ANNALES
POLONICI MATHEMATICI
XVII (1966)

On the convergence of successive approximations in the Darboux problem

J. S. W. Wong (Edmonton)

1. Introduction. Recently B. Palczewski [6] has proved the convergence of successive approximations in the Darboux problem under the uniqueness conditions of Krasnosielski and Krein type ([1]). These uniqueness conditions have been generalized together with other conditions for uniqueness by the present author [8]. In the present paper, we wish to show that the above-mentioned conditions guarantee not only the uniqueness of solutions of the Darboux problem but also the convergence of successive approximations. Instead of following the usual method of proving convergence of successive approximations, it will be shown that these results all follow as a consequence of certain general theorems concerning mappings defined on some appropriate function spaces. This approach was first initiated in a paper by W. A. J. Luxemburg [4].

2. Two theorems on contractions. Let X be an abstract set with elements $x, y, z, \ldots$; and let $d(x, y)$ be a non-negative real valued function ($0 \leqslant d(x, y) \leqslant \infty$), defined on the Cartesian product $X \times X$ and satisfying:

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) $d(x, y) = d(y, x)$,

(D3) $d(x, y) \leqslant d(x, z) + d(z, y)$,

(D4) Every d-Cauchy sequence converges to a limit in X, i.e. $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$ implies the existence of an element $x \in X$ such that $\lim\limits_{n\to\infty} d(x, x_n) = 0$.

An abstract set X on which such a distance function is defined is called *generalized complete metric space*. It differs from the usual concept of complete metric space by the fact that not every pair of elements necessarily has a finite distance.

([1]) These uniqueness conditions are first given in [7].

THEOREM 1 (Luxemburg [4]). *Let X be a generalized complete metric space, and T a mapping defined on X into itself satisfying the following conditions:*

(i) *There exists a constant λ, $0 < \lambda < 1$, such that*

$$d(Tx, Ty) \leqslant \lambda d(x, y)$$

for all $x, y \in X$ with $d(x, y) < \infty$.

(ii) *For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+l}) < \infty$ for all $l = 1, 2, \ldots$*

(iii) *If x and y are two fixed points of T, i.e. $Tx = x$ and $Ty = y$ then $d(x, y) < \infty$.*

Then the equation $Tx = x$ has one and only one solution, and every sequence of successive approximations $\{x_n\}$ converges in distance to this unique solution.

THEOREM 2 (Edelstein [2]). *Let X be a complete metric space and T a mapping defined on X into itself satisfying the following conditions:*

(i) *For all $x, y \in X$, $x \neq y$, we have*

$$d(Tx, Ty) < d(x, y).$$

(ii) *For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists a subsequence which converges to a point $x \in X$.*

Then, the equation $Tx = x$ has one and only one solution, and every sequence of successive approximations $\{x_n\}$ converges in distance to this unique solution.

3. The Darboux problem. Let D denote the rectangle: $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ $(a, b > 0)$ and $E = D \times \{-\infty < u < \infty\}$. We are here concerned with the solutions of the following partial differential equation:

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u) \tag{1}$$

where $f(x, y, u)$ is defined and continuous over E, and the solutions $u(x, y)$ satisfy the conditions that: $u(x, 0) = \sigma(x)$ and $u(0, y) = \tau(y)$. Moreover, we assume that the functions $\sigma(x)$ and $\tau(y)$ are of the class C^1 satisfying the condition $\sigma(0) = \tau(0)$ and defined respectively on $[0, a]$ and $[0, b]$.

The Darboux problem is equivalent to solve the following integral equation:

$$u(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\big(s, t, u(s, t)\big)\, ds\, dt \tag{2}$$

where $\varphi_0(x, y) = \sigma(x)+\tau(y)-\sigma(0)$. In term of (2), the sequence of successive approximations of Picard is then defined by:

$$\varphi_{n+1}(x, y) = \varphi_0(x, y)+\int_0^x\int_0^y f\big(s, t, \varphi_n(s, t)\big)\,ds\,dt \tag{3}$$

for $n = 0, 1, 2, \ldots$, where φ_0 is any function defined on D continuous with $(\varphi_0)_x$, $(\varphi_0)_y$ and $(\varphi_0)_{xy}$ which also satisfies $\varphi(x, 0) = \sigma(x)$ and $\varphi(0, y) = \tau(y)$ for $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$ respectively. The set of functions satisfying the above mentioned properties will be denoted by $C^*(D)$.

