.

With this order of multi-indices the matrix W becomes an upper semi-diagonal matrix that is $W_{ij} = 0$ if $i \leqslant j$. Moreover, the diagonal elements are identities. It is a simple matter to verify that such a matrix has a right-hand inverse W^{-1}. Now, if $\hat{f} \in \overline{H}_r$, then

$$(D\hat{f}, \hat{f}) = (DWW^{-1}\hat{f}, WW^{-1}\hat{f}) = \left(W^*DW(W^{-1}\hat{f}), W^{-1}\hat{f}\right).$$

Thus, if W^*DW is positive, then so does D, q. e. d.

Theorem 11.1 generalizes a series of results obtained in [3] and [42].

We say that T_α *double commutes with* T_β $(\alpha \neq \beta)$ if $T_\alpha T_\beta = T_\beta T_\alpha$ and $T_\alpha T_\beta^* = T_\beta^* T_\alpha$. We say that $\{T_\alpha\}$ *double commute* if T_α double commutes with every $\beta \neq \alpha$, for every α.

Suppose now that T_1 double commutes with $T_2, \ldots, T_r$. Then T_1 commutes with $P(T_2, \ldots, T_r)$. Hence

$$\begin{aligned} P(T_1, \ldots, T_r) &= P(T_2, \ldots, T_r) - T_1^* T_1 P(T_2, \ldots, T_r) \\ &= (I - T_1^* T_1) P(T_2, \ldots, T_r). \end{aligned}$$

If T_1 is a contraction, then $(I - T_1^* T_1) \geqslant 0$ and commutes then with $P(T_2, \ldots, T_r)$. It follows that if $P(T_2, \ldots, T_r) \geqslant 0$, then also $P(T_1, \ldots, T_r) \geqslant 0$. We use here the property that the product of symmetric positive commuting operators is positive. A simple proof of this last

statement, valid for real, complex and quaternionic Hilbert spaces was given by F. Riesz in [31].

Using induction on r one proves by applying theorem 11.1 that the following theorem holds true (see [42]):

THEOREM 11.2. *Suppose that T_a $(a \in I)$ is a doubly commuting family of contractions. Then T_a has the Sz.-Nagy-Brehmer unitary dilation.*

Suppose now that all T_a are commuting isometries. Then $T_a^* T_a = I$ $(a \in J)$ and consequently

$$P(T_1, \ldots, T_r)$$

$$= \sum_{u} \pm (T_r^*)^{u_r} \ldots (T_2^*)^{u_2} T_1^* T_1 T_2^{u_2} \quad T_r^{u_r} - (T_r^*)^{u_r} \quad (T_2^*)^{u_2} T_2^{u_2} \quad T_r^{u_r}$$

$$= \sum \pm (0) = \pm 0.$$

Hence $P(T_1, \ldots, T_r) = 0$ in this case.

If T_a are commuting contractions such that

$$\sum_{n=1}^{\infty} |T_{a_n}|^2 \quad I \tag{11.18}$$

for an arbitrary sequence of indices, then $P(T_1, \ldots, T_r) \quad 0$ for every finite collection $T_1, \ldots, T_r$ of elements of the family.

Indeed (11.8) implies that $P(T_1, \ldots, T_r) \leqslant I$ and so $T_{r+1}^* P(T_1, \ldots, T_r) T_{r+1} \leqslant T_{r+1}^* T_{r+1}$. Hence

$$P(T_1, \ldots, T_{r+1}) = P(T_1, \ldots, T_r) - T_{r+1}^* P(T_1, \ldots, T_r) T_{r+1} \tag{11.19}$$

$$\geqslant P(T_1, \ldots, T_r) - T_{r+1}^* T_{r+1}.$$

Observe that $0 \leqslant I - T_1^* T_1 = P(T_1) \leqslant I$ provided T_1 is a contraction. Suppose that $I - \sum_{i=1}^{r} T_i^* T_i \leqslant P(T_1, \ldots, T_r)$. Then

$$I - \sum_{i=1}^{r+1} T_i^* T_i \leqslant P(T_1, \ldots, T_r) - T_{r+1}^* T_{r+1}.$$

By (11.19) we get

$$I - \sum_{i=1}^{r+1} T_i^* T_i \leqslant P(T_1, \ldots, T_{r+1}).$$

It follows from (11.18) that $0 \leqslant I - \sum_{i=1}^{r+1} T_i^* T_i$. We have just proved by induction on r that (11.18) implies $P(T_1, \ldots, T_r) \geqslant 0$ for every r.

Summing up and using theorem 11.1 we get

THEOREM 11.3 (see [3]). *Suppose that T_a are commuting contractions. Each of the conditions below is sufficient for the existence of the Sz.-Nagy-Brehmer unitary dilations of T_a:*

(11.20) *All T_a are isometries* (1).

(11.21) *For every sequence $T_1, \ldots, T_n$ of contractions of the family T_a the inequality $\sum_{i=1}^{n} |T_i|^2 \leqslant 1$ holds true.*

Remark. It may be shown (see [15] and [44]) that the positivity condition (11.8) is not affected by discarding from $\{T_a\}$ isometries commuting with all the other elements of $\{T_a\}$ or by discarding contractions which double commute with other contractions of $\{T_a\}$.

A considerable part of theorems known for dilations of a single contraction can be extended to Sz.-Nagy-Brehmer dilations. We mention here the geometrical interpretation of condition (11.8) and the conditions which ensure that Sz.-Nagy-Brehmer dilations are bilateral shifts (see [16]). For the function-theoretical significance of Sz.-Nagy-Brehmer dilations see [23]. For unitary dilations of more then one contraction see also [1], where the dilations are not necessarily of the Sz.-Nagy-Brehmer type.

(1) See also [19].

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