4. The generalized Krasnosielski and Krein condition.

THEOREM 3. *If $f(x, y, u)$ is defined, continuous and bounded on E, and it satisfies in addition the following*:

$$|f(x, y, u_1)-f(x, y, u_2)| \leqslant \frac{k}{xy}|u_1-u_2|, \qquad k > 0, \tag{4}$$

$$|f(x, y, u_1)-f(x, y, u_2)| \leqslant \frac{C}{x^\beta y^\beta}|u_1-u_2|^a, \qquad C > 0 \tag{5}$$

with $0 < a < 1$, $\beta < a$, and $k(1-a)^2 < (1-\beta)$ for all $(x, y, u) \in E$, then there exists one and only one solution $u(x, y)$ of the Darboux problem for equation (3), *and furthermore the sequence of successive approximations of Picard, as defined by* (3) *for any function φ_0 in $C^*(D)$, converges uniformly on D to this unique solution.*

We now show how Theorem 1 may be applied to prove the above result. For this purpose we have to construct a suitable complete metric space X and a mapping T from X into itself, and prove that conditions (i), (ii), (iii) of Theorem 1 are indeed satisfied. A natural choice for X is the space $C^*(D)$ and for every pair of elements $\varphi_1, \varphi_2 \in X$, we define the following distance function on $X \times X$:

$$d(\varphi_1, \varphi_2) = \sup_D \frac{|\varphi_1(x, y)-\varphi_2(x, y)|}{(xy)^{p\sqrt{k}}} \tag{6}$$

where $p > 1$ and $pk(1-a)^2 < (1-\beta)^2$, which is possible since we always have $k(1-a)^2 < (1-\beta)^2$. Clearly this function $d(\varphi_1, \varphi_2)$ satisfies the requirements (D1), (D2), (D3) for a metric. Moreover since

$$(ab)^{-p\sqrt{k}}d_1(\varphi_1, \varphi_2) \leqslant d(\varphi_1, \varphi_2) \tag{7}$$

where $d_1(\varphi_1, \varphi_2) = \operatorname*{Max}_D|\varphi_1(x, y)-\varphi_2(x, y)|$ denotes the metric of uniform convergence; it follows that d-convergence implies uniform convergence. To show that condition (D4) is also satisfied for $d(\varphi_1, \varphi_2)$, we let $\varphi_n \in X$, $n = 1, 2, \ldots$ be a d-Cauchy sequence, i.e. $\lim_{n,m\to\infty} d(\varphi_n, \varphi_m) = 0$. Hence there exists a subsequence ψ_n such that $d(\psi_{n+1}, \psi_n) \leqslant 2^{-n}$. Now since X

is complete under the uniform metric, therefore the subsequence $\{\psi_n\}$ converges to some element $\varphi \in X$. The fact that $\varphi - \psi_n = \sum_{k=n}^{\infty} (\psi_{k+1} - \psi_k)$ and hence $d(\varphi, \psi_n) \leqslant 2^{-n+1}$ implies $d(\varphi, \varphi_n) \leqslant \lim_{k\to\infty} d(\varphi, \psi_k) + \lim_{k\to\infty} d(\psi_k, \varphi_n)$ which tends to zero as n tends to infinity, proving (D4).

The natural choice for the mapping T is the following:

$$T\varphi(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\big(s, t, \varphi(s, t)\big)\, ds\, dt \tag{8}$$

which is easily seen to be a mapping of X into itself. Furthermore, the solution of the Darboux problem in its equivalent form (2) corresponds to the fixed point of T and conversely. For an arbitrary element $\varphi_0 \in C^*(D)$, the successive approximations of Picard as defined by (3), are simply the sequence $\{\varphi_n : \varphi_n = T\varphi_{n-1},\ n = 1, 2, \ldots\}$.

Proof of (i). Let φ_1, φ_2 be two arbitrary elements of X. Then by (4) we obtain:

$$\begin{aligned} |T\varphi_1 - T\varphi_2| &\leqslant \int_0^x \int_0^y \big| f\big(s, t, \varphi_1(s, t)\big) - f\big(s, t, \varphi_2(s, t)\big)\big|\, ds\, dt \\ &\leqslant k \int_0^x \int_0^y \frac{|\varphi_1(s, t) - \varphi_2(s, t)|}{st}\, ds\, dt \\ &= k \int_0^x \int_0^y (st)^{p\sqrt{k}-1} \frac{|\varphi_1(s, t) - \varphi_2(s, t)|}{(st)^{p\sqrt{k}}}\, ds\, dt\,. \end{aligned}$$

If $d(\varphi_1, \varphi_2) < \infty$, then we conclude

$$|T\varphi_1 - T\varphi_2| \leqslant d(\varphi_1, \varphi_2) \frac{(st)^{p\sqrt{k}}}{p}$$

and hence by definition of d, $d(T\varphi_1, T\varphi_2) \leqslant \lambda d(\varphi_1, \varphi_2)$ where $\lambda = 1/p$, proving (i).

Proof of (ii). Let $M = \sup_E |f(x, y, u)|$, and $\varphi_n = T\varphi_{n-1}$, $n = 1, 2, \ldots$, where φ_0 is an arbitrary element in X. We obtain from (2) that

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqslant 2Mxy\,. \tag{9}$$

It follows from (9) that:

$$\begin{aligned} |\varphi_3(x, y) - \varphi_2(x, y)| &\leqslant \int_0^x \int_0^y \big| f\big(s, t, \varphi_2(s, t)\big) - f\big(s, t, \varphi_1(s, t)\big)\big|\, ds\, dt \\ &\leqslant C \int_0^x \int_0^y \frac{(2M)^\alpha x^\alpha y^\alpha}{x^\beta y^\beta}\, dx\, dy \\ &\leqslant C(2M)^\alpha (xy)^{(1-\beta)+\alpha} \end{aligned}$$

and successively we obtain:

$$(10)\quad |\varphi_{n+3}(x, y)-\varphi_{n+2}(x, y)| \leqslant C^{1+a+\ldots+a^n}(2M)^{a^{n+1}}(xy)^{(1-\beta)(1+a+\ldots+a^n)+a^{n+1}} .$$

By hypothesis: $pk(1-a)^2 < (1-\beta)^2$, hence there exists a $N(p)$ such that $n \geqslant N(p)$ we have $(1-\beta)(1+a+\ldots+a^n)+a^{n+1} > p\sqrt{k}$. This shows in particular that $d(\varphi_{n+1}, \varphi_n) < +\infty$ for $n \geqslant N(p)+2$.

Proof of (iii). Assume that both $u_1, u_2 \in X$ are fixed points of T, i.e. $Tu_1 = u_1$ and $Tu_2 = u_2$. Using the argument just presented for equation (10), we conclude easily $d(u_1, u_2) < \infty$.

After these verifications on conditions (i), (ii), and (iii) of Theorem 1 the conclusion of Theorem 3 follows immediately from Theorem 1.

5. The generalized Nagumo-Perron-Van Kampen condition.

THEOREM 4. *If $f(x, y, u)$ is defined and continuous on E, and it satisfies in addition the following*:

$$(11)\qquad |f(x, y, u)| \leqslant A(xy)^p , \quad p > -1 , \ A > 0 ,$$

$$(12)\qquad |f(x, y, u_1)-f(x, y, u_2)| \leqslant \frac{C}{(xy)^r}|u_1-u_2|^q , \quad q \geqslant 1 , \ C > 0$$

where

$$q(1+p)-r = p , \qquad \varrho = \frac{C(2A)^{q-1}}{(p+1)^q} < 1$$

for all $(x, y, u) \in E$. Then there exists one and only one solution $u(x, y)$ of the Darboux problem for equation (3), *and furthermore the sequence of successive approximations of Picard $\{\varphi_n\}$ as defined by* (3) *converges uniformly on D to this unique solution.*

The proof of this result follows exactly the same pattern as that of Theorem 3. Here we choose the same space $C^*(D)$ for X and same mapping T as defined by (8). In this case, we define the distance function $d(\varphi_1, \varphi_2)$ on $X\times X$ by:

$$(13)\qquad d(\varphi_1, \varphi_2) = \sup_D \frac{|\varphi_1(x, y)-\varphi_2(x, y)|}{(xy)^{p+1}} .$$

Following the same procedures as in the previous section, it can be easily shown that d satisfies the conditions (D1), (D2), (D3), (D4) for a complete metric.

Proof of (i). Let φ_1, φ_2 be two arbitrary elements of X. Then from (2) and (11) we obtain:

$$(14)\qquad |u(x, y)-v(x, y)| \leqslant \frac{2A}{p+1}(xy)^{p+1} .$$

Moreover, by (12) and (14) it follows:

$$|T\varphi_1(x,y)-T\varphi_2(x,y)| \leqslant C\int_0^x\int_0^y \frac{|\varphi_1(s,t)-\varphi_2(s,t)|^q}{(st)^r}\,ds\,dt$$

$$\leqslant Cd(\varphi_1,\varphi_2)\int_0^x\int_0^y (st)^{p+1-r}|\varphi_1(s,t)-\varphi_2(s,t)|^{q-1}\,ds\,dt$$

$$\leqslant Cd(\varphi_1,\varphi_2)\left(\frac{2A}{p+1}\right)^{q-1}\int_0^x\int_0^y (st)^{p+1-r+(p+1)(q-1)}\,ds\,dt$$

$$= Cd(\varphi_1,\varphi_2)\left(\frac{2A}{p+1}\right)^{q-1}\frac{(xy)^{p+1}}{p+1}$$

$$= \varrho d(\varphi_1,\varphi_2)(xy)^{p+1}.$$

Hence by the definition of d we have $d(T\varphi_1, T\varphi_2) \leqslant \varrho d(\varphi_1,\varphi_2)$.

Proof of (ii) and (iii). The proofs are trivial in this case, as the required estimate is already given in (14).

6. The classical Nagumo condition.

THEOREM 5. *If $f(x,y,u)$ is defined continuous and bounded on E, and it satisfies the following:*

$$|f(x,y,u_1)-f(x,y,u_2)| \leqslant \frac{k}{xy}|u_1-u_2| \tag{15}$$

with $k \leqslant 1$ for all $(x,y,u)\in E$, then there exists one and only one solution of the Darboux Problem for equation (3) and furthermore the sequence of successive approximations $\{\varphi_n\}$ as defined by (3) converges uniformly on D to this unique solution.

We shall apply the result of Theorem 2 to prove the above statement. To do this, we again choose the space $C^*(D)$ as the underlying space X and T is the mapping defined by (8). However, in this case, the distance function $d(\varphi_1,\varphi_2)$ will be defined by:

$$d(\varphi_1,\varphi_2) = \sup_D \frac{|\varphi_1(x,y)-\varphi_2(x,y)|}{xy}. \tag{16}$$

Clearly d satisfies all four conditions (D1), (D2), (D3), (D4) for a complete metric, provided that $d(\varphi_1,\varphi_2)<\infty$ for any pair of elements $\varphi_1,\varphi_2\in X$. Since $f(x,y,u)$ is continuous, we note that $\big|f(x,y,u(x,y))-f(x,y,v(x,y))\big| \leqslant M_{xy}$ where M_{xy} is bounded on D and tends to zero as x or y tends to zero, or both. Therefore it follows easily that $d(\varphi_1,\varphi_2)$ is finite. To obtain the desired conclusion of the above theorem we need only to check that conditions (i) and (ii) of Theorem 2 are satisfied for T.

Proof of (i). Let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ two distinct functions from $C^*(D)$ and define $B(x, y) = \frac{|\varphi_1(x, y) - \varphi_2(x, y)|}{xy}$. From the previous argument, we have shown that $B(x, y)$ is continuous over D and hence attains its maximum at some point $(x_0, y_0) \in D$, i.e. $B(x_0, y_0) = d(\varphi_1, \varphi_2)$. Consider the following estimate:

$$(17)\quad |T\varphi_1(x, y) - T\varphi_2(x, y)| \leqslant \int_0^x \int_0^y \frac{|\varphi_1(s, t) - \varphi_2(s, t)|}{st}\, ds\, dt \leqslant d(\varphi_1, \varphi_2)xy .$$

From the definition of d we obtain $d(T\varphi_1, T\varphi_2) \leqslant d(\varphi_1, \varphi_2)$. We claim however that the equality cannot occur in (17), i.e. we must have $d(T\varphi_1, T\varphi_2) < d(\varphi_1, \varphi_2)$. Assume contrary, then there exists points (x_0, y_0), $(x_1, y_1) \in D$ such that

$$d(\varphi_1, \varphi_2) = \frac{|\varphi_1(x_0, y_0) - \varphi_2(x_0, y_0)|}{x_0 y_0} = \frac{|T\varphi_1(x_1, y_1) - T\varphi_2(x_1, y_1)|}{x_1 y_1} = d(T\varphi_1, T\varphi_2) .$$

It follows from (8) and (2) that

$$d(T\varphi_1, T\varphi_2) \leqslant \frac{1}{x_1 y_1} \int_0^{x_1} \int_0^{y_1} \frac{|\varphi_1(s, t) - \varphi_2(s, t)|}{st}\, ds\, dt .$$

Since $\varphi_1(0, y) = \varphi_2(0, y)$ and $\varphi_1(x, 0) = \varphi_2(x, 0)$, we obtain

$$d(\varphi_1, \varphi_2) < \frac{1}{x_1 y_1} d(\varphi_1, \varphi_2) \int_0^{x_1} \int_0^{y_1} ds\, dt = d(\varphi_1, \varphi_2)$$

which is the desired contradiction.

Proof of (ii). Let $\varphi(x, y)$ be an arbitrary function of $C^*(D)$, and we consider the family of iterates $\{T^n\varphi\}$. Since $f(x, y, u)$ is continuous and bounded over E, we can easily show that the sequence of iterates form a family of uniformly bounded equi-continuous functions. Hence as a consequence of Ascoli's Theorem ([1], p. 5), there exists a subsequence $\{T^{n_i}\varphi\}$ which converges in the sense of uniform metric d_1 to some element $\bar{\varphi}(x, y) \in C^*(D)$.

Observe for $\varepsilon > 0$, there exists (on account of the continuity of f) $\delta > 0$, such that for all i we have $\sup\limits_{E} \frac{|T^{n_i}\varphi(x, y) - \bar{\varphi}(x, y)|}{xy} < \varepsilon$, where $E = \{(x, y) : (x, y) \in D$ and either $0 \leqslant x \leqslant \delta$ or $0 \leqslant y \leqslant \delta\}$. Next, we may choose $N(\delta)$ such that for $i \geqslant N(\delta)$, $\sup\limits_{D-E} |T^{n_i}\varphi - \bar{\varphi}| < \delta^2\varepsilon$. Hence we obtain:

$$d(T^{n_i}\varphi, \bar{\varphi}) \leqslant \mathrm{Max}\left\{\sup_{E} \frac{|T^{n_i}\varphi - \bar{\varphi}|}{xy};\ \sup_{D-E} \frac{|T^{n_i}\varphi - \bar{\varphi}|}{xy}\right\} \leqslant \varepsilon$$

for all $i \geqslant N(\delta)$. Therefore $d(T^{n_i}\varphi, \bar{\varphi}) \to 0$ as $i \to \infty$.

7. Remarks.

(i) The statement originally given by M. Edelstein in [2] is somewhat different from Theorem 2. However the modification is evident.

(ii) By taking $\beta = 0$ in Theorem 3, we obtain the result of B. Palczewski [6].

(iii) In case that the rectangle D degenerates into an interval, say $[0, a]$, Theorem 3 and 4 reduce to the corresponding result for ordinary differential equations [4], [5]. Similarly, Theorem 5 reduces to the classical result of van Kampen [3].

(iv) Our results not only prove the uniqueness of solutions and convergence of successive approximations, but also the existence of solutions to the Darboux problem.

Acknowledgement. The author wishes to express his sincere thanks to Professor W. A. J. Luxemburg for introducing him to this subject and for several helpful discussions.

References

[1] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, New York 1955.

[2] M. Edelstein, *On fixed and periodic points under contractive mappings*, Journal London Math. Soc. 37 (1962), pp. 74-79.

[3] E. R. van Kampen, *Notes on systems of ordinary differential equations*, American Journal of Math. 63 (1941), pp. 371-376.

[4] W. A. J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations II*, Indag. Math. 20 (1958), pp. 540-546.

[5] — *On the convergence of successive approximations in the theory of ordinary differential equations III*, Nieuw Archief voor Wiskunde (3), VI (1958), pp. 93-98.

[6] B. Palczewski, *On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem under the conditions of the Krasnosielski and Krein type*, Ann. Polon. Math. 14 (1964), pp. 183-190.

[7] B. Palczewski and W. Pawelski, *Some remarks on the uniqueness of solutions of the Darboux problem with conditions of the Krasnosielski-Krein type*, Ann. Polon. Math. 14 (1964), pp. 97-100.

[8] J. S. W. Wong, *Remarks on the uniqueness theorem of solutions of the Darboux problem* Canad, Math. Bull. (to appear).

Reçu par la Rédaction le 30. 11. 1964